An Introduction to Hyperbolic Geometry

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Lecture - 10

Hyperbolic Trigonometry: Sine and Cosine Rules in the Disc Model

Hello, and welcome to this lecture on hyperbolic geometry! In our previous session, we explored the disk model of the hyperbolic plane. Building on that foundation, we will now delve into hyperbolic trigonometry. Specifically, we will be proving the sine rule and two versions of the cosine rule.

These hyperbolic analogs of the sine and cosine rules closely mirror the sine and cosine rules you are familiar with from Euclidean geometry. Furthermore, by utilizing these sine and cosine tools, we will also demonstrate the Pythagorean theorem within the context of the hyperbolic plane. So, let us begin.

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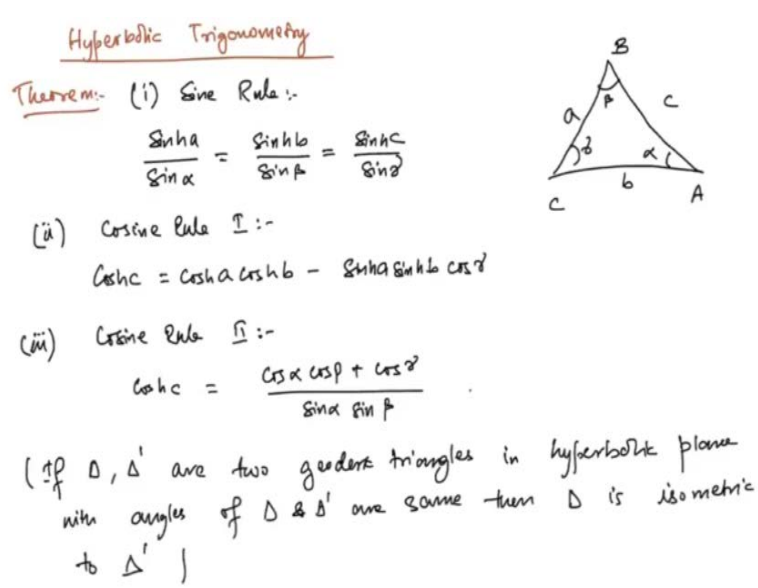

Let me begin by outlining the sine and cosine rules. First, we have the sine rule. Consider a geodesic triangle within the unit disk equipped with the hyperbolic metric. This triangle has vertex angles denoted by α, β, and γ, with sides represented as a, b, and c. The sine rule states that:

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}$$

Now, let’s move on to the first cosine rule. The first cosine rule can be expressed as:

$$\cosh(c) = \cosh(a) \cdot \cosh(b) - \sinh(a) \cdot \sinh(b) \cdot \cos(\gamma)$$

Next, we have the second cosine rule, which is formulated as:

$$\cosh(c) = \frac{\cos(\alpha) \cdot \cos(\beta) + \cos(\gamma)}{\sin(\alpha) \cdot \sin(\beta)}$$

Now, I’d like to emphasize the significance of this second cosine rule. Suppose we have two geodesic triangles, Δ and Δ', within the hyperbolic plane. You can utilize either the disk model or the upper half-plane model, as both are isometric to one another; thus, all these sine and cosine rules hold true in both frameworks.

If the triangles Δ and Δ' have the same angles, then Δ is isometric to Δ'. This application of the second cosine rule is crucial because, when the angles α, β, and γ are identical, we find that $\cosh(c) = \cosh(c')$. Consequently, this leads to the conclusion that c = c', indicating that the lengths of the sides are also equal. Thus, the two triangles must be isometric to one another.

This property does not hold in Euclidean geometry. In the Euclidean plane, it is indeed possible to have two triangles with the same vertex angles but differing side lengths, which illustrates a fundamental difference between the two geometries.

