An Introduction to Hyperbolic Geometry

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Lecture – 12

Uniform Slimness of Geodesic Triangles in Hyperbolic Geometry

Hello, and welcome to this lecture on hyperbolic geometry! In today's session, we will demonstrate that triangles in the hyperbolic plane possess a unique property: they are uniformly slim. But what do we mean by this?

When we refer to a geodesic triangle in the hyperbolic plane, we assert that any side of this triangle lies within a fixed δ-neighborhood of the union of the other two sides. Importantly, this δ is a constant that applies uniformly to all triangles in the hyperbolic plane, and it is a positive number specific to this geometric setting.

To establish this fact, we will need to explore certain geometric objects intrinsic to the hyperbolic plane, such as horodisks and horocycles. So, let us begin.

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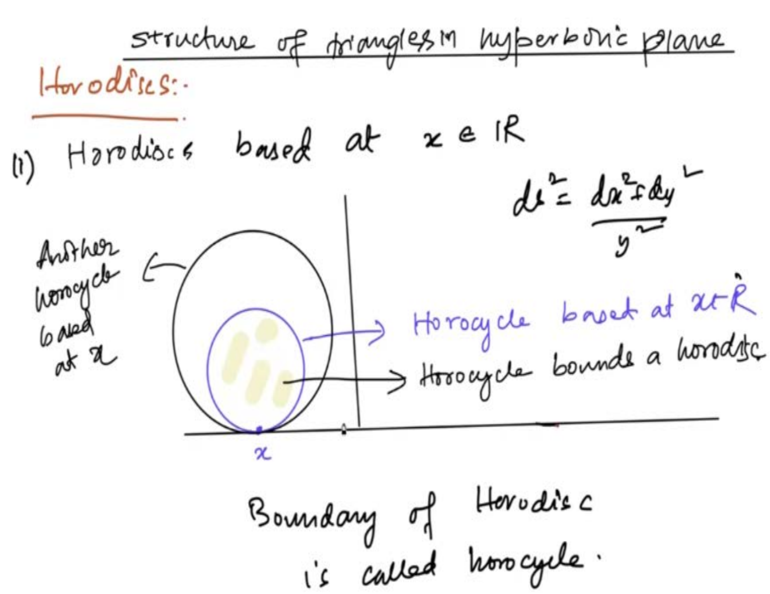

To comprehend the structure of triangles in the hyperbolic plane, let's begin by exploring the concept of a horodisc based at a real number. For our discussion, we will utilize the upper half-plane model of hyperbolic geometry.

First, let's clarify what we mean by a horodisc or horocycle based at a point x in the real numbers. Consider x as a specific real number. Now, imagine any circle that is tangent to the real axis, or the x-axis, at the point x. This means that the x-axis just touches the circle at this point.

For instance, take the blue circle that touches the x-axis precisely at point x. We refer to this circle as a horocycle based at the point x. It's important to note that there are infinitely many horocycles based at this point. You can create numerous circles that all touch the x-axis at the same point, x, without intersecting it anywhere else.

This circle will enclose a disc; for example, if we consider the blue circle, it will bound a certain disc. We refer to this disc as a horodisc. Consequently, the boundary of the horodisc, which is the circle itself, is termed a horocycle.

Once we grasp the notion of horodiscs based at a real number, we can then extend our discussion to the concept of a horodisc based at infinity, as well as a horocycle based at infinity.

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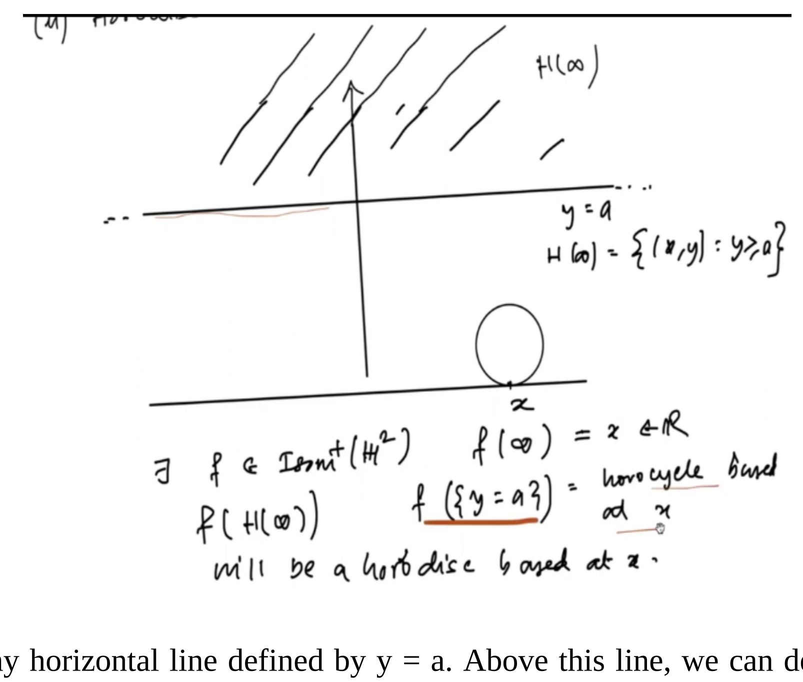

Let's consider any horizontal line defined by y = a. Above this line, we can describe the region where H_∞ is represented as the set of points (x, y) such that $y \geq a$. We refer to this space, H_∞, as the horodisc based at infinity. Meanwhile, the line y = a is termed the horocycle based at the point at infinity.

