

An Introduction to Hyperbolic Geometry

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Lecture – 14

Fuchsian Groups: Discrete Subgroups of Hyperbolic Isometries

Hello! In our last lecture, we explored the fascinating classification of isometries in the hyperbolic plane. Today, we will delve into the intriguing world of discrete subgroups of these isometries. These discrete subgroups are known as Fuchsian groups. With that, let's get started!

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Fuchsian Group

Let X be a topological space and $A \subset X$. The subspace A is said to be a discrete subspace of X if for any sequence (x_n) in X such that $x_n \rightarrow a$ for some $a \in A$ then $\exists n_0 \in \mathbb{N}$ s.t. $x_n = a \forall n \geq n_0$
(A does not have any limit points in X)

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

\mathbb{R}^4 has usual topology

$$SL(2, \mathbb{R}) \hookrightarrow \mathbb{R}^4$$

(a, b, c, d)

I will define the Fuchsian group shortly. However, before we dive into that, let's clarify what is meant by a discrete subset of a topological space.

To begin, let X be a topological space and A a subset of X . When I say that A is a subspace of X , it implies that the topology on A is the subspace topology induced from X .

Now, the subspace A is considered a discrete subspace of X if the following condition holds: for any sequence x_n in X that converges to a point in A , the sequence x_n must eventually be constant. This means there exists a natural number n_0 such that $x_n = a$ for all $n \geq n_0$.

This definition of a discrete subspace implies that A does not have any limit points in X . In other words, if we take any sequence in X that converges to a point in A , then that sequence must be eventually constant. Thus, we can conclude that A has no limit points in X .

Now, let's transition to discussing Fuchsian groups. We begin with the group $SL(2, \mathbb{R})$. This group consists of all 2×2 real matrices with a determinant equal to 1. We can think of each matrix, represented as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as a point in \mathbb{R}^4 . With the standard topology of \mathbb{R}^4 , it follows that $SL(2, \mathbb{R})$ naturally forms a subspace of \mathbb{R}^4 via this mapping. Consequently, $SL(2, \mathbb{R})$ inherits its subspace topology from \mathbb{R}^4 . Thus, we have a topology on the group $SL(2, \mathbb{R})$ that is induced from \mathbb{R}^4 .

(Refer Slide Time: 04:38)

$PSL(2, \mathbb{R}) = \overline{SL(2, \mathbb{R})} / \{\pm Id\}$ $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 $SL(2, \mathbb{R}) \xrightarrow{q} PSL(2, \mathbb{R})$
 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
 The topology on $PSL(2, \mathbb{R})$ is the quotient topology
 $PSL(2, \mathbb{R}) \leq \text{Isom}(\mathbb{H}^2)$
Fuchsian Group: A subgroup $\Gamma \leq PSL(2, \mathbb{R})$ (or $PSU(1,1)$) is said
 to be Fuchsian group if Γ is a discrete subspace
 of $PSL(2, \mathbb{R})$
observation: $\Gamma (\leq PSL(2, \mathbb{R}))$ is discrete if and only if
 (no limit points convergence)

Now, let's discuss $PSL(2, \mathbb{R})$, as we will focus on proving certain aspects related to this group. Specifically, $PSL(2, \mathbb{R})$ is the quotient group $SL(2, \mathbb{R})/\{\pm I\}$, where I is the identity matrix. There exists a natural quotient map q from $SL(2, \mathbb{R})$ to $PSL(2, \mathbb{R})$.

So, what about the topology on $PSL(2, \mathbb{R})$? The topology on $PSL(2, \mathbb{R})$ is defined as the quotient topology. In essence, $PSL(2, \mathbb{R})$ serves as a quotient space, and when we endow it with this quotient

topology, the map q becomes continuous. Thus, we have established a topology on $\text{PSL}(2, \mathbb{R})$ that is specifically a quotient topology.

Moreover, we know that $\text{PSL}(2, \mathbb{R})$ corresponds to an index two subgroup of the isometry group of the upper half-plane model. We will primarily be working with this upper half-plane model, as the disk model is isometric to it. If we consider the isometry group of the unit disk model, we obtain $\text{PSU}(1, 1)$.

Now, let's define what we mean by a Fuchsian group. A subgroup Γ of either $\text{PSL}(2, \mathbb{R})$ or $\text{PSU}(1, 1)$ is termed a Fuchsian group if Γ is a discrete subspace of $\text{PSL}(2, \mathbb{R})$. This is the crux of the definition: if we have a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ or $\text{PSU}(1, 1)$, we classify that group as a Fuchsian group.

