An Introduction to Hyperbolic Geometry

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Lecture – 13

Classification of Orientation-Preserving Isometries in Hyperbolic Geometry

Hello, and welcome to today’s lecture! In this session, we will delve into the classification of orientation-preserving isometries within the hyperbolic plane. To illustrate this concept, let’s consider the upper half-plane model of the hyperbolic plane.

In this context, we know that the group $\mathrm{PSL}(2, R)$ acts as an orientation-preserving isometric group on the upper half-plane. The classification of an isometry depends significantly on its fixed points. Therefore, as we examine each element of $\mathrm{PSL}(2, R)$, we will determine how these isometries can be classified based on their respective fixed points. So, let us begin.

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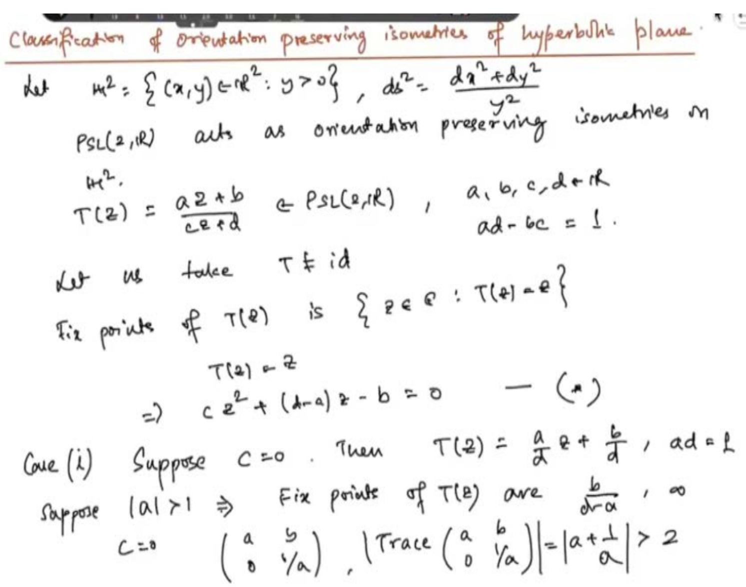

Let us consider the upper half-plane model of the hyperbolic plane, denoted as H^2. In this context, we have the hyperbolic metric given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

We know that the group $\mathrm{PSL}(2, R)$ acts on this upper half-plane, and this action is characterized by orientation-preserving isometries. Now, if we take any element T(z) in $\mathrm{PSL}(2, R)$, we denote this element as T, where a, b, c, and d are real numbers satisfying the condition ad - bc = 1.

Let's focus on a non-identity element T of $\mathrm{PSL}(2, R)$. Our initial interest lies in identifying the fixed points of T(z). To find these fixed points, we need to solve the equation:

$$T(z) = z.$$

By substituting our transformation into this equation, we obtain the following quadratic equation:

$$az^2 + (d - a)z - b = 0.$$

Now, let's examine two cases for c.

Case 1: c = 0

If c = 0, then the transformation T(z) simplifies to

$$T(z) = \frac{a}{d}z + \frac{b}{d}.$$

Given that ad - bc = 1 and c = 0, we find that ad = 1. Now, assuming that |a| > 1, we can derive the fixed points of T(z). From the quadratic equation, the fixed points will be

$$\frac{b}{d-a} \quad \text{and} \quad \infty.$$

The first fixed point, $\frac{b}{d-a}$, arises from the quadratic equation when we set c = 0 and |a| > 1. Additionally, we see that ∞ is also a fixed point under these conditions.

Now, consider the case when |a| = 1. In this scenario, d must also equal 1, which implies d - a = 0. Since |a| > 1 implies that d cannot equal a, we conclude that $\frac{b}{d-a}$ remains valid as a complex number. Thus, the fixed points of T(z) are

$$\frac{b}{d-a} \quad \text{and} \quad \infty.$$

Moreover, since we have established c = 0, the matrix representing the transformation takes the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with ad = 1 leading to $d = \frac{1}{a}$. Consequently, the trace of this matrix is given by

$$\mathrm{tr} = a + \frac{1}{a}.$$

This trace is always greater than 2 when we consider the modulus, i.e.,

$$|\mathrm{tr}| = \left|a + \frac{1}{a}\right| > 2.$$

So, we have established an important observation: when |a| > 1, it holds that

$$\left|a + \frac{1}{a}\right| > 2.$$

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Suppose $|a| = 1$, $T(z) = \frac{a}{d} z + \frac{b}{d}$, $ad = 1$, $a \in \mathbb{R}$