Now, why is the horizontal line y = a called the horocycle based at infinity? To understand this, we can invoke an orientation-preserving isometry of the upper half-plane model. Specifically, there exists a Mobius transformation, denoted as f, which maps the point at infinity to some point x. This isometry can be represented as an element of $PSL(2, R)$.

As a result of this transformation, the line y = a is taken to a circle or possibly remains a straight line. Given that f(∞) = x, the only viable outcome is that the image of this horizontal line becomes a circle that intersects the real axis, or x-axis, at the point x exclusively.

Thus, we conclude that the image of the set defined by y = a under the transformation f corresponds to the horocycle based at x. This provides the rationale for referring to the line y = a as the horocycle based at infinity. In summary, the space where $y \geq a$ is appropriately designated as the horodisc based at infinity.

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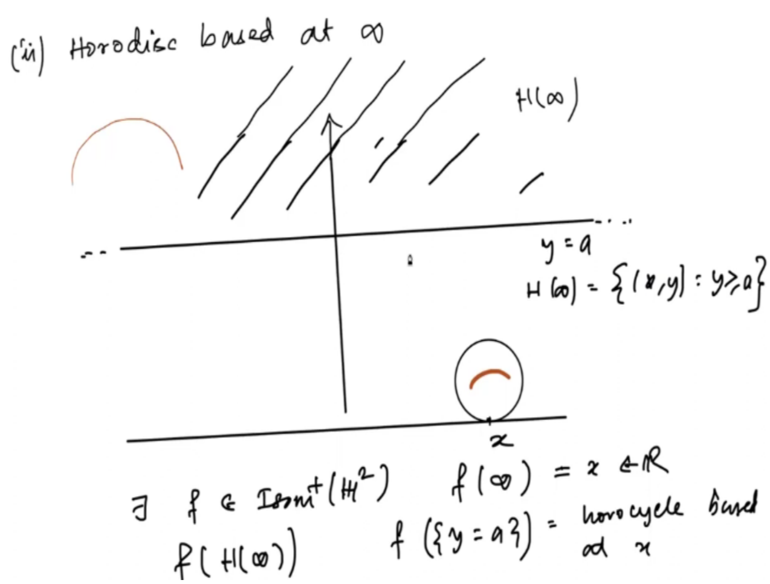

Now, let us explore some additional properties of the horodisc. This space, the horodisc, is indeed a convex set; one could also refer to it as a totally geodesic space. But what does this mean? It implies that if we select any two points within the horodisc, the geodesic connecting these two points will also lie entirely inside the horodisc.

So, how can we prove this assertion? The proof is quite straightforward. Specifically, for H_∞, the horodisc based at infinity, this property holds true and is rather simple to demonstrate. When we take any geodesic that joins two points in H_∞, it will be contained within this H_∞. Thus, we can conclude that H_∞ is indeed a convex set or a totally geodesic space.

Furthermore, we have an isometry that maps H_∞ to H_x. Consequently, it follows that H_x will also inherit this property, meaning it too is a convex set or a totally geodesic space. This highlights the robust geometric structure inherent in these horodiscs.

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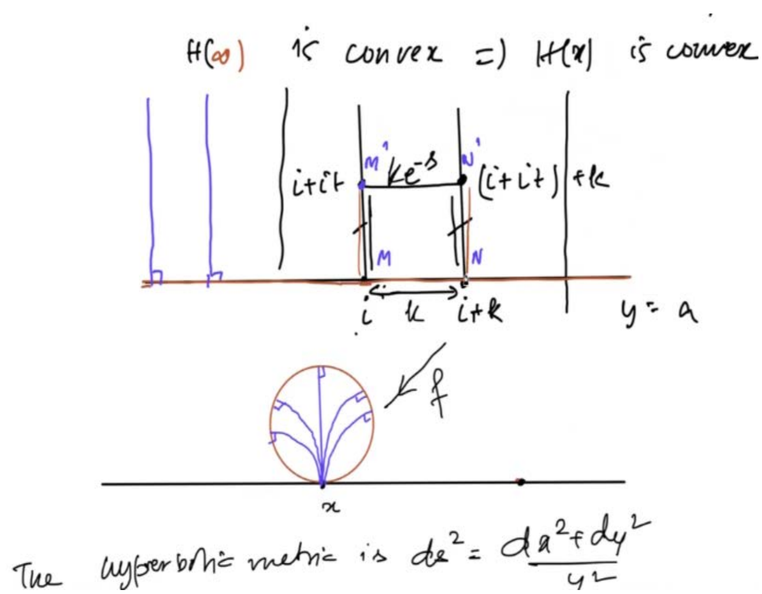

Now, let's delve into another intriguing property of the horodisc, specifically, the horodisc based at infinity and the horodisc based at a point x. Here we have H_∞, and within this space, let’s consider two vertical geodesics. These vertical geodesics originate from the points i and i + k, where k represents the length of the horizontal segment connecting i and i + k.

