An Introduction to Hyperbolic Geometry
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Module - 1
Lecture - 1
Origin of Hyperbolic Geometry and Course Summary

Hello everyone, and welcome to this lecture. Today, I will begin by providing a brief overview of the origins of hyperbolic geometry, followed by a summary of the course and its key details. Let's dive in. Now, hyperbolic geometry is a branch of non-Euclidean geometry, which might sound a bit unfamiliar at first. To grasp the concept of hyperbolic geometry, it's crucial to first have a solid understanding of Euclidean geometry, which most of us are already familiar with from our school days. So, before we delve deeper into hyperbolic geometry, let's take a moment to revisit the fundamentals of Euclidean geometry.

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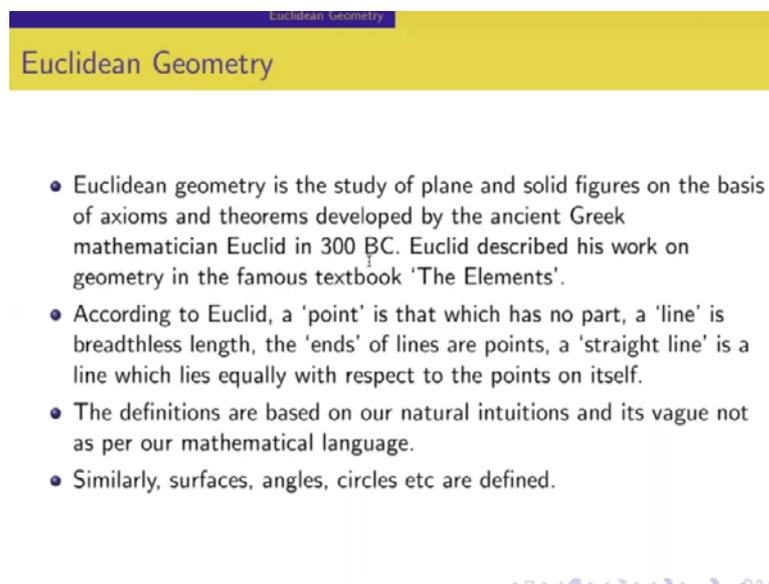
Euclidean geometry, as the name suggests, originates from the ancient Greek mathematician Euclid. It forms the foundation for the study of plane and solid figures, all based on the axioms that Euclid meticulously developed. Euclid compiled his groundbreaking work on geometry in the famous textbook The Elements, which remains a cornerstone of mathematical education even today.

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According to Euclid's definitions, a point is something that has no parts or dimensions, essentially, it's dimensionless. A line, as he described it, is a breadthless length, meaning it extends infinitely in both directions without any width. The ends of lines are defined as points, and a straight line is one that maintains equal distance with respect to all points along its length. These definitions, though abstract, align with our natural intuitions about how we perceive space and shapes. In a similar manner, Euclid went on to define surfaces, angles, circles, and other geometric figures, all based on simple yet profound principles.

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Euclidean geometry is built upon five fundamental axioms, or postulates, that were established by Euclid. Let's go through these one by one.

The first axiom is straightforward: if you are given two points, you can always draw a straight line between them. Essentially, it states that a straight line can be drawn from any point to any other point.
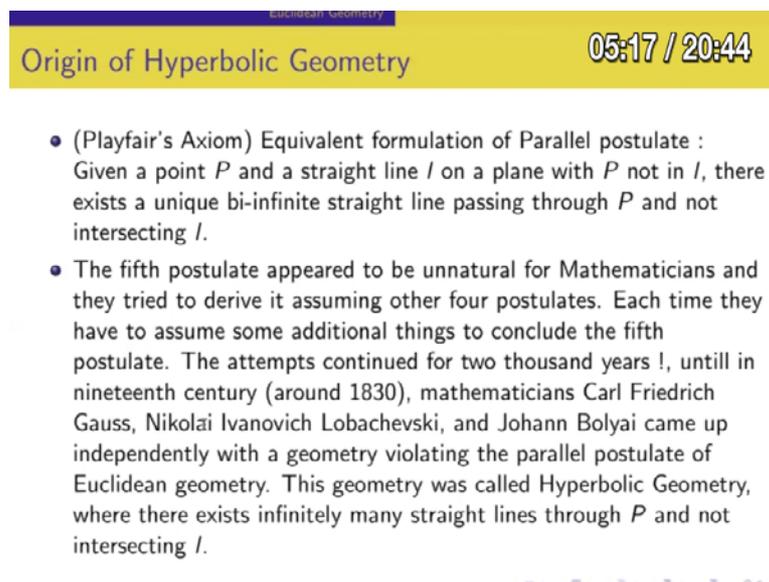
The second axiom extends this idea by stating that a finite straight line can be extended indefinitely in a straight line. In other words, you can continue a straight line as far as you wish without changing its direction.

