Measure Theoretic Probability 1
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Lecture 28
Properties of Measure Theoretic Integration (Part 2)

Welcome to this lecture. Before proceeding forward, let us first quickly recall what we have been doing in his week. In the first lecture, we defined the integration of measurable functions with respect to a given measure on the domain side. And in the second lecture, we started discussing the properties of this integration procedure. In particular, we have discussed two special properties, one was multiplication by constants of scalars. And then another property was a comparison-type inequality.

So now, in all of these, our idea is to get some nice properties of this integration procedure, which will make our job in computing these integrations easier. So, this is what we are targeting. So, in particular, one of the major properties that is still not proved, is the linearity of the integration by that I mean, if you have two functions, let us say g and h, then addition of g and h, if you consider the integral of that, that should be computable as $\int g\, d\mu$ plus $\int h\, d\mu$.

So, this is something we are here to prove, so before proving that, we cannot, in fact use this property in all arguments. So, we will have to be careful with this. So anyways, so, let us start off with the discussion in this lecture. So, we move on to looking at certain very special type of integration procedure.

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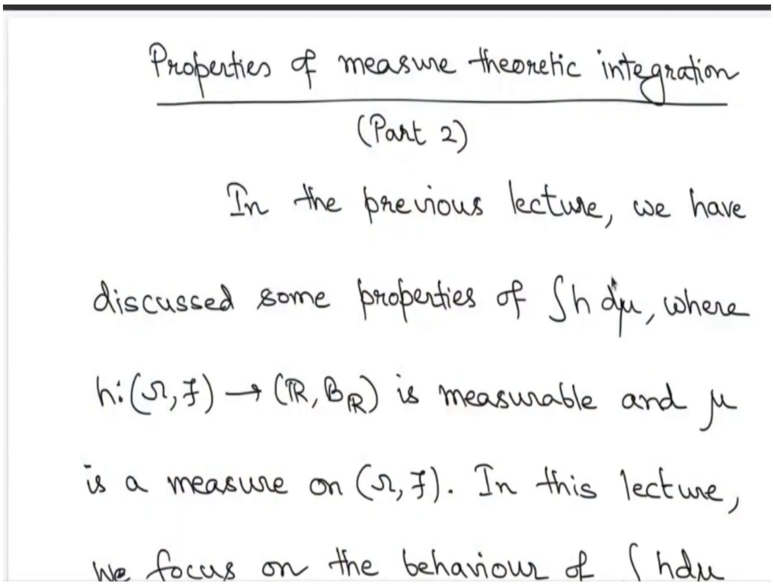

So, in the previous lecture, we have continued the discussions about $\int h \, d\mu$, when h is a measurable function. And we measure, measure on this domain side.

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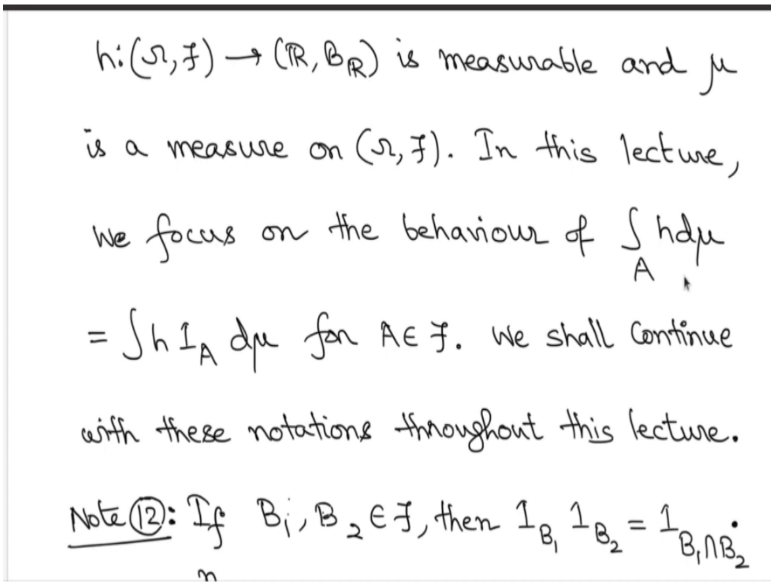

