An Introduction to Hyperbolic Geometry
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Lecture – 16
Properties of Commuting Elements in PSL₂($\mathbb{R}$)

Hello, in our previous lectures, we explored the concept of quotient groups and their actions on the hyperbolic plane. In today's lecture, we will shift our focus to the properties of commuting elements within the group PSL(2, R). This topic will deepen our understanding of group dynamics in hyperbolic geometry, revealing interesting relationships between elements. So, let us begin.

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First, we will prove this lemma. This lemma states the following: Suppose we have two elements T and S in PSL(2, R) that commute with each other, meaning that S ∘ T = T ∘ S. We aim to demonstrate that S maps the fixed points of T to the fixed points of T itself. In other words, the fixed points of T are invariant under the mapping by S. The proof is quite straightforward.

Let's denote α as a fixed point of T. By definition, this means T(α) = α. Now, if we apply S to this equation, we get ST(α) = S(α). Since we know ST = TS, we can also write this as T(S(α)) = S(α).

This implies that S(α) is also a fixed point of T. Thus, S(α) belongs to the fixed point set of T, which leads us to conclude that the image of the fixed points of T under S is a subset of the fixed points of T.

Since both fixed point sets are finite, we note that the fixed point set of T can contain at most two points. This is because both T and S are Möbius transformations, and hence bijections. Therefore, the cardinality of S applied to the fixed points of T is equal to the cardinality of the fixed point set of T. This gives us the desired equality, thus proving the lemma.

Now, let's move on to the theorem. We will consider two non-identity elements of PSL(2, R) and show that if they commute with each other, they share the same fixed point sets. Furthermore, we will prove the converse: if the fixed point sets of two elements in PSL(2, R) are the same, then they commute with each other.

Assuming S and T are two elements of PSL(2, R) that commute, meaning ST = TS, and that both are non-identity elements, our claim is that the fixed point set of S is the same as the fixed point set of T.

(Refer Slide Time: 04:58)

Now, the fixed point set of T can contain at most two points, so we will consider different cases. First, let's assume the fixed point set of T consists of only a single point, say the singleton set {α}.

Since T(α) = α and S commutes with T, by the lemma we have S(α) = α. Therefore, α is also a fixed point of S.

What we are going to demonstrate is that the cardinality of the fixed point set of S is precisely 1. This means that the fixed point of S is a singleton set. We have already established that α belongs to the fixed point set of S, and now we will prove that the fixed point set of S is solely the singleton set {α}.

Let's suppose there exists another point β in the fixed point set of S such that β is not equal to α. By utilizing the previous lemma, we can show that the image of the fixed point set of S under T equals the fixed point set of S. This implies that T(β) must also belong to the fixed point set of S.

Now, since β is in the fixed point set of S and is distinct from α, and given that S is a Möbius transformation, the cardinality of the fixed point set of S must be less than or equal to 2. Thus, if both α and β belong to the fixed point set of S, we would conclude that the fixed point set of S is exactly the set {α, β}.

However, since T is an isometry, it follows that the image of the fixed point set of S under T remains within the fixed point set of S. Therefore, the transformation T({α, β}) must equal {α, β}. This leads us to the conclusion that the fixed point set of T is a singleton set.

Thus, we have T(α) = α and T(β) cannot equal β, since the fixed point set of T is exactly the singleton set {α}. Consequently, we conclude that T(β) = α. This outcome indicates that T is not a one-to-one map, resulting in a contradiction. Hence, this contradiction confirms that the fixed point set of S is indeed the singleton set {α}, which aligns with the fixed point of T.

Now, let us consider the case where the cardinality of the fixed point set of T equals 2. This means that T fixes two points. If T fixes two points, it must be a hyperbolic isometry. We can identify an isometry F such that FTF^{-1} fixes both 0 and infinity. This approach has been established in previous discussions. Therefore, without loss of generality, we can assume that T fixes the points 0 and infinity, taking the form T(z) = λ z.

Let's reiterate: assuming T is a hyperbolic isometry, it possesses an axis along which this hyperbolic transformation operates. We can utilize some isometry F such that FTF^{-1} is again a

hyperbolic isometry, fixing the imaginary axis. This imaginary axis corresponds to the axis of the conjugate transformation FTF^{-1}, which can be expressed in the form $z \mapsto \lambda z$.

Thus, we will work with this conjugate transformation. Without loss of generality, we can assert that $T(z) = \lambda z$ where $\lambda > 1$. Consequently, the fixed points of T are at 0 and ∞. Since S maps the fixed points of T to the fixed points of T, we recall that we are assuming ST = TS. This implies, according to the lemma, that S applied to the fixed points of T equals the fixed points of T.

Therefore, when we consider the transformation S, we find that S(0, ∞) results in the set {0, ∞} once again. However, a possibility arises: what if S(0) = ∞ and S(∞) = 0? If we assume that S(0) is equal to ∞ and S(∞) equals 0, then we can represent S(z) in the form:

$$S(z) = \frac{az + b}{cz + d}$$

where the condition ad - bc = 1 must hold, as S(0) evaluates to ∞.