Now, if we take any two points along these vertical lines, let’s denote them as i + it and i + it + k. These coordinates correspond to two points, m' and n', positioned on this vertical line. To find the length of the line segment joining m' and n', we can compute it, and this length will ultimately turn out to be ke^{-s}.

So, what exactly is this s? This value s plays a crucial role in understanding the geometry within the hyperbolic space.

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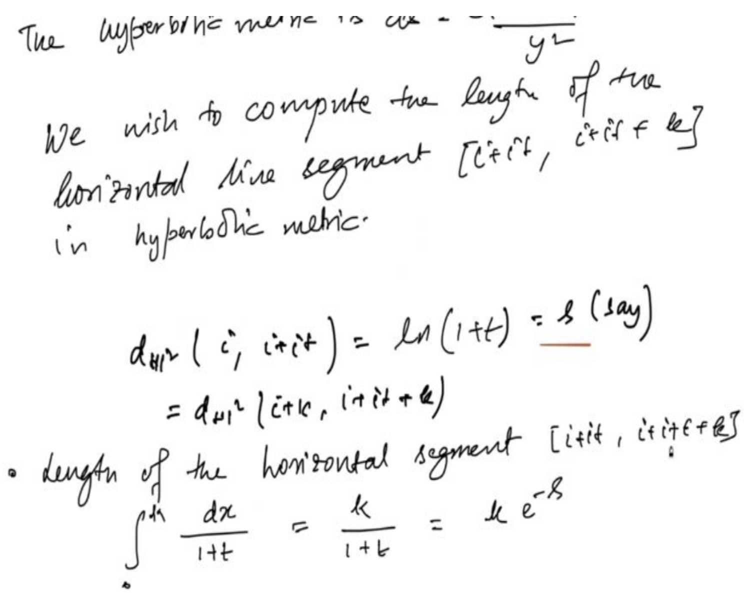

In this context, s represents the hyperbolic distance from i to i + it. Let's take a closer look at this concept. We aim to compute the length of the horizontal line segment joining the points i + it and i + it + k with respect to the hyperbolic metric.

First, let's observe the distance between i and i + it. We know that this distance can be expressed as $\log\left(\frac{i+it}{i}\right)$, which simplifies to log(i + t). Consequently, we find that this is equal to s. Interestingly, this is also the hyperbolic distance between i + k and i + it + k. Thus, we can conclude that these two distances are indeed the same, and we can denote this common value as s.

Now, let's determine the length of the horizontal segment connecting i + it and i + it + k. This length can be computed as the integral

$$\int \frac{dx}{1+t}.$$

It is important to note that here, y = 1 + t remains constant while we are integrating along this path, which means that dy = 0. Therefore, this integral simplifies to

$$\frac{k}{1+t}.$$

Since we have established that $\log(1+t) = s$, we can conclude that the length of the horizontal segment is $k \times e^{-s}$.

To visualize this, if we take any horizontal line segment and move upwards by a distance of s, the length of that segment changes to $k \times e^{-s}$. As we let s tend to infinity, this distance approaches zero, indicating that these two lines are asymptotic to each other.

Furthermore, we know that there exists an isometry that maps infinity to a point x. In this scenario, the red horocycle defined by the equation y = a represents a circle whose tangent at the point x coincides with the real axis.

Now, what about the images of all these vertical lines under the mapping f? These images will become geodesics, infinitely extending from the point x. Importantly, these geodesics will intersect the horocycle, or the red circle, orthogonally. This is because the map f is controllable, preserving the angles and maintaining orthogonality at these vertical axes.

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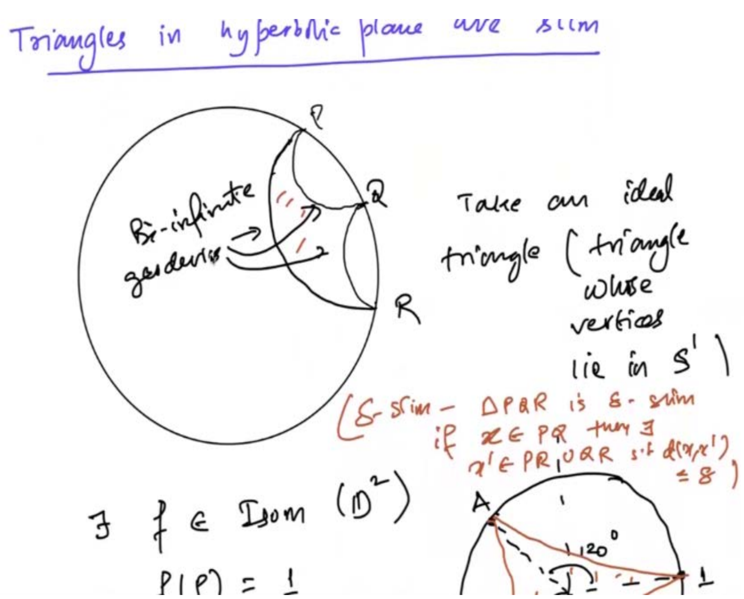

Now, let's transition to the next segment of our discussion, where we will prove that triangles in hyperbolic planes exhibit a characteristic known as "slimness." But what exactly do we mean by "slim"? The definition of slimness is quite specific.

