

# An Introduction to Hyperbolic Geometry

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Lecture – 11

## Equivalence of Hyperbolic and Euclidean Topologies on the Upper Half-Plane

Hello everyone! In today's lecture, we will demonstrate that the topology on the upper half-plane with respect to the hyperbolic metric is indeed equivalent to the topology derived from the Euclidean metric. To establish this equivalence, we will show that hyperbolic circles can also be represented as Euclidean circles and, conversely, that Euclidean circles correspond to hyperbolic circles. However, it's important to note that while both types of circles will maintain their circular form, their centers and radii will differ. So, let us begin.

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Topology on  $\mathbb{H}^2$  coming from hyperbolic metric & Euclidean metric are equivalent

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Hyperbolic circle,  $\mathbb{H}^2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$   
 $d_{\mathbb{H}^2} \rightarrow$  hyperbolic distance function

Circle of radius  $r$  with center  $a \in \mathbb{H}^2$   
 $\{z \in \mathbb{H}^2 : d_{\mathbb{H}^2}(z, a) = r\}$  (Hyperbolic circle)

Euclidean circle  $\{z \in \mathbb{H}^2 : d_{\mathbb{E}^2}(z, a) = r\}$   $d_{\mathbb{E}^2}$  is the Euclidean metric on  $\mathbb{H}^2$ .

Proposition: Hyperbolic circles are Euclidean circles.

Proof: Let  $C$  be a hyperbolic circle in  $\mathbb{H}^2$  with center  $a \in \mathbb{H}^2$  & radius  $r$   
 $d_{\mathbb{H}^2}(z, a) = r$  if  $z \in C$ .

To prove that the topologies derived from the hyperbolic metric and the Euclidean metric on the upper half-plane are equivalent, we will begin by demonstrating that hyperbolic circles correspond to Euclidean circles.

First, let's define what we mean by a hyperbolic circle. We are working within the upper half-

plane, which consists of all complex numbers where the imaginary part is greater than zero. We utilize the hyperbolic metric, expressed as the Riemannian metric on this upper half-plane, which allows us to define a distance function that we will denote as  $d_H$ . This represents the hyperbolic distance.

Now, what exactly is a circle in this context? A circle of radius  $r$  with center  $a$  in the upper half-plane is defined as the set of all points  $z$  in the upper half-plane such that the hyperbolic distance from  $z$  to  $a$  is equal to  $r$ . This is what we refer to as a hyperbolic circle.

In addition to this, we also have the Euclidean metric on the upper half-plane. So, how do we define a Euclidean circle? Similar to the hyperbolic case, a Euclidean circle with center  $a$  and radius  $r$  comprises the set of points  $z$  in the upper half-plane such that the Euclidean distance  $d_E$  from  $z$  to  $a$  equals  $r$ .

Now, let's state our first claim:

Proposition: Hyperbolic circles are Euclidean circles.

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$$d_H(z, a) = \ln \left( \frac{|z-\bar{a}| + |z-a|}{|z-\bar{a}| - |z-a|} \right) = r$$

$$\frac{|z-\bar{a}| + |z-a|}{|z-\bar{a}| - |z-a|} = e^r$$

$$\frac{|z-\bar{a}|}{|z-a|} = \frac{e^r + e^{-r}}{e^r - e^{-r}} = f(r), \text{ say}$$

$\downarrow$   
constant  
 $f(r) > 1$

$$z = x+iy$$

$$a = a_1 + ia_2$$

$$|x+iy - (a_1 - ia_2)|^2 = f(r)^2 |x+iy - (a_1 + ia_2)|^2$$

$$|(x-a_1) + i(y+a_2)|^2 = f(r)^2 |(x-a_1) + i(y-a_2)|^2$$

$$\therefore (x-a_1)^2 + (y+a_2)^2 = f(r)^2 ((x-a_1)^2 + (y-a_2)^2)$$

$\therefore x^2 + y^2 + 2y a_2 + a_2^2 + k = 0 \quad (*)$

Since  $f(r) > 1$   $(*)$  represent a circle in  $H^2$  (Euclidean)

So, how do we prove this? Let  $C$  be a hyperbolic circle in the upper half-plane with center  $a$  and radius  $r$ . This implies that the hyperbolic distance between  $z$  and  $a$  is equal to  $r$  for all points  $z$  that

belong to  $\mathbb{C}$ . From this, we can derive the distance formula between  $z$  and  $a$ .

The distance between the points  $z$  and  $a$  in hyperbolic space is defined as follows:

$$d_H(z, a) = \log \left( \frac{|z - \bar{a}| + |z - a|}{|z - \bar{a}| - |z - a|} \right) = r$$

This means that we can express the equation as:

$$\frac{|z - \bar{a}| + |z - a|}{|z - \bar{a}| - |z - a|} = e^r$$

Now, let's rearrange this. We have:

$$|z - \bar{a}| = f(r) \cdot |z - a|$$

where  $f(r)$  is a constant defined by:

$$f(r) = \frac{e^r + e^{-r}}{e^r - e^{-r}}$$

It is important to note that  $f(r) > 1$ .

