An Introduction to Hyperbolic Geometry

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Lecture – 15

Properly Discontinuous Actions and Discrete Orbits in Fuchsian Groups

Hello, and welcome to today's lecture. We will continue our discussion on quotient groups, with a particular focus on defining what it means for a group to act properly discontinuously. Our main goal will be to prove that a group is Fuchsian if and only if it acts properly discontinuously on the hyperbolic plane. Additionally, we will demonstrate that a group is Fuchsian if and only if its orbit is discrete. So, let us begin.

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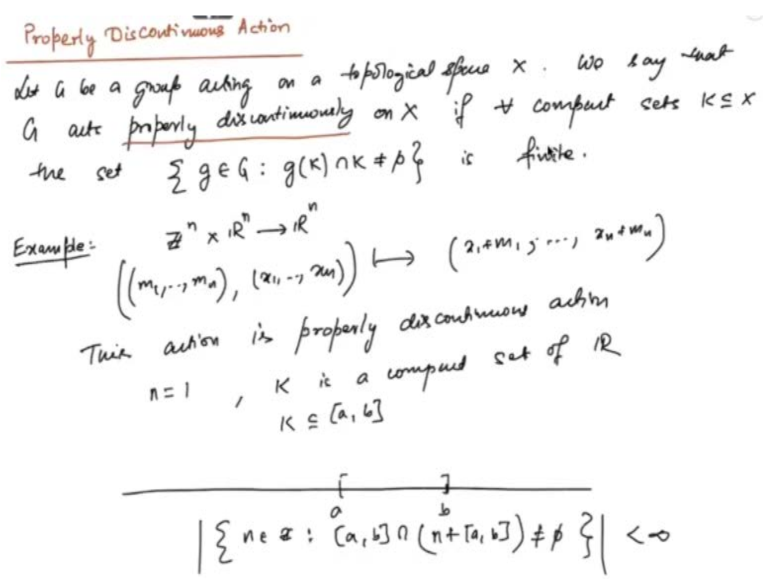

Let's begin by defining what we mean by a properly discontinuous action. Consider a group G acting on a topological space X (although, for the sake of clarity, we could take X to be a metric space as well). We say that G acts properly discontinuously on X if, for any compact set $K \subset X$, the set of group elements $g \in G$ such that the intersection $g(K) \cap K$ is non-empty, is finite. This forms the essence of the definition of a properly discontinuous action.

Importantly, this definition does not require X to be a metric space; it holds more generally for topological spaces. Now, let's explore this concept through an example to understand it better.

Consider the group Z^n acting on R^n by translation. This action is straightforward, it's simply a translation of points in R^n. To check if this action is properly discontinuous, let's take the case when n = 1 and let K be a compact subset of R. So, K will be contained in some closed, bounded interval [a, b].

Now, if we consider the set of integers $n \in Z$ such that the interval [a, b] intersects with its translated counterpart, n + [a, b], this intersection will only be non-empty for finitely many n. Intuitively, only a limited number of translates of the interval [a, b] will overlap with the original interval, resulting in a finite set of overlaps. Consequently, the set of such n's has finite cardinality.

This concept can be easily extended for n > 1 as well. Therefore, we conclude that the action of Z^n on R^n is an example of a properly discontinuous action.

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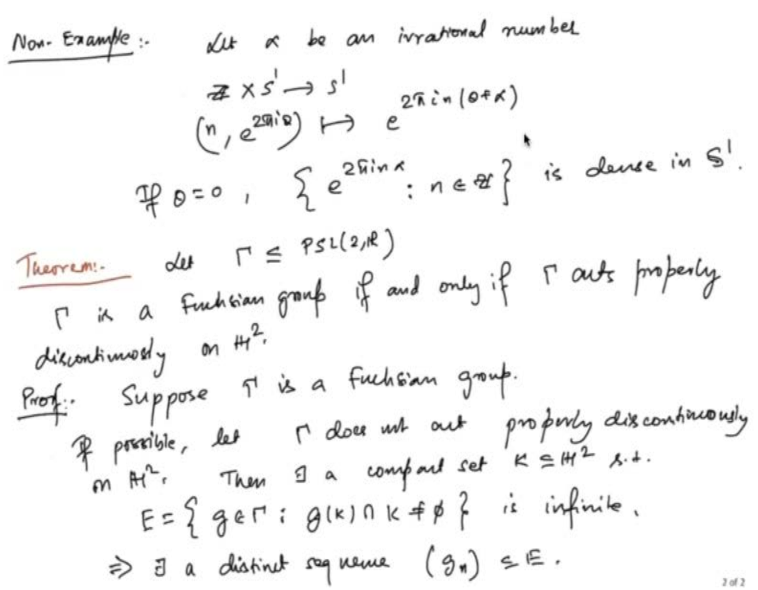

Let me provide you with a well-known example to illustrate the concept. Consider an irrational number α, and let's define an action where Z acts on the unit circle S^1. Every point on S^1 can be expressed in the form $e^{2\pi i\theta}$. The action we define here is given by $e^{2\pi i(\theta+n\alpha)}$, where $n \in Z$. Now, if θ = 0, this action generates the set $e^{2\pi in\alpha}$, with $n \in Z$. It can be shown that this set is dense in the circle S^1.