Consider any triangle PQR situated in the upper half-plane or within a unit disc. We define this triangle to be δ-slim if, for any point x located on the side PQ, there exists a point x' that belongs to the union of the sides PR and QR. Here, PR is the geodesic connecting points P and R, while QR is the geodesic connecting points Q and R. The crucial condition is that the distance between x and x' must be less than or equal to δ.

In the illustration of triangle PQR I have presented, it may appear that the triangle's vertices lie on the boundary, but it's essential to note that the triangle can also exist entirely within the upper half-plane or the unit disc. Our goal is to demonstrate that there exists a specific value of δ, in fact, this δ will be log(3), indicating that all triangles in the upper half-plane are log(3)-slim.

This implies that if we take any side of the triangle PQR, that side is contained within a δ-neighborhood of the union of the other two sides, which we will rigorously prove.

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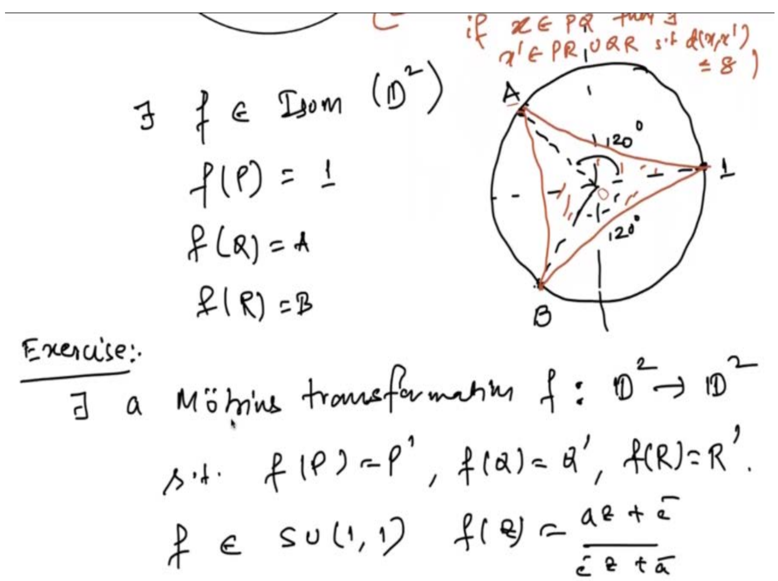

To begin, we will first establish the slimness property for what we call an "ideal triangle." An ideal triangle is defined as one where the vertices of the triangle lie on the boundary of the unit disc. For instance, in the diagram I have provided, the triangle PQR is positioned on the unit circle. Consequently, the geodesic connecting points P and Q will form a bi-infinite geodesic. Similarly, the geodesics connecting Q and R, as well as P and R, will also be bi-infinite geodesics. Thus, we classify this triangle as an ideal triangle.

Now, let's discuss another important observation concerning the properties of Möbius transformations. There exists an isometry of the unit disc that maps the point P to 1. By this mapping, we denote that f(P) = 1, which implies that point Q is mapped to point A.

So, what exactly is point A? Point A lies on the unit circle. To visualize this, let's establish our origin, denoted as O. We can draw a line segment from point 1 to the origin O, and another line segment from O to A.

Point A is positioned on the inner circle such that the angle $\angle 1OA$ measures $120°$. This means that the angle between the line segments is precisely 120°. Hence, we have f(Q) = A, and for point R, we will designate it to be B.

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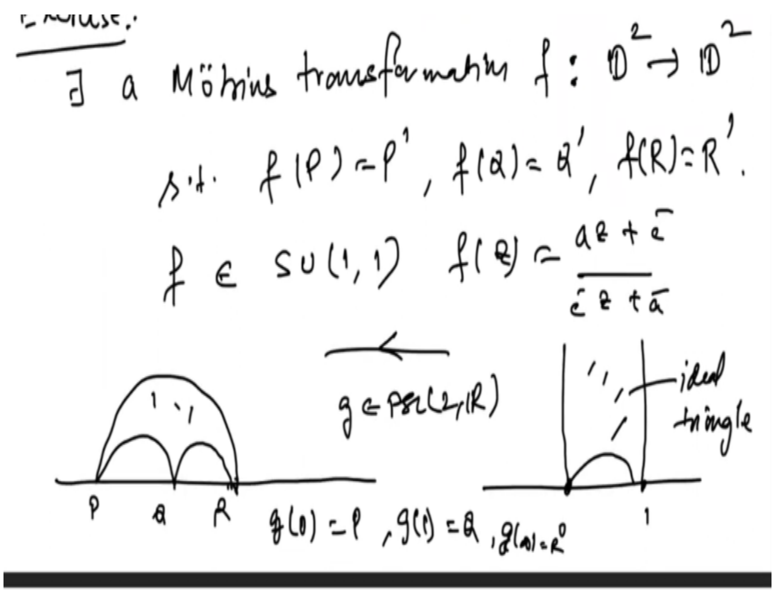

Now, what is point B in the context of the unit circle? It is derived from the segments AO and OB, where the angle ∠AOB also measures 120°. Naturally, this results in ∠1OB also being 120°. Thus, the points 1, A, and B are three equally spaced points on the inner circle.