So, with that at hand, let us first focus our attention to this kind of a set function. So, we had defined this in the first lecture of this week, that once you are considering our function h, then look at $\int_A h \, d\mu$. And how did we define it, we considered $h \, 1_A$ which is also a measurable function. So therefore, you can consider $\int h\, 1_A\, d\mu$ whatever that integration value is considered it to be $\int_A h\, d\mu$.

And now, given this measurable function and given this measure fixed them, then try to vary this set A from the domain side, once you keep varying this set A, you are going to get different, different values for this integration, provided these integrations exist. If you now vary these sets, you are going to get all these values. And then what you end up having is a set function for each set you get some extended real number if the integration exists. And hence, you can look at this set function for each set A in the domain side.

Then, what we do is that we are going to look at the properties of this specific type of an integration in this lecture. So, before proceeding forward, we need to understand what kind of structures does this function have $h\, 1_A$, so let us see.

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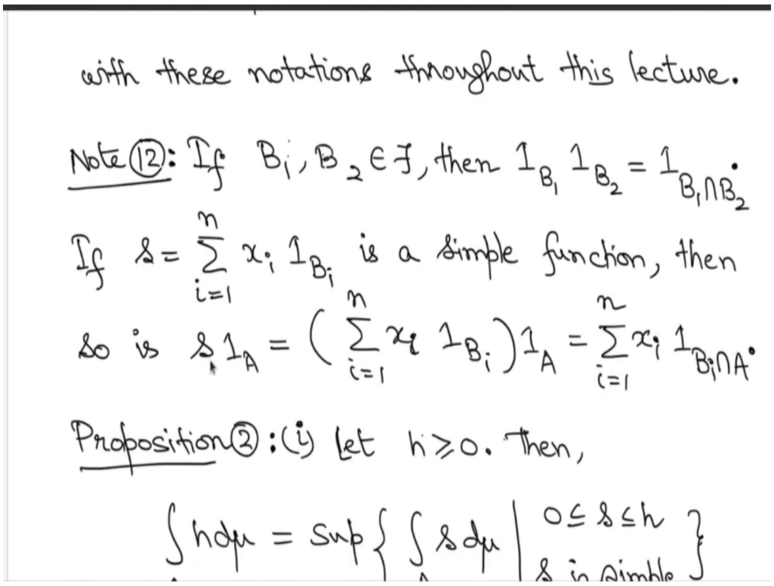

So, if you take two sets from your domain side, look at the product of their indicators. It is a easy observation that you can actually write it as the index indicator function of the intersection of two sets. And using this observation, you can now make this nice observation that starts with a simple function, which is a linear combination of indicators, then you multiply the simple function by this some indicator.

You can rewrite this in terms of again certain indicators, what is happening here is that you have this product of indicators coming up here and then these products of indicators by the observation above can be written as the indicator of the intersection of the sets.

So, therefore, the simple function multiplied by an indicator still remains a simple function, we are going to make use of this observation in our next result. In this lecture, we have assumed that h is a measurable function and μ is a measure.