Therefore, this action of Z on S^1 is not a properly discontinuous action, because the orbit fills the circle densely. Proving this rigorously would require a detailed argument, but this is the essential intuition.

Now, let's move to an important theorem. Let Γ be a subgroup of PSL(2, R). The theorem states that Γ is a Fuchsian group if and only if Γ acts properly discontinuously on the upper half-plane H^2.

Let's outline the proof. Suppose Γ is indeed a Fuchsian group. Our goal is to prove that Γ acts properly discontinuously on H^2. To establish this, we will use a proof by contradiction.

Assume, for the sake of contradiction, that Γ does not act properly discontinuously on H^2. This would imply the existence of a compact set $K \subset H^2$ such that the set of group elements $g \in \Gamma$ satisfying $g(K) \cap K \neq \emptyset$, is infinite.

In other words, there would exist an infinite, distinct sequence $g_n \in \Gamma$ such that $g_n(K) \cap K \neq \emptyset$, for all n.

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$g_n K \cap K \neq \phi$

$\Rightarrow \exists\, x_n \in K$ & $g_n(x_n) \in K$

Without loss of generality, $x_n \to x$ in K, $x \in K$

$g_n(x_n) \to y$; $y \in K$

Take $\epsilon > 0$

$\forall$ large n, $d_{\mathbb{H}^2}(x_n, x) < \epsilon/2$

$g_n \in \mathrm{Isom}(\mathbb{H}^2) \Rightarrow d_{\mathbb{H}^2}(g_n(x_n), g_n(x)) < \epsilon/2$

For all large n, $d_{\mathbb{H}^2}(g_n(x_n), y) < \epsilon/2$

$d_{\mathbb{H}^2}(g_n(x), y) < \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow g_n(x) \to y$

Take $\overline{B}(y;\epsilon)$. $\exists\, n_0 \in \mathbb{N}$ s.t.

$g_n(x) \in \overline{B}(y;\epsilon)$ $\forall\, n \geq n_0$

Compact set in $\mathbb{H}^2$.

Now, because each g_n belongs to Γ, and the intersection $g_n(K) \cap K$ is nonempty, this implies that there exists a point $x_n \in K$ such that $g_n(x_n) \in K$ as well. To prove this, let's proceed step by step.

Suppose y_n is a point in the intersection $K \cap g_n(K)$, meaning $y_n \in K$ and $y_n \in g_n(K)$. Since $y_n \in g_n(K)$, there must exist some $x_n \in K$ such that $y_n = g_n(x_n)$. Thus, we have both $x_n \in K$ and $g_n(x_n) \in K$, which gives us the desired relation.

Now, because K is compact, it follows that K is closed and bounded. Therefore, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence of $\{x_n\}$ that converges to a point in K. Without loss of generality, we can assume that the sequence x_n converges to some point $x \in K$. Similarly, we can assume that $g_n(x_n)$ converges to some point $y \in K$, where y is also in K.

Next, consider an arbitrary $\epsilon > 0$. Since x_n converges to x, for sufficiently large n, the hyperbolic distance $d(x_n, x)$ will be less than ϵ. Now, because g_n is an isometry of the upper half-plane, it preserves distances. Hence, the distance between $g_n(x)$ and $g_n(x_n)$ will also be less than $\epsilon/2$.

Moreover, since $g_n(x_n)$ converges to y, for sufficiently large n, the hyperbolic distance $d(g_n(x_n), y)$ will be less than $\epsilon/2$.

Applying the triangle inequality, we now have for sufficiently large n:

$$d(g_n(x), y) \leq d\big(g_n(x), g_n(x_n)\big) + d(g_n(x_n), y)$$

Since both terms on the right-hand side are less than $\epsilon/2$, the total distance $d(g_n(x), y)$ is less than ϵ. Therefore, we have shown that $g_n(x)$ converges to y.

Now, consider a closed ball around y in the hyperbolic plane. Since $g_n(x)$ converges to y, there exists a natural number n_0 such that for all $n \geq n_0, g_n(x)$ will lie inside this closed ball.