This concept is a standard result in complex analysis, where we assert that f(P) = 1, f(Q) = A, and f(R) = B.

So, what is the standard fact in complex analysis? There exists a Möbius transformation f that maps the unit disc to itself, such that f(P) = P', f(Q) = Q', and f(R) = R'. Notably, this transformation f belongs to the group SU(1,1).

Specifically, the form of the transformation can be expressed as:

$$f(z) = \frac{az + \bar{c}}{\bar{c}z + \bar{a}}$$

where it is essential that $|a|^2 - |c|^2 = 1$.

Interestingly, a similar property holds true in the upper half-plane as well; you can verify this for yourself. This indicates the versatility and consistency of Möbius transformations across different geometric contexts in complex analysis.

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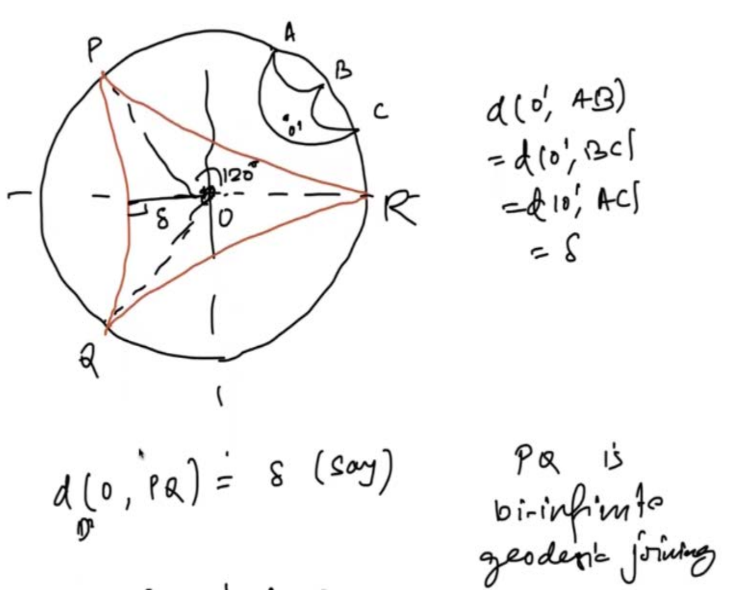

Now, what we aim to demonstrate is that for this triangle, whose sides are bi-infinite geodesics, each side of the triangle lies within a log 3 neighborhood of the union of the other two sides. This is the assertion we need to prove. Without loss of generality, based on our earlier discussions, we can position points P, Q, and R such that R = 1, with the angle between line segments 1 OP and 1 OQ each measuring 120 degrees.

Thus, we will prove that this triangle PQR is indeed slim. To begin, let us define δ as the hyperbolic distance from the origin O to the line segment PQ, where P and Q are connected by the bi-infinite geodesic. We will take this distance as our δ.

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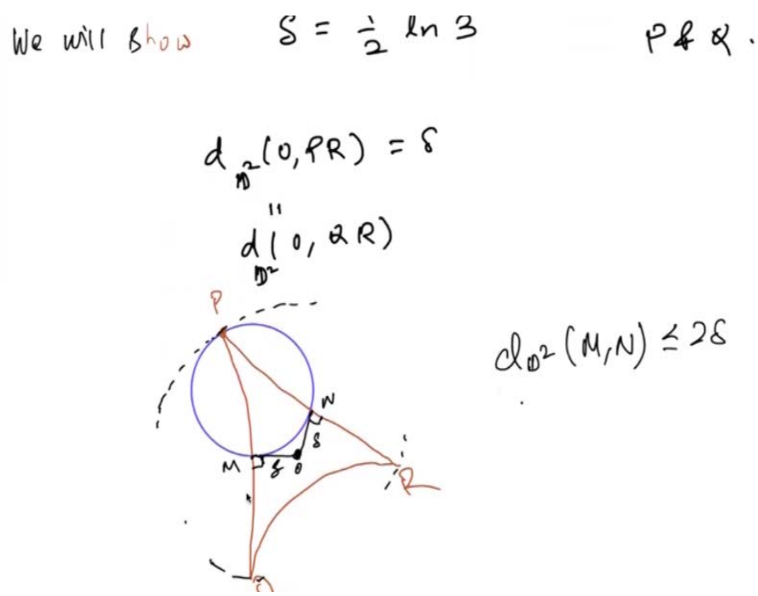

We will demonstrate that δ is actually equal to half of log 3. Furthermore, due to the symmetry in the distances, the distance from the origin O to the line segment PR and the distance from O to the line segment QR are also equal to δ.