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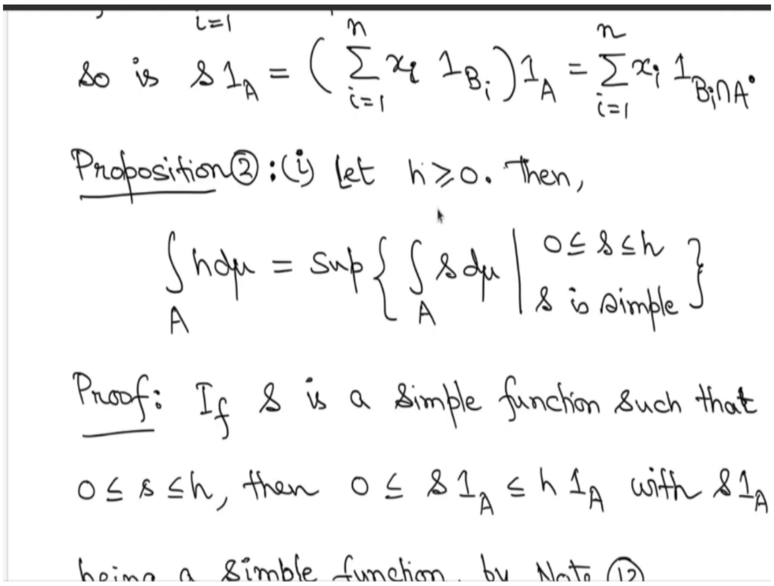

Now, assume that h is also non-negative. So, now, here h is non-negative and measurable. Then consider $\int_A h \, d\mu$, as per definition it is $\int h \, 1_A \, d\mu$. So, therefore, as far definition what you need to consider to compute this integral is to look at all simple functions below $h \, 1_A$. But we are saying there is another way of doing this. So, that is what this right hand side suggests.

So, what is in the right hand side, forget about $h \, 1_A$ look at all simple function s below the function h and look at the simple functions integrations over the set A. So that is all you need to check. So, you do not have to go to simple functions below $h \, 1_A$, but look at all simple functions below h, consider their integration over the set A. If you can manage to compute all these quantities, consider the supremum, that supremum will agree with the left hand side. So that is the statement here.

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$$\int_A h\,d\mu = \sup\left\{\int_A s\,d\mu \;\middle|\; \begin{array}{l} 0 \le s \le h \\ s \text{ is simple} \end{array}\right\}$$

Proof: If s is a simple function such that $0 \le s \le h$, then $0 \le s\,1_A \le h\,1_A$ with $s\,1_A$ being a simple function, by Note 12.

By Proposition 1 (ii),

$$\int_A s\,d\mu = \int s\,1_A\,d\mu \le \int h\,1_A\,d\mu = \int_A h\,d\mu$$

So, how do you prove this. So, you have to assume that h is a non-negative measurable function. Now, if you take a simple function that is below this function h, then observe that by multiplying this inequality on each term by 1_A, you end up with this relation, that $s\,1_A$ is still a non-negative simple function and $s\,1_A \le h\,1_A$. So, you have that, that is following from the previous observation, but then, you have these inequalities and we had proved these inequalities for the corresponding integrations in the previous lecture.

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being a simple function, by Note 12.

By Proposition 1 (ii),

$$\int_A s\,d\mu = \int s\,1_A\,d\mu \le \int h\,1_A\,d\mu = \int_A h\,d\mu$$

and hence $\int_A h\,d\mu \ge \sup\left\{\int_A s\,d\mu \;\middle|\; \begin{array}{l} 0 \le s \le h \\ s \text{ is simple} \end{array}\right\}$.

On the other hand, if s is a simple function

So, following that result, you can now look at $\int_A s\, d\mu$ that is nothing but $\int s\, 1_A\, d\mu$. Now, $s\, 1_A$ is smaller than $h\, 1_A$ and therefore, $\int_A s\, d\mu \leq \int_A h\, d\mu$. So, that is all it requires to prove this inequality.

Now, take supremum over all such simple functions below h, so that is what is appearing on a left-hand side. So therefore, supremum of all such simple functions that are below h, if you consider the integrations of that over the set A, and consider the supremum that will be upper bounded by $\int_A h\, d\mu$. But we want to show the equality here. So, we want to somehow get the other sided inequality.

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and hence $\int_A h\, d\mu \geq \sup\left\{ \int_A s\, d\mu \;\middle|\; \begin{array}{l} 0 \leq s \leq h \\ s \text{ is simple} \end{array} \right\}$.

on the other hand, if s is a simple function such that $0 \leq s \leq h\, 1_A$, then $s = s\, 1_A$

and hence $\int s\, d\mu = \int s\, 1_A\, d\mu = \int_A s\, d\mu$.