Now, the third axiom introduces the concept of circles. It says that given any point, you can use it as the center, and with a given distance (or radius), you can construct a circle. The point becomes the center, and the chosen distance defines the radius of the circle.

The fourth axiom is equally important, stating that all right angles are equal to one another. This is a foundational property of right angles and plays a critical role in geometric reasoning.

Now, let's move on to the fifth axiom, which is the most significant of all, it's known as the parallel postulate. This postulate says the following: If the sum of two interior angles, let's call them α and β, is less than 180°, then two lines, let's label them l and m, when extended indefinitely, will eventually meet. To illustrate this, imagine two lines, l and m, and a third line n intersecting both l and m. If the sum of the interior angles α and β formed by n is less than 180°, then the lines l and m will meet at some point when extended. This postulate is what distinguishes Euclidean geometry from non-Euclidean geometries and has been the subject of much exploration and study throughout the history of mathematics.

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An alternative formulation of the parallel postulate was provided by the mathematician Playfair. According to Playfair's version, given a point P and a straight line l on a plane, where P does not lie on l, there exists exactly one bi-infinite straight line passing through P that does not intersect l. This is an equivalent, yet slightly more intuitive way to express Euclid's fifth postulate.

However, historically, mathematicians found this fifth postulate to be somewhat unnatural. For centuries, they attempted to prove it using the other four axioms, but each time they were forced to introduce additional assumptions in order to reach a conclusion that upheld the fifth postulate.

This effort spanned nearly 2000 years, until the 19th century, around the 1830s. During this period, three mathematicians, Carl Friedrich Gauss, Nikolai Lobachevsky, and Johann Bolyai, independently developed a new type of geometry that directly contradicted the parallel postulate of Euclidean geometry. In this alternative geometry, known as hyperbolic geometry, there exist infinitely many straight lines passing through the point P that do not intersect the line l. This discovery marked a significant departure from traditional Euclidean thinking and led to the birth of non-Euclidean geometries, with hyperbolic geometry being a prime example.

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So far, we've discussed the Euclidean axioms established by Euclid. Now, let's look at a model for the Euclidean plane where these axioms hold. In this model, we can clearly define concepts like points, straight lines, distances, angles, and circles. This approach, which many of you encountered during your school years, is known as analytic geometry. It was introduced by René Descartes and Pierre de Fermat around 1637.

To begin, we consider the Euclidean plane as R^2, the set of all ordered pairs (x, y), where both x and y are real numbers. Next, we define the distance between two points. Given points (x_1, y_1) and (x_2, y_2), the distance between them is calculated as:

$$d = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}$$

This is known as the Euclidean distance formula.

Now, what is a straight line in this context? A straight line can be described using a map α from

any interval to R^2. Specifically, $\alpha(t)$ is the convex combination of two points (x_1, y_1) and (x_2, y_2), expressed as:

$$\alpha(t) = (1 - t)(x_1, y_1) + t(x_2, y_2)$$

Similarly, we can define circles and angles. Angles between vectors are defined using the concept of an inner product. For two vectors (x_1, y_1) and (x_2, y_2), the inner product in R^2 is given by:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + y_1 y_2$$

Using this inner product, we can define norms, and subsequently, the angle between two vectors. With these tools, we can verify that all of Euclid's axioms are satisfied in this model.

Furthermore, we can compute the length of any differentiable path in R^2 using this metric. If you have a path α from a closed interval [a, b] to R^2, you can calculate its Euclidean length. Suppose $\alpha(t) = (x(t), y(t))$, the length of the path α is given by the following integral:

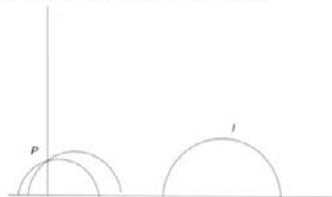$$\text{Length of } \alpha = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

This formula allows us to compute the length of any differentiable path within the Euclidean plane R^2.