Next, let's express  $z$  in terms of its real and imaginary components:  $z = x + iy$  and let  $a = a_1 + ia_2$ .

Substituting these into our earlier equation yields:

$$|x + iy - (a_1 - ia_2)| = f(r) \cdot |x + iy - (a_1 + ia_2)|$$

We can manipulate this expression further. Taking the square of both sides, we obtain:

$$(x - a_1)^2 + (y - a_2)^2 = f(r)^2((x - a_1)^2 + (y - a_2)^2)$$

This results in an equation of the form:

$$x^2 + y^2 + 2gx + 2fy + h = 0$$

Here,  $f(r)$  being greater than 1 indicates that this equation represents a circle in the upper half-plane, specifically a Euclidean circle.

It is crucial to recognize that the center and radius of this circle will differ from those of the hyperbolic circle. Thus, what we have demonstrated so far is that hyperbolic circles indeed correspond to Euclidean circles. Now, let us proceed to prove the converse: that Euclidean circles are also hyperbolic circles.

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Proposition Euclidean circles in the hyperbolic plane are same as hyperbolic circles.

Proof  $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$ ,  $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$

Let  $C = \{z \in \mathbb{D}^2 \mid |z-a|=r\}$  be a circle in  $\mathbb{D}^2$ ,  $a \in \mathbb{D}^2$

Let  $w_0$  be an element on the geodesic such that  $d_{\mathbb{D}^2}(w_0, p) = d_{\mathbb{D}^2}(w_0, q)$  ( $w_0$  is the hyperbolic mid point of the geodesic segment joining  $p$  &  $q$ )

$\exists$  an isometry  $T \in \text{SU}(1,1)$  s.t.  $T(w_0) = 0$ ,  $T(p), T(q)$  lies on real axis

Proposition: Euclidean circles in the hyperbolic plane correspond to hyperbolic circles. To demonstrate this, we will utilize the unit disc model of the hyperbolic plane, as it simplifies the computations significantly. Thus, we will work within this model.

In the unit disc model, the hyperbolic metric is given by:

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

Now, let's consider a Euclidean circle  $C$  defined by the equation  $|z - a| = r$ , where  $z$  belongs to this disc, and the center  $a$  also resides within the unit disc.

To visualize this, imagine our unit disc, where we have drawn a Euclidean circle with center  $a$  and radius  $r$ .

Next, let's denote the center of the unit disc as  $O$ . We will connect  $O$  to  $a$  with a geodesic and

extend this geodesic until it intersects the circle  $C$ . This geodesic will represent a straight line, specifically a segment of the diameter of the unit disc. As a result, the geodesic intersects the circle  $C$  at two distinct points, which we will call  $p$  and  $q$ . Notably, the angle formed at the intersection is a right angle, confirming that this geodesic intersects the circle orthogonally.

Now, let's identify the intersection points  $p$  and  $q$  as the points where our geodesic meets circle  $C$ . Let's introduce a point  $w_0$  located on the blue geodesic segment such that the hyperbolic distance from  $w_0$  to  $p$  is equal to the distance from  $w_0$  to  $q$ . In other words,  $w_0$  serves as the hyperbolic midpoint of the segment joining  $p$  and  $q$ .

Positioned somewhere along this segment,  $w_0$  will act as the center of the hyperbolic circle, which is what we aim to establish.

Now, let's concentrate on the geodesic segment connecting points  $p$  and  $q$ . We know that there exists an isometry  $T$  belonging to the group  $SU(1,1)$  that acts by isometry on the unit disc. As previously noted in an exercise, this isometry  $T$  can be chosen such that  $T(w_0) = 0$ , and both  $T(p)$  and  $T(q)$  lie on the real axis.

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$$d_{\mathbb{D}^2}(0, \tau(p)) = d_{\mathbb{D}^2}(0, \tau(q))$$

$$\Rightarrow d_{\mathbb{E}^2}(0, \tau(p)) = d_{\mathbb{E}^2}(0, \tau(q))$$

$T(C)$  is a curve passing through  $\tau(p)$  &  $\tau(q)$ .

$T$  is a Möbius transformation

$T(C)$  is a circle (Euclidean)

$\forall w \in T(C), d_{\mathbb{E}^2}(0, w)$  is constant

$T(C)$  is also a hyperbolic circle.

$$C = T^{-1}(T(C))$$

$T$  is an isometry,  $T \in \text{Isom}(\mathbb{D}^2)$

$T^{-1}$  is an isometry

$$\Rightarrow C \text{ is a hyperbolic circle.}$$

Consequently, through this isometric transformation, the Euclidean circle  $C$  is mapped to a new curve in the hyperbolic plane. The point  $T(p)$  corresponds to one intersection point, while  $T(q)$

corresponds to the other intersection point. Thus, the geodesic segment connecting  $p$  and  $q$  is transformed by this isometry into a geodesic that passes through point  $O$ .