But,

$\int_A h\, d\mu = \int h\, 1_A\, d\mu = \sup\left\{ \int s\, d\mu \;\middle|\; \begin{array}{l} 0 \leq s \leq h\, 1_A \\ s \text{ is simple} \end{array} \right\}$

Now, what do you do, you will now start with a simple function that is below $h\, 1_A$, so our target is to compare with the integration of this function. So therefore, you start with a simple function, which is now below $h\, 1_A$. Now, observe that on the complement of the set A, this relation will tell you that s is 0. Why, because if you take any point ω on A^c, $h\, 1_A$ is 0.

So therefore, s on those points on A^c is also 0. So therefore, you can identify s as $s\, 1_A$ itself. So, this is a nice observation. Therefore, in this case, when s is a simple function below $h 1_A$, therefore, integration of s will be the same as integration of $s\,1_A$. But that is nothing but in our notation, $\int_A s\, d\mu$..

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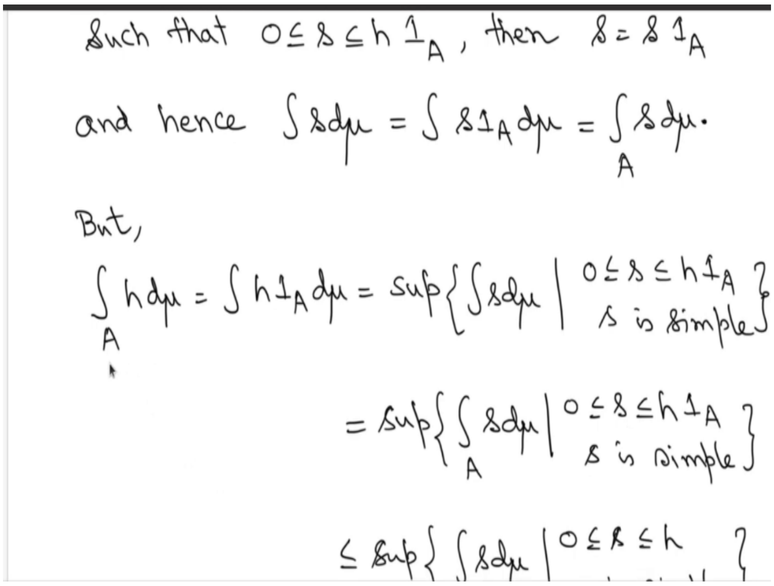

Now consider $\int_A h\, d\mu$., that is nothing but $\int h\,1_A\, d\mu$., so this function, but this function integration, follow the definition, you have to consider all simple functions that are below $h\,1_A$, consider their integrations and consider their supremum. So that is what this definition suggests.

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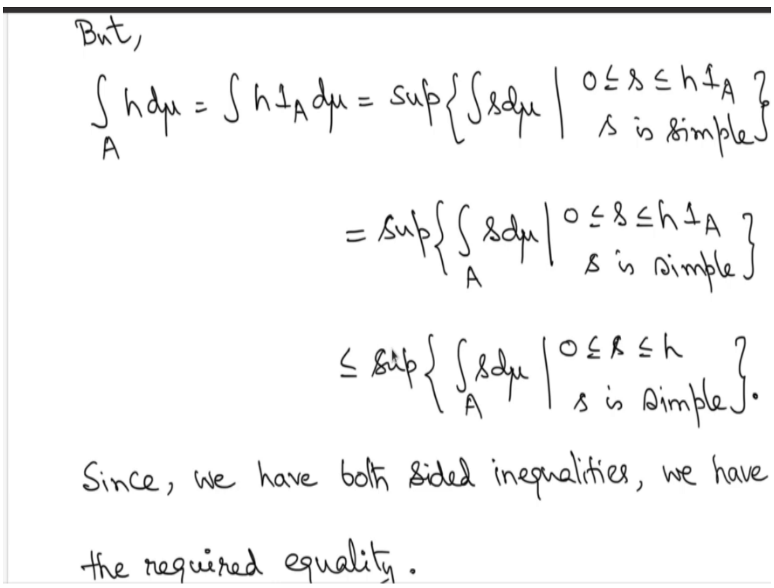