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Now that we have established a model for the Euclidean plane, let's shift our focus to constructing a model for the hyperbolic plane, where the axioms of hyperbolic geometry will hold.

We begin with the upper half-plane model. This upper half-plane consists of ordered pairs $(x, y) \in R^2$, where $y > 0$. To define the metric on this plane, we use the following formula for the line element:

$$ds^2 = (dx^2 + dy^2)/y^2$$

This is a brief overview, and we will go into much greater detail later. However, using this formula, you can compute the length of any differentiable path α in the upper half-plane by integrating along the path. This formula is referred to as the hyperbolic metric. Don't worry, we will cover all these details thoroughly later.

In hyperbolic geometry, the concept of straight lines is replaced by geodesics. Geodesics are defined as the paths of shortest length between two points, and in the context of this hyperbolic plane, geodesics are vertical lines or semicircular arcs whose centers lie on the real axis.

For example, in the diagram provided, you can see a vertical line, which is a geodesic. Similarly, any semicircle centered on the real axis also represents a geodesic. What's interesting here is that this model directly violates the fifth postulate of Euclidean geometry, also known as the parallel postulate.

To illustrate this violation, let's consider a point P in the upper half-plane and a geodesic l that does not intersect P (i.e., P is not on l). Through the point P, you can draw a vertical geodesic as well as multiple semicircles centered on the real axis. All of these geodesics pass through P but do not intersect the given line l. In fact, there are infinitely many such geodesics passing through P that do not intersect l, which clearly violates Euclid's fifth postulate. This is a key feature of hyperbolic geometry and demonstrates its departure from traditional Euclidean concepts.

Now, why is hyperbolic geometry so intriguing? Let me pose a question to you: Can you find a simply connected metric space with constant curvature? If you've encountered the concept of a fundamental group, you'll know that a space is simply connected when it is both path-connected and has a trivial fundamental group.

In terms of surface geometry, curvature refers to the bending of a surface or space. The challenge here is to identify a simply connected surface with constant curvature. Let's start with a few examples to help clarify.

Take the Euclidean plane, R^2. It is flat, which means the curvature at every point is zero, making it a constant curvature surface. On the other hand, consider the unit sphere embedded in R^3. This surface has a constant curvature of +1 at every point. So, we now have two examples:

- The Euclidean plane with curvature 0 at each point.
- The unit sphere with curvature +1 at each point.

But what about the hyperbolic plane? The hyperbolic plane, often modeled as the upper half-plane equipped with the hyperbolic metric, has a constant curvature of -1 at every point. Proving this is a bit more complex, but it's crucial to note that this type of curvature cannot be easily visualized.

When I say "it cannot be visualized," we need to clarify what we mean by that. The renowned mathematician David Hilbert proved that the hyperbolic plane cannot be isometrically embedded in R^3. In other words, we cannot represent the hyperbolic plane in our typical 3-dimensional space in a way that preserves its geometric properties. So, when we talk about "visualization" here, we mean something that can be embedded in R^3, and hyperbolic geometry defies this.

Unfortunately, in this course, we won't delve into the details of Hilbert's proof, as it is highly intricate. However, let's move on to another fascinating question: Can we identify surfaces with constant curvature?

A surface, in this context, is simply a two-dimensional manifold. Let's consider the torus, which is topologically equivalent to $S^1 \times S^1$. It can be shown that the torus admits a metric such that its curvature at every point is zero. This metric naturally arises from the Euclidean plane, meaning the torus can be viewed as having zero curvature across its surface.

Now, consider the case where the genus g is greater than or equal to 2. That is, take a connected sum of g tori (i.e., a surface made by connecting multiple tori together). For $g \geq 2$, we can prove that this connected sum of tori admits a hyperbolic metric. This is an important result that we will explore in this course.

Furthermore, it can be shown that there exists a covering map from the upper half-plane to this surface. By pushing the hyperbolic metric from the upper half-plane onto this surface, the curvature at each point of the surface becomes -1, meaning the surface inherits the constant negative curvature of the hyperbolic plane.

This result emphasizes the power of hyperbolic geometry in describing spaces and surfaces with negative curvature, providing a stark contrast to the flatness of Euclidean geometry and the positive curvature of spherical geometry.