$\Rightarrow a = d$

$T(z) = z + b/d$

$T \neq id \Rightarrow b/d \neq 0$

$T(\infty) = \infty$

$\mathrm{Fix}(T) = \{\infty\}$

Case (ii) Let $c \neq 0$. $cz^2 + (d-a)z - b = 0$ — $(*)$

Discriminant, $D = (d-a)^2 + 4bc$

$= (a+d)^2 - 4(ad-bc)$

$= (a+d)^2 - 4$

$\left|\mathrm{Trace}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = |a+d|$

(a) $|a+d| > 2$, $D > 0 \Rightarrow T$ has exactly two fixed po

Now, let's consider the scenario where |a| = 1. In this case, we have already established that c = 0, which leads us to the transformation $T(z) = \frac{a}{d}z + \frac{b}{d}$. Since ad = 1, we find that a is a real number. Consequently, if |a| = 1, then a must equal either 1 or -1.

When a = -1, it follows that d must also be -1. Thus, we conclude that a = d. Under these circumstances, the transformation T(z) can be expressed as

$$T(z) = z + \frac{b}{d}.$$

It is essential to note that T is not an identity transformation; therefore, $\frac{b}{d}$ cannot equal 0. As a result, we see that T acts merely as a translation. This leads us to conclude that T(∞) = ∞, which means the only fixed point of T is the singleton set {∞}.

Now, let's move on to Case 2, where we assume $c \neq 0$. In this scenario, we revisit our quadratic equation, given by

$$cz^2 + (d - a)z - b = 0.$$

Next, we need to analyze the discriminant, denoted as D. The discriminant is computed as

$$D = (d - a)^2 + 4bc.$$

This can be simplified to

$$D = (a + d)^2 - 4(ad - bc).$$

Since we know ad - bc = 1, we can further express the discriminant as

$$D = (a + d)^2 - 4.$$

The modulus of the trace of the transformation matrix, represented as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is given by |a + d|. Now, if we assume that |a + d| > 2, it follows that the discriminant D > 0. A positive discriminant indicates that the quadratic equation has exactly two roots in R. Consequently, this implies that the transformation T possesses exactly two fixed points on the real line R.

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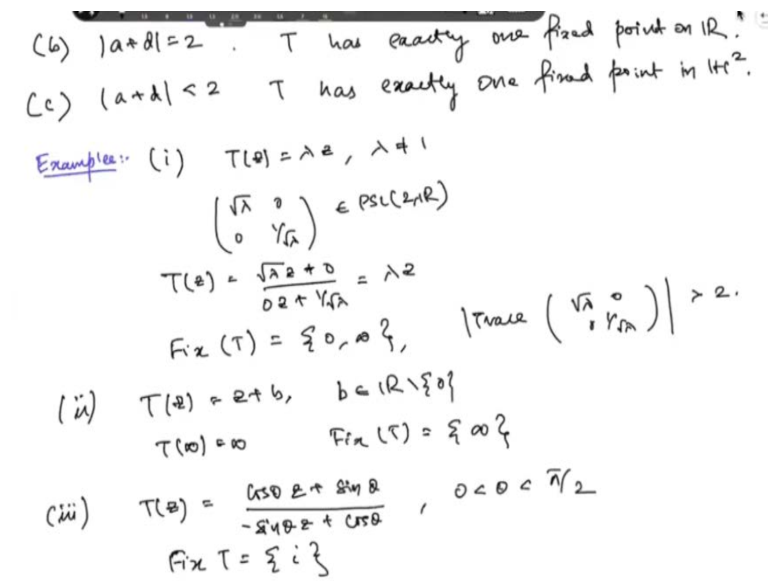

In the second sub-case, let us assume that |a + d| = 2. This condition indicates that the quadratic equation has two equal real roots. Consequently, we can conclude that the transformation T has exactly one fixed point on the real line R.