$$\int_A s\, d\mu = \int s\, 1_A \, d\mu \le \int h\, 1_A \, d\mu = \int_A h\, d\mu$$

and hence $\int_A h\, d\mu \ge \sup\left\{ \int_A s\, d\mu \,\middle|\, \begin{array}{l} 0 \le s \le h \\ s \text{ is simple} \end{array} \right\}$.

on the other hand, if s is a simple function such that $0 \le s \le h\, 1_A$, then $s = s\, 1_A$ and hence $\int s\, d\mu = \int s\, 1_A \, d\mu = \int_A s\, d\mu$.

But then you have just said that for such simple functions, $\int s\, d\mu$ is nothing but $\int_A s\, d\mu$. So, I have just made use of this equality, that $s = 0$ on A^c, so it will not contribute there. So that is all we are using here. But then these are for any simple function that is below $h\, 1_A$. But any simple function that is below $h\, 1_A$ is also below the function h and therefore, this is a bigger class now. So, any simple function that is below $h\, 1_A$ is also below the function h because h is non-negative.

So, therefore, you are considering a larger set of functions here and considering their integrations and these are all non-negative quantities. So therefore, what you get is a supremum over a smaller set is less equal to the supremum over the larger set. So that is all you were using to write down this inequality.

So, therefore, what happens at the end is this last inequality is giving us the other side of the inequality that we wanted. So, we had earlier proved that $\int_A h\, d\mu \ge \sup\left\{\int_A s\, d\mu \,/\, 0 \le s \le h,\ s \text{ is simple}\right\}$.

Now, we are saying $\int_A h\, d\mu \ge \sup\left\{\int_A s\, d\mu \,/\, 0 \le s \le h,\ s \text{ is simple}\right\}$, and therefore, once you have this both sided inequalities, you get the required equality. So, this is very easy to prove. You just have to keep track of which measurable functions you are working with and you have to appropriately choose the corresponding simple functions.

So, once you go through this argument, you have manged to show that $\int_A h\, d\mu$ can be computed very easily as integrations of all simple functions below h itself, and you just have to consider the integrations of these simple functions over the set.

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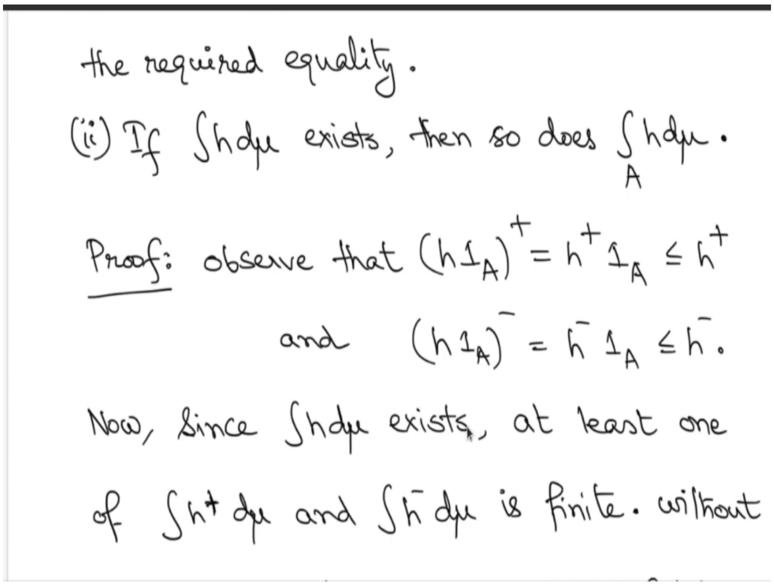