Next, let's make an important observation regarding Γ , which we will take as a subgroup of $\text{PSL}(2, \mathbb{R})$. The group Γ is discrete if and only if, for a sequence T_n in Γ that converges to the identity element, this convergence must imply that T_n is equal to the identity for all sufficiently large n . This is a straightforward result.

If we assume that Γ is discrete and T_n converges to the identity, then since the identity is in Γ , we can conclude that Γ has no limit points. Consequently, the sequence T_n must be eventually constant.

Now, for the converse part, we want to establish that if Γ is discrete, and we take a sequence T_n that converges to a point $T \in \Gamma$, then we can analyze $T^{-1}T_n$ which converges to the identity. This is valid because $\text{PSL}(2, \mathbb{R})$ is indeed a group. By our assumption, $T^{-1}T_n$ converging to the identity means that $T^{-1}T_n = I$ for all sufficiently large n . This implies that $T_n = T$ for those same large n , thereby completing our proof.

Now, let's move on to the next significant point. We have an important lemma to prove: if we take any Fuchsian group, it will act on the upper half-plane or the unit disk, such that any orbit will form a discrete set in either the upper half-plane or the unit disk. This lemma will be crucial for characterizing any discrete subgroup by the discrete subsets of the upper half-plane.

Let us consider a point z_0 in the upper half-plane and let K be a compact subset of this upper half-

plane. We define the set E to consist of all elements of $\text{PSL}(2, \mathbb{R})$ such that $T(z_0)$ belongs to K . It follows that E is a compact set within $\text{PSL}(2, \mathbb{R})$.

(Refer Slide Time: 11:35)

Lemma: Let $z_0 \in \mathbb{H}^2$ and K be a compact subset of \mathbb{H}^2 .
 Then $E = \{ T \in \text{PSL}(2, \mathbb{R}) \mid T(z_0) \in K \}$ is compact.

Proof: $E_1 = \{ T \in \text{SL}(2, \mathbb{R}) \mid T(z_0) \in K \}$
 We will prove that E_1 is compact.
 $q: \text{SL}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$
 q is a continuous map & $q(E_1) = E$
 E_1 compact $\Rightarrow E$ is compact

Claim: E_1 is compact
 We will prove E_1 is closed and bounded.
 (Any closed & bounded subset of \mathbb{R}^n is compact)

Let $T_n \in E_1$ s.t. $T_n \rightarrow T$ (pointwise)

Now, what is the proof of this claim? We will demonstrate the following: Let E_1 be defined as the set of all T in $\text{SL}(2, \mathbb{R})$ such that $T(z_0)$ belongs to K . We will prove that E_1 is compact.

Once we establish that E_1 is compact, we can utilize the fact that there is a quotient map from $\text{SL}(2, \mathbb{R})$ to $\text{PSL}(2, \mathbb{R})$. Given that q is a continuous map, we have that $q(E_1) = E$. Therefore, if we can prove that E_1 is compact, it will follow that E is also compact.

Now, our goal is to show that E_1 is compact. We can view E_1 as a subset of \mathbb{R}^4 . To demonstrate that E_1 is compact, we need to show that it is both closed and bounded. According to the Heine-Borel theorem, any closed and bounded subset of \mathbb{R}^n is compact.

Thus, our first task is to establish that E_1 is closed. Let us take a sequence T_n belonging to E_1 such that T_n converges to T pointwise. Each T_n can be expressed in the form

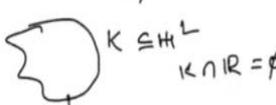
$$T_n(z) = \frac{a_n z + b_n}{c_n z + d_n}$$

Since T_n belongs to E_1 , we know that E_1 is a subset of $\text{SL}(2, \mathbb{R})$, which means that the determinant

$a_n d_n - b_n c_n = 1$ for all n . Furthermore, since $T_n(z_0)$ belongs to the compact set K , we can proceed with our proof.