Now, let’s consider the third sub-case, where we assume that |a + d| < 2. Under this scenario, the discriminant D will be less than 0, which implies that our quadratic equation will yield two imaginary roots that are conjugates of each other. One of these roots will lie in the upper half-plane, while the other will be located in the lower half-plane. Therefore, since only one root exists in the upper half-plane, we can conclude that T has exactly one fixed point in that region.

Now, let’s explore some examples to illustrate these concepts. For the first example, consider the transformation defined by $T(z) = \lambda z$, where λ is not equal to 1 and can be greater than 1. If we examine the matrix

$$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \dfrac{1}{\sqrt{\lambda}} \end{pmatrix},$$

we find that this matrix belongs to PSL(2, R). By applying this transformation, we see that

$$T(z) = \frac{\sqrt{\lambda}z + 0}{0 \cdot z + \dfrac{1}{\sqrt{\lambda}}} = \lambda z.$$

In this case, the fixed point of the transformation is at ∞, and notably, the modulus of the trace of this matrix is greater than 2.

For our second example, let's consider the transformation T(z) = z + b, where b is a nonzero real number. Here, we observe that T(∞) = ∞, which indicates that the fixed point of T is simply the singleton set {∞}.

In the third example, we analyze the transformation given by

$$T(z) = \frac{\cos\theta \ z + \sin\theta}{-\sin\theta \ z + \cos\theta}.$$

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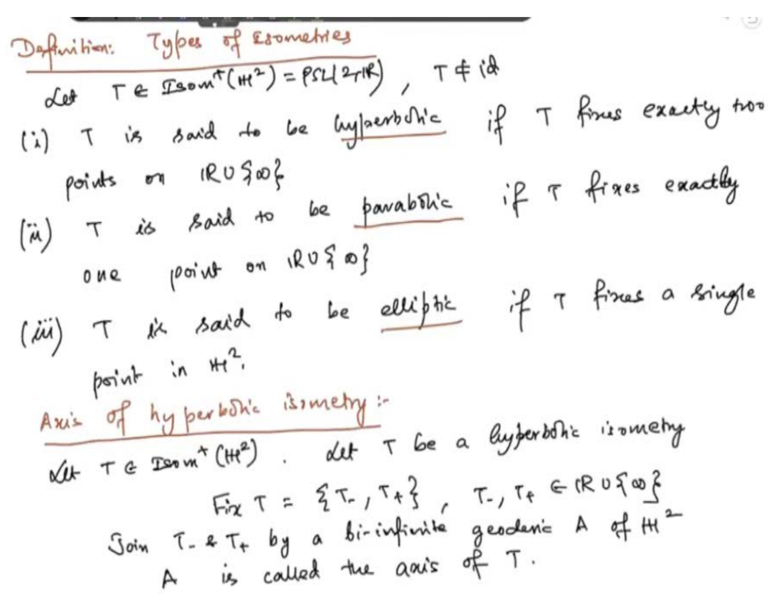

In this scenario, the fixed point of the transformation is the singleton set $\{i\}$. Now, let me provide further definitions to clarify these concepts.

Definition: Types of Isometries

Let T be an orientation-preserving isometry of the upper half-plane. This group is precisely equal to PSL(2, R), and we denote the identity element by $T_0 = \text{id}$. We classify T(z) as a hyperbolic isometry if T fixes exactly two points in the set $R \cup \infty$. Conversely, T is said to be parabolic if it fixes exactly one point in $R \cup \infty$. Finally, T is classified as elliptic if it fixes a single point in the upper half-plane, without touching the boundary of this region.

To summarize the characteristics of these isometries: a transformation T is hyperbolic if the modulus of the trace of its corresponding matrix is greater than 2; it is parabolic if the modulus of the trace equals 2; and it is elliptic if the modulus of the trace is less than 2.

Next, we will explore the axis of a hyperbolic isometry. Again, let T be an orientation-preserving isometry of the upper half-plane, and assume that T is a hyperbolic isometry. This means that T fixes exactly two points in the set $R \cup \{\infty\}$. We can denote these fixed points of T as T^- and T^+, which belong to $R \cup \{\infty\}$.