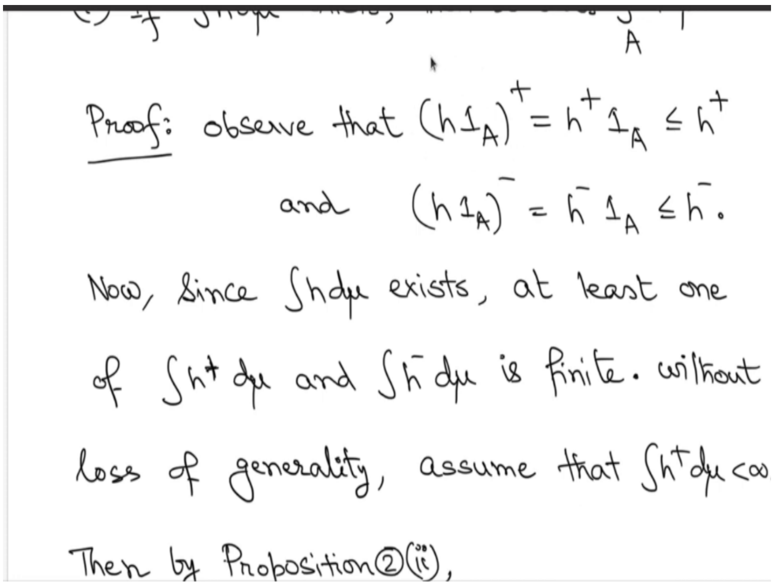

But with that at hand, we can now make very nice statements, what is this, so start with the fact that $\int h\, d\mu$ exists, so assume this. So, once you assume that $\int h\, d\mu$ exists, then you can immediately claim that $\int_A h\, d\mu$ also exists. So, why is this, here we are going to make use of comparison type inequalities.

So, here start with the positive part of $h\,1_A$. So here note, we have not assumed h to be non-negative, so h could be taking any real values. So here $h\,1_A$ the positive part of that is nothing but $h^+\,1_A$, indicator is always taking non-negative values. So therefore, that is what happens, but then you immediately observe that $h^+\,1_A < h^+$.

Similarly, look at the negative part of $h\,1_A$. So that you first write it as $h^-\,1_A$, but $h^-\,1_A \le h^-$. So, once you have these two inequalities, and provided $\int h\, d\mu$ exists, you can now claim that one of the $\int h^+\, d\mu$ and $\int h^-\, d\mu$.

So, if both are finite, then h is integrable. If exactly one of them is finite, the other is infinite, then h is quasi integrable. In either case, at least one of them is finite. And hence, without loss of generality, let us assume first that h^+ has a finite integral.

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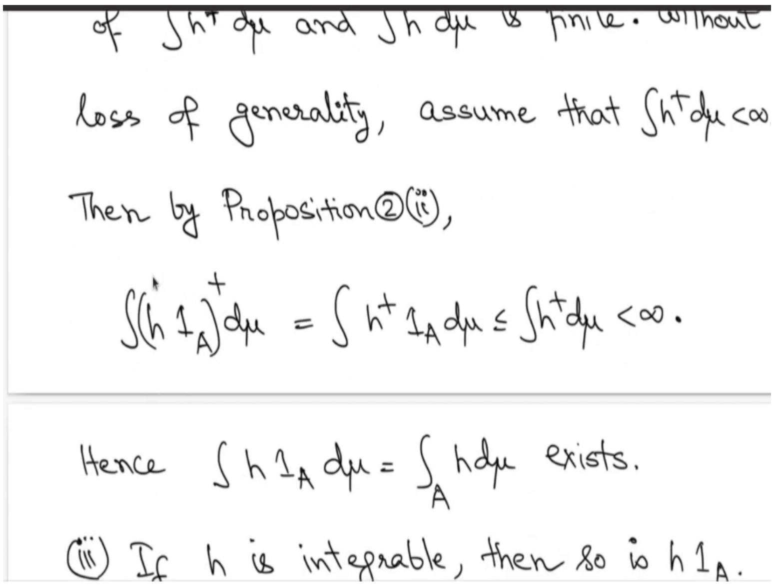