(Refer Slide Time: 15:48)

$T_n(z_0) \rightarrow T(z_0)$
 As K is compact $\Rightarrow K$ is closed in \mathbb{H}^2
 Therefore, $T(z_0) \in K \Rightarrow T \in E_1$
 E_1 is closed
 Claim: E_1 is bounded
 Let $T(z) = \frac{az+b}{cz+d} \in E_1$
 $T(z_0) \in K \Rightarrow \exists M_1 > 0$ s.t. $|T(z_0)| < M_1$
 i.e. $\left| \frac{az_0+b}{cz_0+d} \right| < M_1$
 $K \rightarrow \mathbb{R}_{>0}$ } continuous
 $w \mapsto |\operatorname{Im}(w)|$ } map
 K is compact in \mathbb{H}^2 } $K \subseteq \mathbb{H}^2$
 $K \cap \mathbb{R} = \emptyset$



Thus, $T_n(z_0)$ converges to $T(z_0)$. Since K is compact, it is also a closed subset of the upper half-plane, meaning that K is closed within this space. Therefore, if $T(z_0)$ belongs to K , it implies that T must also belong to E_1 . This is in accordance with the definition of E_1 : T is an element of $SL(2, \mathbb{R})$ if $T(z_0)$ is within that collection. Hence, we can conclude that T indeed belongs to E_1 , which proves that E_1 is closed.

Now, our next objective is to demonstrate the claim that E_1 is bounded. Consider an element $T(z) = \frac{az+b}{cz+d}$ that belongs to E_1 . We know that $T(z_0)$ is contained in the compact set K . Because K is a closed and bounded set, it follows that there exists a real number M_1 such that

$$|T(z_0)| < M_1$$

Specifically, we have

$$\left| \frac{az_0 + b}{cz_0 + d} \right| < M_1$$

Let's denote this as Equation (1).

Next, consider the following important aspect: we can define a map from K to $R \geq 0$. This map takes any w belonging to K and maps it to the absolute value of the imaginary part of w . Notably, this map is continuous. Since K is compact in the upper half-plane, we can illustrate this concept with a diagram. Picture K as lying entirely within the upper half-plane, ensuring that it does not intersect the real axis.

Given that K is compact, we can find a number M_2 such that the distance from K to the real line is greater than M_2 . Therefore, there exists a value $M_2 > 0$ such that

$$\text{Im}(w) > M_2 \quad \text{for all } w \in K.$$

Moreover, since K is a subset of the upper half-plane, the imaginary part of w is always greater than 0. Thus, we can simplify our expression by removing the absolute value, leading us to the conclusion that

$$\text{Im}(w) > M_2 \quad \text{for all } w \in K.$$

(Refer Slide Time: 21:05)

$$\begin{aligned} \frac{az_0+b}{cz_0+d} &= T(z_0) \in K \\ \text{Im}\left(\frac{az_0+b}{cz_0+d}\right) &> M_2 > 0 \\ &\parallel \\ \frac{\text{Im}(z_0)}{|cz_0+d|^2} & \\ |cz_0+d| &< \sqrt{\frac{\text{Im}(z_0)}{M_2}} \quad \text{--- (2)} \end{aligned}$$

From (1) & (2)

$$|az_0+b| < M_1 \sqrt{\frac{\text{Im}(z_0)}{M_2}} \quad \text{--- (3)}$$

From (2) & (3), we have
 a, b, c, d are bounded
 bounded

Now, we have established that $T(z_0)$ belongs to K , which is equivalent to the expression $\frac{az_0+b}{cz_0+d}$.

Consequently, the imaginary part of $\frac{az_0+b}{cz_0+d}$ is greater than M_2 . We can delve deeper into this

relationship. The imaginary part of $\frac{az_0+b}{cz_0+d}$ can be expressed as

$$\operatorname{Im}\left(\frac{az_0 + b}{cz_0 + d}\right) = \frac{\operatorname{Im}(z_0)}{|cz_0 + d|^2}.$$

From this, we can derive that

$$|cz_0 + d|^2 < \frac{\operatorname{Im}(z_0)}{M_2}.$$

Now, by taking the square root, we can simplify our expression; let's denote this result as Equation (1). From Equations (1) and (2), we conclude that

$$|az_0 + b| < M_1 \sqrt{\frac{\operatorname{Im}(z_0)}{M_2}}.$$

Let's call this Equation (3).

From Equations (2) and (3), we can see that we have taken T to be an arbitrary element of E_1 , where a, b, c, d are varying parameters. This leads us to the conclusion that a, b, c, d must be bounded.

Thus, we have defined T as an arbitrary Möbius transformation from E_1 :

$$T(z) = \frac{az + b}{cz + d},$$

with a, b, c, d being bounded. Therefore, we can conclude that E_1 is also bounded.

In summary, we have demonstrated that E_1 is both closed and bounded. Given that E is the image of E_1 under the quotient map q , which is continuous, we can assert that since E_1 is compact, it follows that E must also be compact.

In our next lecture, we will explore the concept of properly discontinuous actions and prove that for a Fuchsian group, this is equivalent to stating that any orbit in the upper half-plane is finite.