Additionally, note that since the upper half-plane is a compact set, and this ball is closed and bounded, it is also a compact set in the upper half-plane. Hence, we have demonstrated that this behavior holds in the context of compact sets in the upper half-plane.

Now, let's take an index $m \geq n_0$. We can express $g_m(x)$ as:

$$g_m(x) = g_m g_{n_0}^{-1} g_{n_0}(x)$$

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$m \geq n_0$

$$g_m(x) = g_m g_{n_0}^{-1}(g_{n_0}(x)) \in \overline{B}(y;\epsilon)$$

- $(g_m g_{n_0}^{-1})_{m \geq n_0}$ is an infinite sequence.

$$(g_m g_{n_0}^{-1}) \subset \underbrace{\{T \in \Gamma : T(g_{n_0}(x)) \in \overline{B}(y;\epsilon)\}}_{\text{is compact in } PSL(2,\mathbb{R})} = A$$

Γ is Fuchsian $\Rightarrow$ Γ is discrete in $PSL(2,\mathbb{R})$

Compact + discrete $\Rightarrow$ the set A is finite.

$\Rightarrow$ $\{g_m g_{n_0}^{-1} : m \geq n_0\}$ is a finite set

$(g_m g_{n_0}^{-1})$ is not an infinite sequence.

We have a contradiction

Γ acts properly discontinuously on $\mathbb{H}^2$

We know that $g_m(x)$ lies within the closed ball centered at y, with a radius of ϵ. Notice that as we vary m, where $m \geq n_0$, we are generating an infinite sequence. This sequence is a subset of the set:

$$\{T \in \Gamma \mid T\left(g_{n_0}(x)\right) \in \text{closed ball around } y\}$$

Now, observe that the closure of this ball is compact. Since $T \in \Gamma$ and $T\left(g_{n_0}(x)\right)$ lies within this compact set $\overline{B}$, we recall from the previous lecture that this set is compact in PSL(2, R).

Furthermore, because we have assumed that Γ is Fuchsian, it follows that Γ is discrete in PSL(2, R). The combination of compactness and discreteness implies that the set:

$$A = \{T \in \Gamma \mid T\left(g_{n_0}(x)\right) \in \overline{B}\}$$

is finite. Therefore, this sequence, which we initially assumed to be infinite, must actually be finite, leading to a contradiction.

The contradiction arises from our assumption that Γ does not act properly discontinuously on the upper half-plane. Thus, we have proven that Γ must act properly discontinuously on the upper half-plane.

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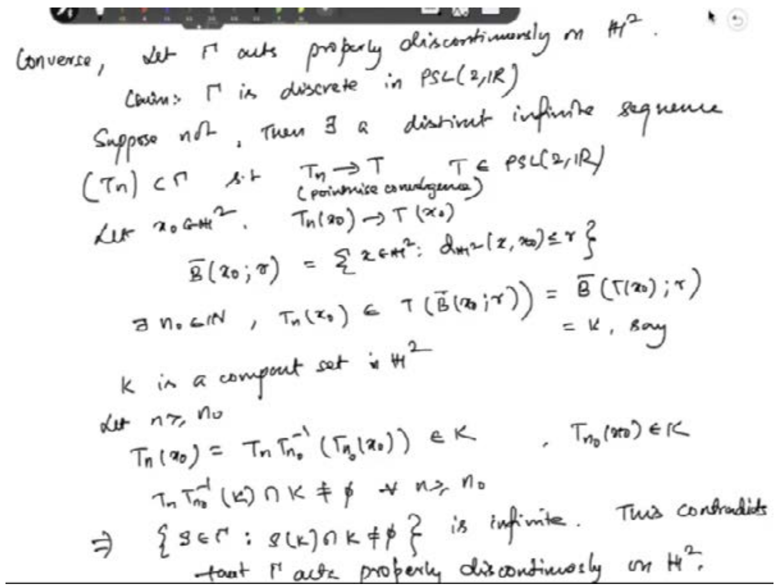

Now, let’s prove the converse.

Conversely, let’s assume that Γ acts properly discontinuously on the upper half-plane. What we want to prove is that Γ is discrete in PSL(2, R). To do this, let's proceed by contradiction. Suppose Γ is not discrete. This would mean there exists a distinct infinite sequence $\{T_n\}$ in Γ such that $T_n \to T$ for some $T \in PSL(2, R)$.

Now, take a point $x_0 \in H^2$ (the upper half-plane). Since $T_n \to T$, we have $T_n(x_0) \to T(x_0)$, meaning that the convergence is pointwise. Consider a closed ball around x_0, say with hyperbolic radius R, such that the hyperbolic distance between any $x \in H^2$ and x_0 is less than or equal to R.