Then what you will claim is that integration of the positive part of $h\,1_A$ that will also have a finite integral. Why, because $h\,1_A$ the positive part of that is dominated from above by h^+ and h^+ if it is finite integral then $h\,1_A$ positive part of that will have a finite integral. Therefore, you have managed to show that if you look at the measurable functions $h\,1_A$ then at least one of the positive part or the negative part will have a finite integral.

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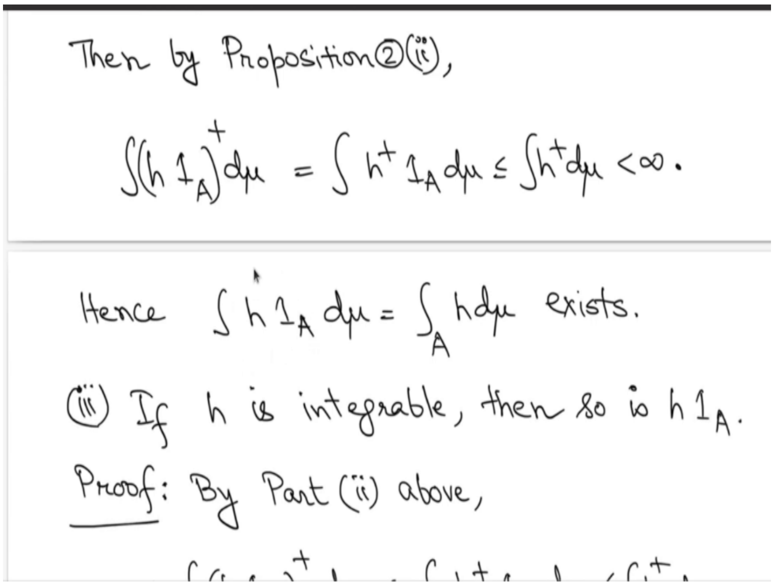

And hence, $\int h\, 1_A\, d\mu$ will exist and that will suggest that $\int_A h\, d\mu$ exists. This is simply rewriting the condition in terms of the comparison type inequalities.

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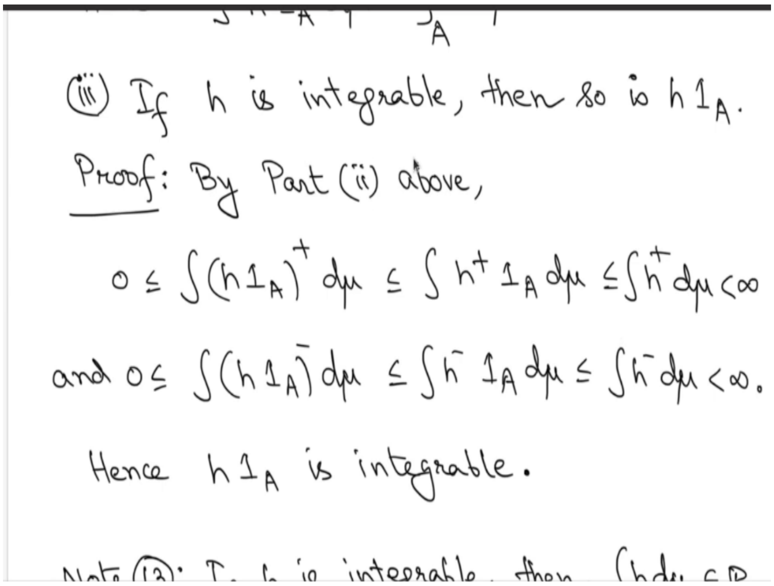

Now, you can make a stronger statement in the case when h is integrable. So, if h is integrable, then let us go back to these integrations once more and consider the inequalities. So, here you assume that both h^+ and h^- finite integral, here what is happening is that the positive part of $h\,1_A$, we will have the integration to be finite, because it is dominated from above by $\int h^+\,d\mu$ which is finite.