Once we establish this, let's consider the point O. Let M be a point on the line segment PQ and N be a point on the line segment PR, such that the distance from O to M is δ and the distance from O to N is also δ. Naturally, by the triangle inequality, the distance between the points M and N will be less than or equal to twice δ.

Let's take a moment to explore an important observation. We have already discussed the horodisc based at points x and infinity in the upper half-plane. Now, if we apply the Möbius transformation given by

$$\phi(z) = \frac{z - i}{z + i},$$

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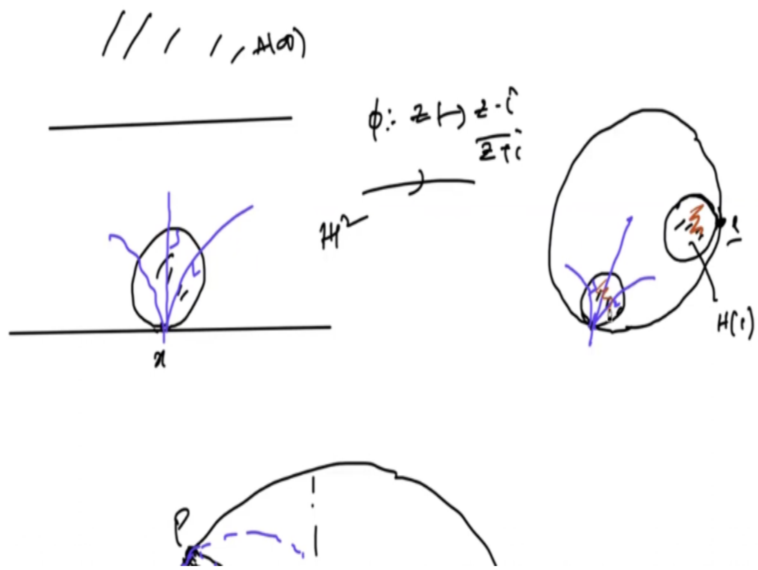

we can see that φ is indeed a Möbius transformation. Notably, this transformation will map circles to circles, and straight lines to circles, or vice versa.

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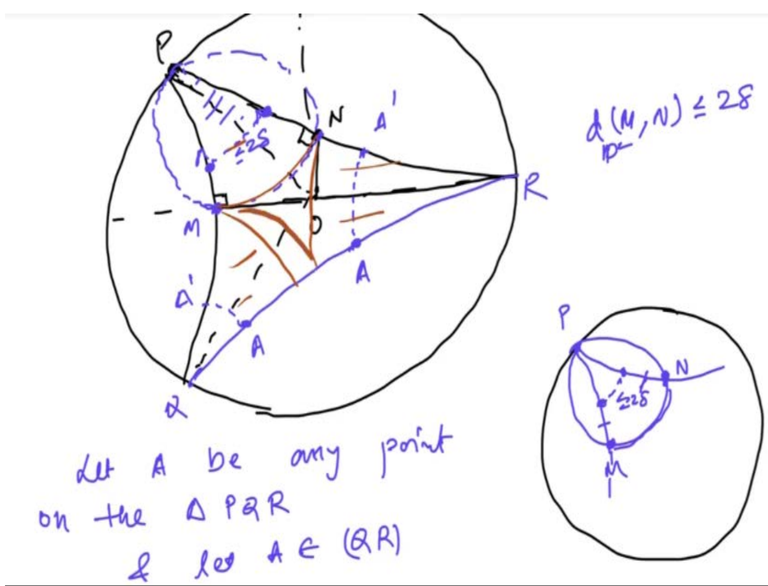

When we examine the horocycle based at x under the transformation φ, we find that it will also be transformed into a circle. Importantly, this newly formed circle will intersect the boundary circle at the point z = 1 only at a single location, which is a straightforward result to demonstrate. Since φ(∞) = 1, the image of this horizontal line will again be a circle located at the point 1. Therefore, we can conclude that both the horodisc and the horocycles are represented in this manner within the unit disc.

Now, let’s return to our illustration. We have triangle PQR situated within the unit disc. So far, we have established that the distance between points M and N is less than or equal to 2δ. Now, if we consider this distance s in the upper half-plane model, we can move upward by a distance s on both sides. Consequently, the length will become e^{2-s} times the length of this horocycle segment.

This implies that this distance will naturally be less than or equal to 2δ because it approaches zero as we move towards point P. Given the triangle PQR, we can neatly divide this triangle into sections. For instance, we can partition it in this manner, creating similar segments in the corresponding areas. All of these portions are indeed slim portions.