Through this process, we have demonstrated that Euclidean circles in the hyperbolic plane indeed correspond to hyperbolic circles, completing our proof.

Now, let's clarify our findings. We know that the hyperbolic distance from point  $O$  to  $T(p)$  is equivalent to the hyperbolic distance from  $O$  to  $T(q)$ . Since we are using the disc model, this means that the distances are indeed the same. Consequently, we can conclude that the Euclidean distance from  $O$  to  $T(p)$  is also equal to the Euclidean distance from  $O$  to  $T(q)$ .

Next, we examine the image of the Euclidean circle  $C$  under the transformation  $T$ . The transformed curve  $T(C)$  passes through the points  $T(p)$  and  $T(q)$ . Since  $T$  is a Möbius transformation, we can verify that  $T(C)$  will also be a circle. It is important to note that the geodesic segment from  $p$  to  $q$  intersects circle  $C$  orthogonally. Since  $T$  is a conformal map,  $T(C)$  will be a curve that intersects the geodesic segment joining  $T(p)$  and  $T(q)$  at right angles. Thus,  $T(C)$  must be a circle.

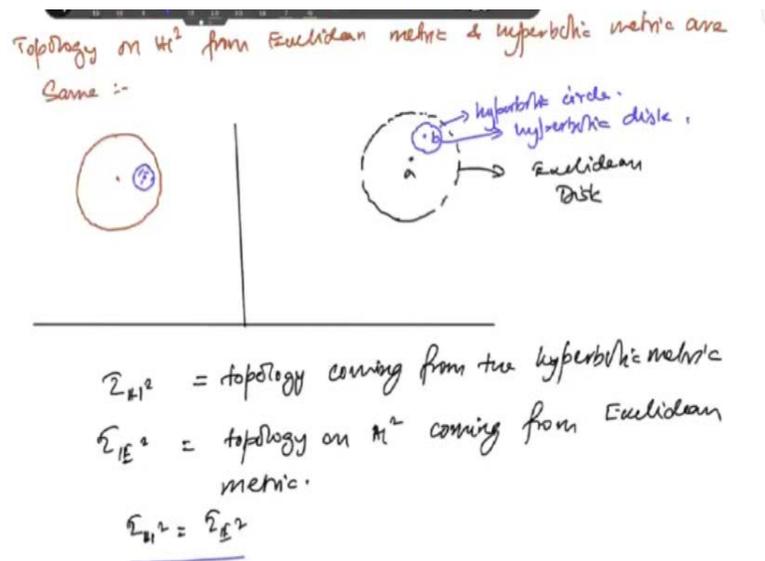
Having established that  $T(C)$  is a circle, we can now consider any point  $w$  that lies on  $T(C)$ . The hyperbolic distance from  $O$  to  $w$  remains constant—this computation is one we have performed previously. Therefore, we have shown that  $T(C)$  is an Euclidean circle. Furthermore, for any point  $w$  on this circle, the distance from the origin to that point is again constant, a fact we discussed in our last lecture.

This leads us to conclude that  $T(C)$  is also a hyperbolic circle. Now, since  $C$  is defined as  $T^{-1}(T(C))$  and since  $T$  is an isometry of the unit disc, it follows that  $T^{-1}$  is also an isometry. Isometries preserve hyperbolic circles, which means that  $C$  must indeed be a hyperbolic circle.

Thus, we have proved that if  $C$  is a Euclidean circle, then  $C$  is also a hyperbolic circle. From this result, it follows easily that the topologies on the upper half-plane, derived from the hyperbolic metric and the Euclidean metric, are equivalent.

Now, let's explore how the topologies on the upper half-plane derived from the Euclidean metric and the hyperbolic metric are, in fact, the same. To demonstrate this, we begin by considering the upper half-plane, which we will denote as our  $H^2$ .

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In this space, we have two distinct topologies: let  $\tau_{\mathbb{H}^2}$  represent the topology induced by the hyperbolic metric, while  $\tau_{\mathbb{E}^2}$  refers to the topology induced by the Euclidean metric. Our goal is to establish that these two topologies are equivalent.

To achieve this, we can focus on the basis elements of each topology. We start with a Euclidean disc, which serves as a fundamental building block for the Euclidean topology on the upper half-plane. Let's denote a point  $b$  within this disc. As we know, there exists a Euclidean disc that contains this point  $b$ . Intriguingly, this same disc will also correspond to a hyperbolic circle.

Thus, we see that within any given Euclidean disc, there exists a hyperbolic disc. The converse is also true: if we begin with a hyperbolic disc and select any point within it, we can find a Euclidean disc that is entirely contained within this hyperbolic disc.

From these observations, it naturally follows that the two topologies are indeed the same. This concludes our discussion on the equivalence of the topologies derived from the Euclidean and hyperbolic metrics in the upper half-plane. Thank you!