Let me present another important corollary of the second cosine rule, which is none other than the Pythagorean theorem. Imagine a geodesic triangle, situated either in the upper half-plane or within the unit disk, where one of the vertex angles measures 90 degrees. We label the sides of this triangle as a, b, and c. According to the first cosine rule, we have:

$$\cosh(c) = \cosh(a) \cdot \cosh(b) - \sinh(a) \cdot \sinh(b) \cdot \cos(90°)$$

Since $\cos(90°) = 0$, it follows that:

$$\cosh(c) = \cosh(a) \cdot \cosh(b)$$

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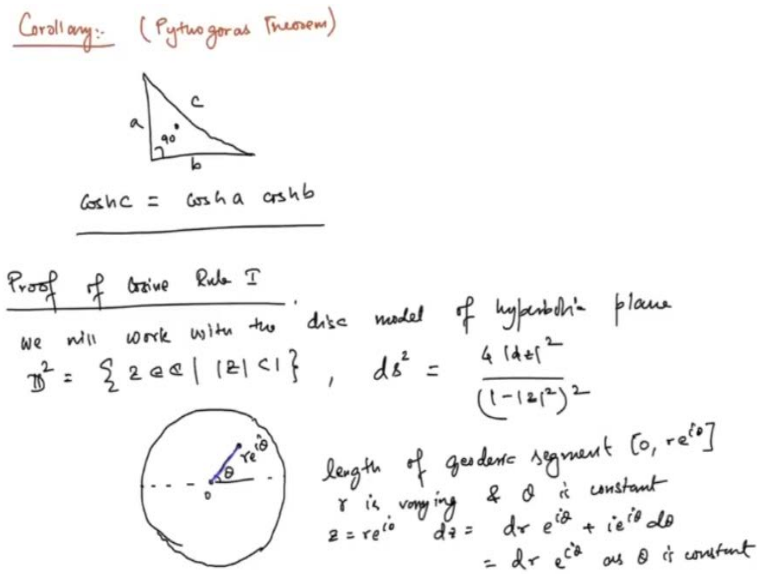

This statement elegantly encapsulates the Pythagorean theorem in the context of hyperbolic geometry. Now, let’s delve into the proof of this theorem. I will begin by proving the first cosine rule, from which we can derive the sine rule, and subsequently demonstrate how these two results lead us to the second cosine rule.

To start, we will work within the disk model of hyperbolic geometry. We consider the unit disk, defined such that $|z| < 1$. The hyperbolic metric here is given by:

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

Now, let’s visualize our unit disk, marking the origin. We will focus on a geodesic segment that connects the origin to a point expressed as $re^{i\theta}$. Our goal is to compute the length of this geodesic segment.

To express this length in terms of the hyperbolic metric, note that in this context, r is a variable while θ remains constant. Thus, we have $z = re^{i\theta}$. The differential dz can be represented as:

$$dz = dr \cdot e^{i\theta} + i \cdot r \cdot e^{i\theta} \cdot d$$

Since θ is constant, dθ = 0, which simplifies our expression to:

$$dz = dr \cdot e^{i\theta}$$

This foundational setup will allow us to continue with our proof and explore the relationships inherent in hyperbolic geometry.

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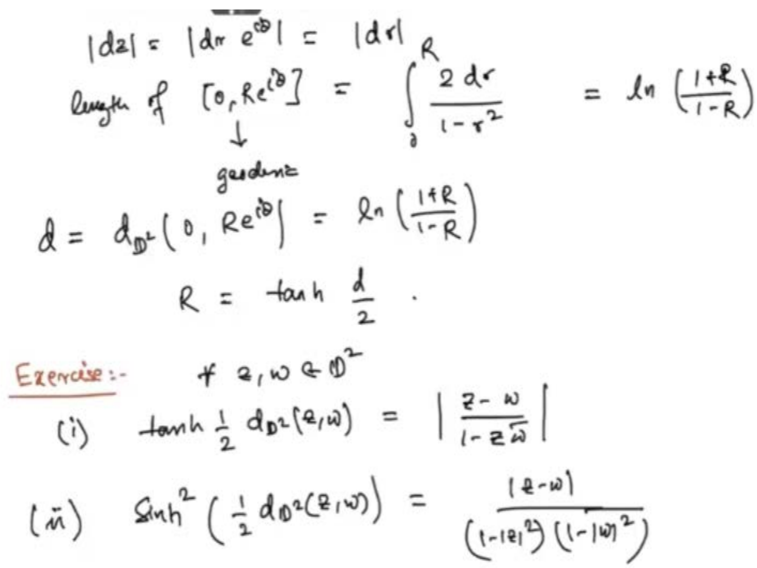