If we substitute S(z) = 0 into this form, we find that T must equal 0. Now, if we examine S(∞) = 0, this leads us to conclude that a = 0.

Consequently, S(z) takes the form:

$$S(z) = \frac{b}{cz}$$

with a = 0 and T being zero. Thus, we arrive at the equation -bc = 1, which simplifies to:

$$S(z) = -\frac{b^2}{z}.$$

Now, if we evaluate S(ib), we discover that:

$$S(ib) = ib.$$

This indicates that the fixed point of S is solely the singleton set {ib}. To summarize, we conclude that the fixed point of S is indeed a singleton set. Moreover, from the earlier lemma, we know that T applied to the fixed point of S equals the fixed point of S.

This implies:

$$T(ib) = ib.$$

Thus, we initially assumed that the fixed points of T were 0 and ∞, but we have now derived another distinct complex number ib. This leads to the conclusion that the cardinality of the fixed point set of T exceeds 2, which is not possible. This inconsistency reveals a contradiction stemming from our original assumption.

Hence, if $S(0) = \infty$ and $S(\infty) = 0$, we find that such an outcome cannot hold. Consequently, we conclude that $S(0)$ must be 0 and $S(\infty)$ must equal ∞. This contradiction reinforces our earlier findings: the fixed point of S is indeed the set $\{0, \infty\}$.

In summary, we have established that $S(0) = 0$ and $S(\infty) = \infty$. Therefore, we conclude that the fixed point set of S is $\{0, \infty\}$, which is equal to the fixed point set of T.

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To encapsulate our findings: we considered two non-identity elements, S and T, which commute with each other. Initially, we examined the case where the fixed point of T was a singleton set. In that scenario, T could either be parabolic or elliptic. However, if the fixed point of T consists of two points, we identified that T must be hyperbolic. Ultimately, we have proven that if two

elements commute with each other, they share the same fixed point set.

Now, let us delve into the converse part of our discussion. We will assume that the fixed points of S and T are equal, and we will take both S and T to be non-identity elements. First, let us consider T as a hyperbolic element. Without loss of generality, we can express T(z) in the form:

$$T(z) = \lambda z,$$

where $\lambda > 1$. We know that the fixed points of T are 0 and ∞, and since we have assumed that the fixed points of S are the same as those of T, it follows that the fixed points of S are also 0 and ∞. Given that S is a non-identity element, we can represent S(z) in the form:

$$S(z) = \mu z,$$

for some $\mu \neq 1$ and $\mu > 0$. Thus, we have T(z) = λ z and S(z) = μ z. Now, let's consider the compositions of these two transformations:

$$T \circ S(z) = T\big(S(z)\big) = T(\mu z) = \lambda(\mu z) = \lambda \mu z,$$

and

$$S \circ T(z) = S\big(T(z)\big) = S(\lambda z) = \mu(\lambda z) = \mu \lambda z.$$

Since these two compositions yield the same result, we conclude that:

$$T \circ S = S \circ T.$$

Next, let us assume T to be a parabolic element. Again, without loss of generality, we can express T(z) in the form:

$$T(z) = z + a,$$

where a is a non-zero real number. The fixed point of T is then the singleton set $\{\infty\}$. Since we have assumed that the fixed points of S are the same, we conclude that the fixed point of S must also be the singleton set $\{\infty\}$. As S is a non-identity isometry, we can represent it as:

$$S(z) = z + b,$$

for some b.

Now, if we compute the compositions, we find:

$$T \circ S(z) = T\big(S(z)\big) = T(z + b) = (z + b) + a = z + (a + b),$$

and

$$S \circ T(z) = S\big(T(z)\big) = S(z + a) = (z + a) + b = z + (a + b).$$

Since a + b = b + a, we see that:

$$T \circ S = S \circ T.$$

Finally, if we assume T to be elliptic, we can proceed in a similar fashion. In this case, we express the transformation T(z) in matrix form as:

$$T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Now, we arrive at an important theorem: let f be an element of PSL(2, R), and denote G as PSL(2, R) as well. The centralizer of f in G is defined as the set of all group elements g in G such that:

$$g f = f g.$$

In other words, g and f commute with each other.

Now, if we assume that f is a hyperbolic isometry, its fixed points form a two-point set. If g is an element from the centralizer, then since g and f commute, the fixed points of g must coincide with those of f, indicating that g is also hyperbolic. Consequently, if f is hyperbolic, the centralizer contains all hyperbolic elements.

If we assume f to be parabolic, then the centralizer will include only the parabolic elements. Conversely, if we assume f to be elliptic, the centralizer will encompass all elliptic elements.

I will pause here. We will apply these concepts to demonstrate some algebraic properties of Fuchsian groups in our next discussion.