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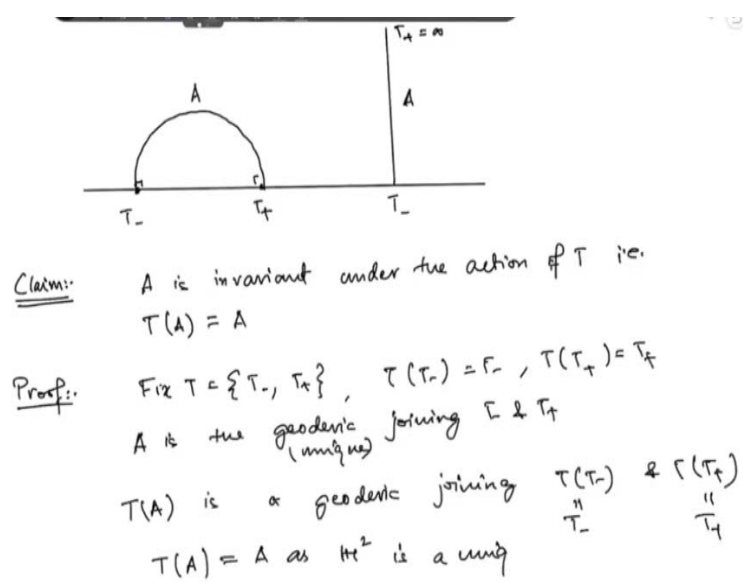

Now, we can connect T^- and T^+ using a bi-infinite geodesic A within the upper half-plane. This geodesic A is referred to as the axis of T, and we denote it as x_A. Thus, we have established the definition of the axis of a hyperbolic isometry.

Let's visualize the situation: we have our upper half-plane, and within this plane, suppose we have an isometry T with fixed points T^- and T^+, both of which lie on the real line R. We can connect these two points with a bi-infinite geodesic, which we will call A. This geodesic A serves as the axis of the hyperbolic isometry T.

Now, if one of these fixed points, say T^- or T^+, approaches infinity, the situation changes slightly. In this case, we find ourselves with a vertical axis where T^- is a finite point and T^+ is equal to infinity. Thus, A continues to represent the axis of the isometry T.

The claim we want to establish is that A is invariant under the action of T; that is, T fixes this axis. In mathematical terms, this means that T(A) = A. The proof of this statement is quite straightforward.

To understand the proof, we note that the fixed points of T are T^- and T^+. Thus, applying T to these points, we have $T(T^-) = T^-$ and $T(T^+) = T^+$. Since A is the unique geodesic connecting T^- and T^+ in the upper half-plane, we leverage the property of the upper half-plane being a unique geodesic space.

Moreover, since T is an isometry and fixes both T^- and T^+, the transformation T(A) must also yield a geodesic that connects $T(T^-)$ to $T(T^+)$. Given that $T(T^-) = T^-$ and $T(T^+) = T^+$, the geodesic T(A) is precisely the same as A.

Therefore, because the upper half-plane possesses the unique geodesic property, we conclude that T(A) = A, confirming that the axis A remains invariant under the action of the isometry T.

Let's take a moment to delve into an important observation regarding orientation-preserving isometries. To begin, consider an orientation-preserving isometry T belonging to the group $\mathrm{PSL}_2(R)$. For the sake of this observation, let's assume that T is not equal to the identity. We'll refer to this as Observation 1.

Now, let's specify that T is indeed a hyperbolic isometry. The claim we want to establish is that there exists an orientation-preserving isometry S within $\mathrm{PSL}_2(R)$ such that the relation

$$STS^{-1}(z) = \lambda z$$

holds for some $\lambda \neq 1$.

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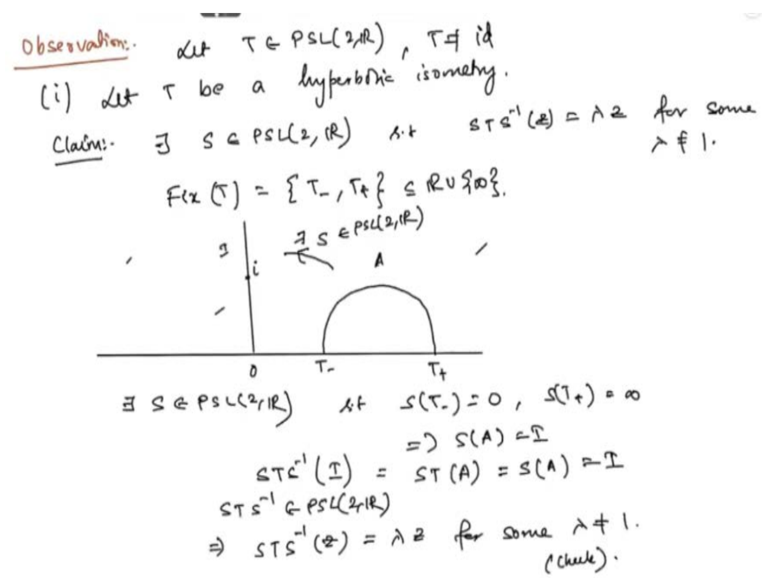