Now, let's analyze the expression for the modulus of z. We have:

$$|z| = \left|dr \cdot e^{i\theta}\right|$$

Since the modulus of $e^{i\theta}$ is equal to 1, we can simplify this to:

$$|z| = |dr|$$

This means that the length of the geodesic segment is given by the integral from 0 to r of |dz|. Thus, we express this as:

$$\text{Length} = \sqrt{\int_0^r |dz|} = \sqrt{\int_0^r \frac{2\,dr}{1-r^2}}$$

To facilitate our calculations, we can replace r with a capital R. Consequently, we find:

$$\text{Length} = \sqrt{\int_0^R \frac{2\,dR}{1-R^2}}$$

This integral evaluates to:

$$\log\left(\frac{1+R}{1-R}\right)$$

Furthermore, since we are dealing with a geodesic segment, we can assert that the distance in the unit disk from 0 to the point $Re^{i\theta}$ is given by:

$$d = \log\left(\frac{1+R}{1-R}\right)$$

It can be shown that this capital R is equivalent to:

$$\tanh\left(\frac{d}{2}\right)$$

You can undertake this exercise to verify it. If we take any two points in this unit disk, the hyperbolic distance between points z and w can be expressed as:

$$\tanh\left(\frac{d}{2}\right) = \frac{|z-w|}{1-z\overline{w}}$$

And for the sine hyperbolic square of half the hyperbolic distance between z and w, we have:

$$\sinh^2\left(\frac{d}{2}\right) = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}$$

Now that we have established these relationships, we are ready to proceed with the proof of the first cosine rule.

Let us consider a triangle ABC within the unit disk. Imagine that this is our triangle. With the aid of an isometry, we can position this triangle such that one vertex lies at the origin, and one side rests along the real axis. This transformation is always achievable. Therefore, we can express this triangle as OAB after applying the isometry.

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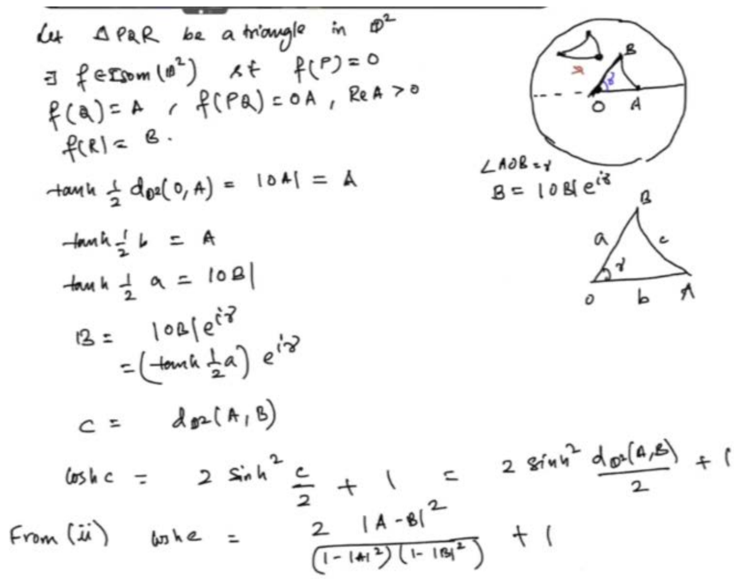

To clarify, let's rename our triangle PQR such that f(P) = O, f(Q) = A, and the geodesic segment PQ aligns with OA. Here, PQ represents the geodesic connecting points P and Q, while OA denotes the geodesic extending from the origin to point A, with f(R) = B. Since f is an isometry, if I successfully prove the cosine rule for triangle OAB, it will also hold true for triangle PQR.

Now, in triangle OAB, the angle at vertex O is denoted as γ, and points A and B are represented as complex numbers. Notably, since OB is a geodesic, it follows a straight line. Thus, the angle AOB measures γ. Point B can be expressed as:

$$B = |OB| \cdot e^{i\gamma}$$

Next, we examine the hyperbolic distance. The hyperbolic tangent of half the distance between points O and A relates to |OA|:

$$\tanh\left(\frac{d}{2}\right) = |OA|$$

From our earlier analysis, we have established that d represents the hyperbolic distance between the origin O and the point R $e^{i\theta}$, which equals r. Thus, we find that:

$$r = \left|Re^{i\theta}\right|$$

Assuming A lies on the real axis, we can safely state that the real part of A is positive. Consequently, we can conclude that:

|OA| = A

Here, A is a positive real number. Now, let's represent the lengths of the triangle's sides as a, b, and c. According to the setup, the hyperbolic distance between points A and B is denoted by c, with angle γ opposite to side b, and the length from O to B represented as a. Thus, we can express:

$$\tanh(b) = A$$

Similarly, we have:

$$\tanh\left(\frac{A}{2}\right) = |OB|$$

Since point B is defined as $|OB| \cdot e^{i\gamma}$, we can relate:

$$|OB| = \tanh\left(\frac{A}{2}\right) \cdot e^{i\gamma}$$

Furthermore, the length c corresponds to the hyperbolic distance between vertices A and B. Using the cosine rule, we can derive that:

$$\cosh(c) = 2\sinh^2\left(\frac{c}{2}\right) + 1$$

To correct a previous oversight, we must note that this relation should be expressed as:

$$2\sinh^2\left(\frac{c}{2}\right) + 1$$

Now, combining the components we've discussed, we find that:

$$2\cosh(c) = \frac{2|A - B|^2}{(1 - |A|^2)(1 - |B|^2)} + 1$$

This relationship leads us to the conclusion, provided we apply Exercise 2 effectively. Thus, we can successfully prove the cosine rule as required.

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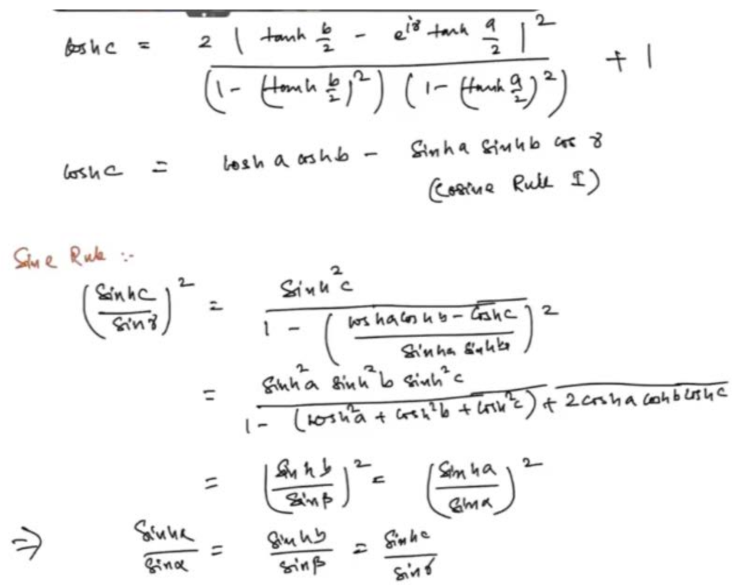

Now, we know that cosh(c) is equal to twice the expression we derived. Specifically, let's consider A and B. Here, A is given by $\tanh\left(\frac{b}{2}\right)$, while B is a complex number expressed as $e^{i\gamma} \cdot \tanh\left(\frac{a}{2}\right)$. This results in the formula:

$$\cosh(c) = 2\frac{\left|\tanh\left(\frac{b}{2}\right) - e^{i\gamma} \cdot \tanh\left(\frac{a}{2}\right)\right|^2}{1 - \tanh^2\left(\frac{b}{2}\right)\left(1 - \tanh^2\left(\frac{a}{2}\right)\right)} + 1$$

Now, applying the formula for tanh, we find that:

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma)$$

Thus, we have established the first cosine rule, which we refer to as Cosine Rule 1.

Now, let's turn our attention to the sine rule. This can be derived through some manipulation of the expressions we have already established. Specifically, if we calculate the ratio:

$$\frac{\sinh(c)}{\sin(\gamma)}$$

we find that this is equivalent to the expression:

$$\frac{\sinh^2(c)}{\sin^2(\gamma)} = \frac{\sinh^2(c)}{1 - \cos^2(\gamma)}$$

Using Cosine Rule 1, we can substitute and manipulate the expression to arrive at:

$$1 - \cosh(a)\cosh(b) - \cosh(c)$$

This expression can be rearranged and simplified, leading us to:

$$\frac{\sinh^2(a)\sinh^2(b)\sinh^2(c)}{1 - (\cosh^2(a) + \cosh^2(b) + \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c))}$$

The right-hand side is symmetric with respect to A, B, and C. Therefore, if you swap a with b, b with c, and c with a, you will obtain the same formula.

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Consequently, we can conclude that:

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}$$

This gives us our Sine Rule. I will leave the proof for Cosine Rule 2 as an exercise for you to explore further.

Now, let's proceed to prove Cosine Rule 2. This result will naturally follow from our earlier derivations of Cosine Rule 1 and the Sine Rule. So, I will stop here.