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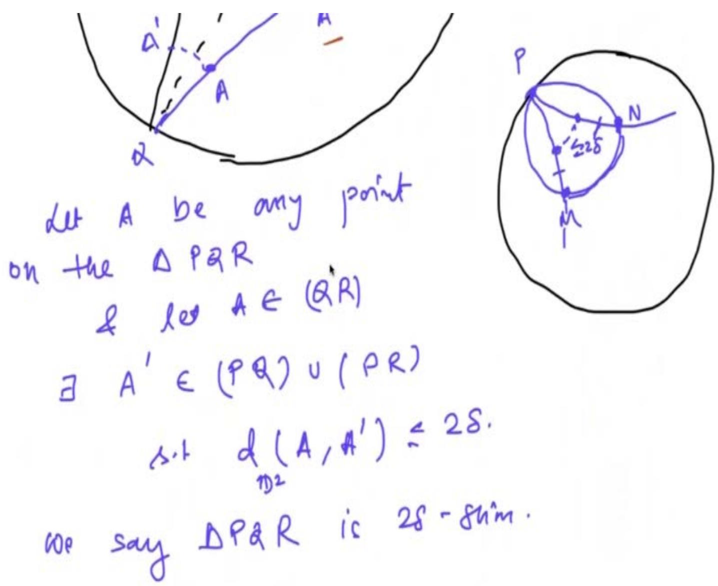

To illustrate further, if we select any point within this triangle, let’s say at some specific location, we will find that the distance from this point to the nearest side of the triangle is at most 2δ. For example, if I take another point here, we will again find that the distance is at most 2δ. This detail has been duly noted in my previous writings.

Let A be any point on the triangle PQR. Suppose we position point A on side QR. In this scenario, there exists a point A' such that the distance between A and A' is less than or equal to 2δ. This is a fact we have previously established. Now, our task is to determine the actual value of this δ.

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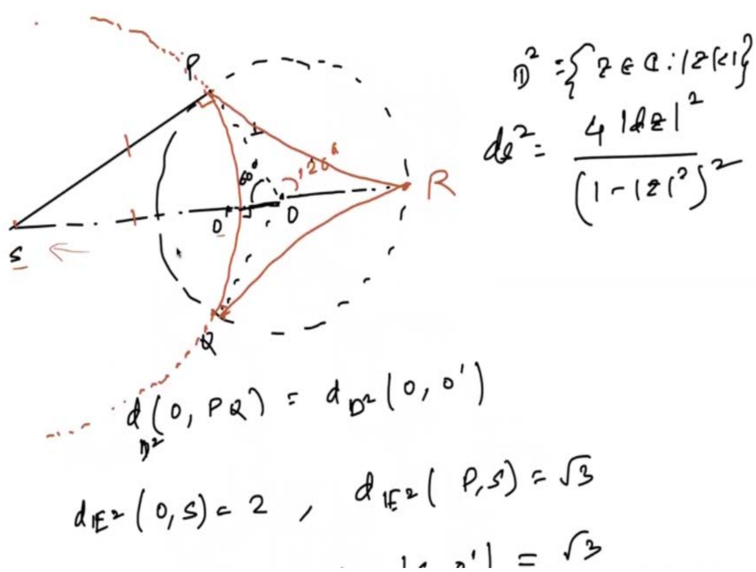

Let's delve into the computation of the length of δ. As illustrated in the accompanying picture, point O' represents the foot of the perpendicular drawn from O to the bi-infinite geodesic PQ. Consequently, the hyperbolic distance from O to PQ is equivalent to the hyperbolic distance between O and O'.

Now, if we extend this real line in the appropriate direction, we observe that the tangent to the unit circle at point P intersects the real axis at point S. The segment SP is, in fact, the radius of that circle. The Euclidean distance between points O and S is calculated to be equal to 2.

Why is the Euclidean distance between O and S equal to 2? It is because the angle between PO and O' measures 60°; this angle is effectively half of 120°. Now, consider triangle OBS, which is a right triangle. The angle OBS is a right angle (i.e., 90°), and we know that the Euclidean distance between O and P equals 1.

The base of the triangle is represented by the distance from O to P, divided by the distance from O to S. This results in a cosine of 60°, confirming that the distance between O and S is indeed 2.

Next, we need to find the Euclidean distance between points P and S. By applying the Pythagorean theorem in this Euclidean context, we find that this distance is $\sqrt{3}$. Lastly, the distance between points S and O' is equal to the radius of the circle centered at S, which is the distance SP.