Since $T_n(x_0) \to T(x_0)$, there exists some natural number n_0 such that for all $n \geq n_0, T_n(x_0) \in T$ applied to the ball, which is just the closed ball around $T(x_0)$ of radius R. Let’s denote this closed ball as K. Clearly, K is a compact set in the upper half-plane.

For $n \geq n_0$, we can write:

$$T_n(x_0) = T_n T_{n_0}^{-1} T_{n_0}(x_0)$$

Since $T_n(x_0) \in K$ and $T_{n_0}(x_0) \in K$, we conclude that:

$$T_n T_{n_0}^{-1}(K) \cap K \neq \emptyset \quad \text{for all } n \geq n_0$$

This implies that the set:

$$S = \{s \in \Gamma \mid s(K) \cap K \neq \emptyset\}$$

is infinite. But this contradicts the assumption that Γ acts properly discontinuously on the upper half-plane, where the intersection of such translates of compact sets should be finite.

Therefore, our initial assumption that Γ is not discrete must be false. Hence, Γ is discrete in PSL(2, R).

Thus, we have proved the theorem: Γ acts properly discontinuously on the upper half-plane if and only if Γ is a Fuchsian group.

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Theorem:- Let $\Gamma \leq PSL(2,\mathbb{R})$.
Γ acts properly discontinuously on $\mathbb{H}^2$ if and only if $\forall z \in \mathbb{H}^2$, $\Gamma.z = \{gz : g \in \Gamma\}$ is a discrete subset of $\mathbb{H}^2$.

Proof:- Let Γ acts properly discontinuously on $\mathbb{H}^2$.
If Γz is not discrete for some $z \in \mathbb{H}^2$, then
$\exists$ an infinite $(g_n) \subset \Gamma$ s.t.

$g_n z \to w \in \mathbb{H}^2$

$K = \overline{B}(w;\epsilon)$, $\epsilon > 0$

$\exists\, n_0 \in \mathbb{N}$ s.t.

$g_n(z) \in K \ \forall n \geq n_0$

$g_{n_0}(z) \in K$, $g_n(z) = g_n g_{n_0}^{-1}(g_n(z)) \in K$

$\Rightarrow g_n g_{n_0}^{-1}(K) \cap K \neq \emptyset \ \forall n \geq n_0$

The next theorem states the following: Let Γ be a subgroup of PSL(2, R). The group Γ acts properly discontinuously on the upper half-plane if and only if, for all points z in the upper half-plane, the orbit $\Gamma \cdot z$, which is the collection of all g(z) where g varies over Γ, forms a discrete subset of the upper half-plane.

Now, let's go through the proof, which follows a similar approach as before.

First, assume that Γ acts properly discontinuously on the upper half-plane. If the orbit is not discrete for some point $z \in H^2$, then there must exist an infinite sequence, in fact, a distinct sequence $\{g_n\} \subset \Gamma$, such that $g_n(z) \to w$ for some $w \in H^2$.

Next, consider a closed ball K around w with radius ϵ, where $\epsilon > 0$ is a small positive number. Since $g_n(z) \to w$, there exists a natural number n_0 such that for all $n \geq n_0$, we have $g_n(z) \in K$.

Now, let's observe that $g_{n_0}(z) \in K$. Using this, we can write:

$$g_n(z) = g_n g_{n_0}^{-1} g_{n_0}(z)$$

Since $g_n(z) \in K$, we have $g_n g_{n_0}^{-1}(K) \cap K \neq \emptyset$, for all $n \geq n_0$.

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$$\underbrace{(g_n g_{n_0}^{-1})}_{\text{infinite sequence}} \subset \underbrace{\{g \in \Gamma : K \cap gK \neq \emptyset\}}_{\text{an infinite set}}$$

$\Downarrow$

Γ does not act properly discontinuously on H^2, which is a contradiction

Converse, Exercise.

This proves that for all $n \geq n_0$, the intersection $g_n g_{n_0}^{-1}(K) \cap K$ is non-empty. Hence, we've shown that if the orbit is not discrete, it contradicts the assumption that Γ acts properly discontinuously on the upper half-plane.

Thus, the theorem is proven: Γ acts properly discontinuously on the upper half-plane if and only if the orbit $\Gamma \cdot z$ is a discrete subset of the upper half-plane for all points $z \in H^2$.

So, this entire sequence is contained within the set, forming an infinite sequence, which implies that it is an infinite set. This leads to the conclusion that Γ does not act properly discontinuously on the upper half-plane. However, this contradicts our initial assumption that Γ acts properly discontinuously on the upper half-plane. As for the converse, I will leave it as an exercise. The proof for the converse follows a similar approach to what we have just demonstrated.