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Now, let's discuss the course summary and what you can expect to gain from it. To start with the fundamentals, you will need some background knowledge in complex analysis, topology, and a bit of algebra. For complex analysis, we will focus on topics such as Möbius transformations, cross-ratio, Cauchy-Riemann equations, and holomorphic functions. You will also need some familiarity with algebraic structures, particularly in the context of group theory, where concepts like the fundamental group and computational methods are important.

Don't worry, though, I will cover these topics as they come up, so you won't feel overwhelmed.

Moving on to hyperbolic geometry, we will delve into various models of hyperbolic space. We've already discussed the upper half-plane model, and we will also explore the unit disc model with the Poincaré metric. Throughout the course, we will touch on topics such as the hyperbolic inner product, geodesics, isometry groups, classification of isometries, areas of triangles, and related trigonometric identities.

Next, we will revisit some topology concepts. We will discuss properly discontinuous actions, cocompact actions, Fuchsian groups, covering spaces, fundamental regions, and tessellations of the upper half-plane. We will also explore some algebraic properties of Fuchsian groups. After that, we will introduce hyperbolic surfaces and state the Poincaré theorem. Later, we will examine closed curves and closed geodesics on hyperbolic surfaces.

Once we cover the theoretical groundwork, we will shift our focus to the applications of hyperbolic geometry in metric spaces and group theory. This is where we introduce hyperbolic groups. In order to define hyperbolic groups, we will first need to understand concepts such as slim and thin triangles, as well as δ-hyperbolic metric spaces. We will prove that in a hyperbolic metric space, geodesics diverge exponentially, a concept known as exponential divergence.
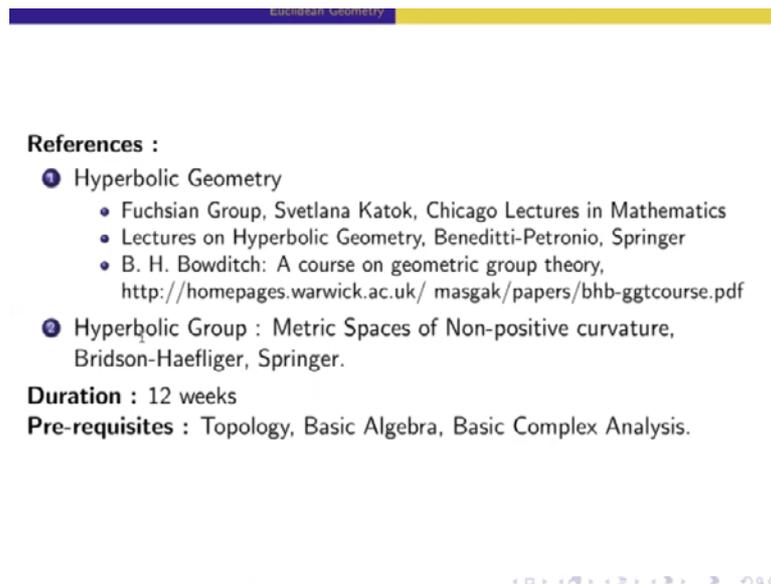
We will also define quasi-geodesics and prove the stability of these curves in hyperbolic spaces. Finally, we will define hyperbolic groups and discuss some of their key properties. This will include addressing the word problem and conjugacy problem, and we will demonstrate that both problems are solvable within the framework of hyperbolic groups.

By the end of the course, you will have a deep understanding of hyperbolic geometry and its fascinating interplay with topology, algebra, and metric spaces.

Here are the references for this course:

For the section on hyperbolic geometry, you can refer to several excellent resources. First, there's Fuchsian Groups by Katok, which is a great foundational text. Another key reference is Lectures on Hyperbolic Geometry by Benedetti and Petronio. Additionally, you can find a highly valuable note by Bowditch available online, titled A Course on Geometric Group Theory.

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For the hyperbolic groups part of the course, we will follow Metric Spaces of Non-Positive Curvature by Bridson and Haefliger. This book provides a deep exploration of the relevant topics in hyperbolic group theory.

The course will run for 12 weeks. As for prerequisites, you'll need a solid background in topology, basic algebra, particularly group theory, and fundamental concepts from complex analysis. Thank you.