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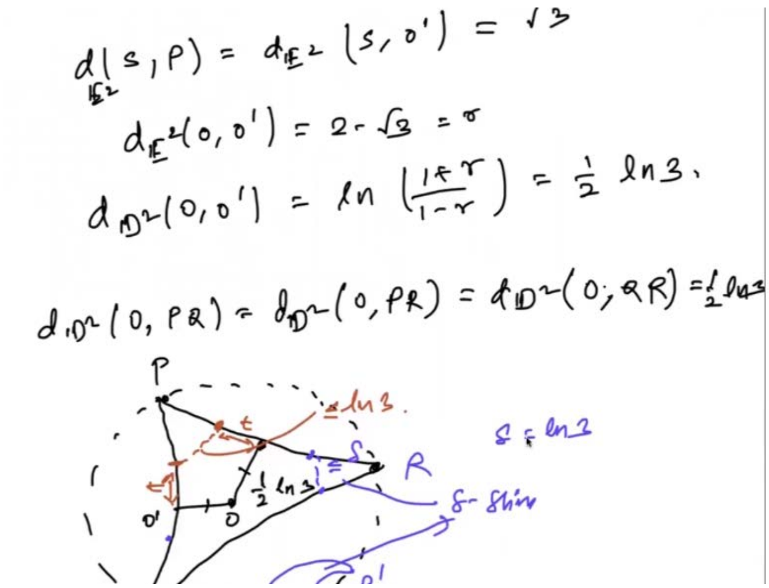

We have now computed the distance between points P and S, which amounts to $\sqrt{3}$. Consequently, the distance between points S and O' is also $\sqrt{3}$. To find the distance between O and O', we can express it as the difference: the distance OS minus the distance O'S. Therefore, we have:

$$\text{Distance}(O, O') = 2 - \sqrt{3}$$

Let's denote this distance as r. Thus, the hyperbolic distance between O and O' can be represented as:

$$\text{Hyperbolic Distance}(O, O') = \log\left(\frac{1+r}{1-r}\right)$$

Upon computation, we find that this simplifies to $\frac{1}{2}\log(3)$.

This means that the distances from point O to the geodesics PQ, PR, and QR are all equal to $\frac{1}{2}\log(3)$. Therefore, we conclude that δ is indeed equal to log(3).

As a result, we have established that triangle PQR is δ-slim. This holds true for any ideal triangle, such as triangle P', Q', R'. When we apply an isometry that maps P' to P, Q' to Q, and R' to R, this transformation ensures that triangle P'Q'R' is also δ-slim. Thus, we have successfully proven our claim.

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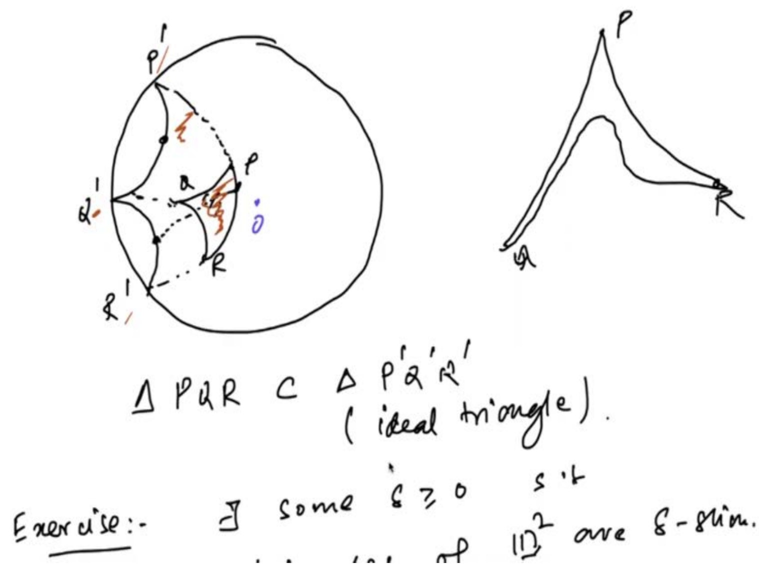

Now, let us consider a triangle positioned inside this unit disc. To illustrate, imagine triangle PQR situated within the unit disc, not touching the boundary. The next step involves extending one side of this triangle. Specifically, we will extend side PR such that this bi-infinite geodesic intersects the unit circle at points P' and R'.

Next, we extend the other side, QR, so that it meets the boundary at point Q'. Now, we can join the points P', Q', and R' to form a larger triangle, which we will refer to as the ideal triangle P'Q'R'.

This ideal triangle contains our original triangle PQR. Having established that the ideal triangle is slim, we can conclude that triangle PQR is also slim. This relationship between the two triangles confirms the slimness of our original triangle, as it is enclosed within an ideal triangle that meets the necessary criteria.

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all the triangles of

A triangle ΔPQR is δ- slim if any side of ΔPQR is contained in δ- neighborhood of union of other two sides.

(i.e. $x \in [PQ] \Rightarrow \exists\, x' \in [PR] \cup [QR]$ s.t. $d(x, x') \leq \delta$)

(triangles in Euclidean plane are "fat")

Let me summarize our findings clearly. We have established that triangle PQR is δ-slim. Specifically, this means that any side of triangle PQR is contained within a δ-neighborhood of the union of the other two sides. For instance, if we take side PQ and any point x that belongs to PQ, there exists a corresponding point x' on either side PR or QR such that the distance between x and x' is less than or equal to δ.

Thus, we conclude that all triangles in the unit disc model of the hyperbolic plane are δ-slim, where δ is equal to log 3. This δ value is uniform across all triangles within this model. However, it's essential to note that this property does not hold in the Euclidean plane, which is why we refer to triangles in Euclidean geometry as "fat."