To understand this in a linear algebraic context, we can represent T by a 2×2 matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are real numbers satisfying the condition ad - bc = 1. This relationship indicates that the matrix can be diagonalized within this context.

Now, let's provide a geometric interpretation of this scenario. Given that T is a hyperbolic isometry, it has two distinct fixed points, which we will denote as T^- and T^+, located on the extended real line $R \cup \{\infty\}$. We can visualize this in the upper half-plane model.

In our diagram, let's denote the axis of the hyperbolic isometry T as A, which is the geodesic connecting the fixed points T^- and T^+. Next, we'll identify the imaginary axis in our upper half-plane representation.

A crucial fact to note is that there exists a Möbius transformation, a specific element of $\mathrm{PSL}_2(R)$, that transforms this geodesic A into the imaginary axis. We can denote this transformation as S. Thus, we can assert that there exists an $S \in \mathrm{PSL}_2(R)$ such that

$$S(T^-) = 0 \quad \text{and} \quad S(T^+) = \infty.$$

This transformation implies that S(A) maps to the imaginary axis, which we will denote as I.

Now, consider the conjugate transformation STS^{-1}. When we apply this transformation to the imaginary axis I, we observe that

$$S^{-1}(I) = A$$

because we established that S(A) = I and since T fixes the axis A, we have

$$T(A) = A.$$

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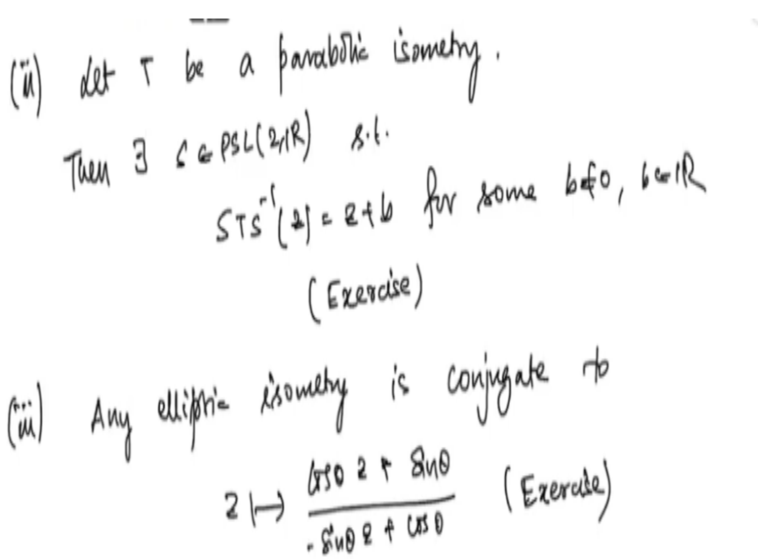

Thus, the transformation behaves such that S(A) = I. Since we've established that STS^{-1} is still a member of $\mathrm{PSL}_2(R)$ and fixes the imaginary axis I, we conclude that

$$STS^{-1}(z) = \lambda z$$

for some $\lambda \neq 1$. This relationship can be verified through direct calculations, solidifying our claim.

Now, let's move on to our next observation. Suppose we take T to be a parabolic isometry. In this case, there exists an S belonging to $\mathrm{PSL}_2(R)$ such that

$$STS^{-1}(z) = z + b$$

for some nonzero b that lies within R. I encourage you to explore this further as an exercise.

Additionally, I'd like to mention a third observation, which I will also leave as an exercise for you to consider. Specifically, any elliptic isometry can be shown to be conjugate to a particular map. This is another important aspect that we can discuss in future tutorials.

These observations are not merely academic; they will prove to be quite useful in our later classes as we delve deeper into the subject matter. So, I will stop here.