Similarly, the negative part of $h\,1_A$ will have its integration finite, because it is less or equals to $\int h^-\,d\mu$ which is given to be finite. And hence, since the function $h\,1_A$ has this interesting property that both its positive part and negative part have finite integration, then $h\,1_A$ becomes integrable. So, that proves it.

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Note (13): If h is integrable, then $\int h\,d\mu \in \mathbb{R}$.

By Proposition (1)(iii), $\int_A h\,d\mu \in \mathbb{R}$.

If $h \geq 0$, then $\int h\,d\mu$ exists and takes some value in $[0,\infty]$. Moreover, we have the inequalities,

$$0 \leq \int h\,d\mu = \int h\,1_A\,d\mu \leq \int h\,d\mu \leq \infty.$$

But then you now make this observation, this is a very minor observation, but quite useful that if h is integrable, then $\int h\,d\mu$ is a real number because, it is the difference of 2 real numbers meaning, it is a difference of integration of h^+ and integration of h^- both being finite in the case when h is integrable.

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(iii) If h is integrable, then so is $h\,1_A$.

Proof: By Part (ii) above,

$$0 \leq \int (h\,1_A)^+ \, d\mu \leq \int h^+ 1_A \, d\mu \leq \int h^+ \, d\mu < \infty$$

and $0 \leq \int (h\,1_A)^- \, d\mu \leq \int h^- 1_A \, d\mu \leq \int h^- \, d\mu < \infty$.

Hence $h\,1_A$ is integrable.

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If $h \geq 0$, then $\int h \, d\mu$ exists and takes some value in $[0, \infty]$. Moreover, we have the inequalities,

Now, using these integrability structures, you can now look at integration of $h\,1_A$. So, the positive part of that will be having finite value once h is integrable. Similarly, the negative part of $h\,1_A$ will have a finite integral when h is integrable.

So, provided this happens, then $\int_A h \, d\mu$, that quantity will also be a real number. So, this is a very interesting fact, when h is integrable then not only the $\int h \, d\mu$ is a real number, integration of h over any set is also a real number. Of course, the set has to be chosen from the domain side σ-function.

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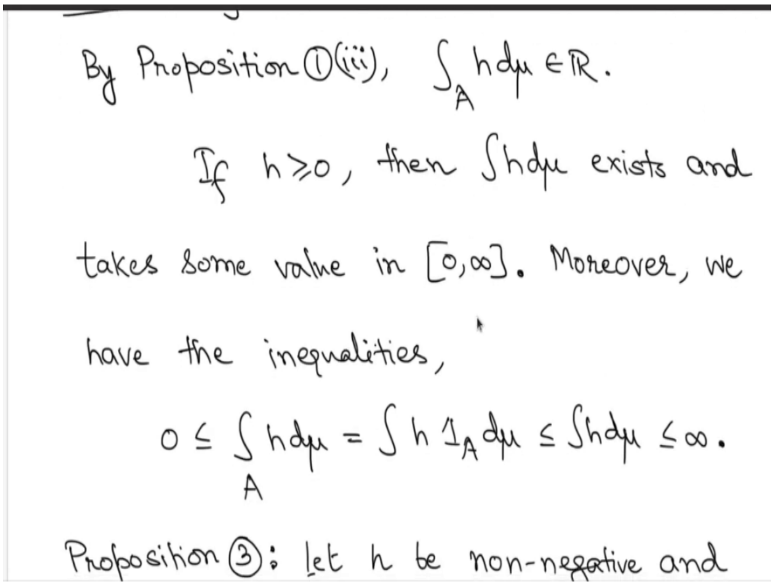

Now, we now restrict our attention to non-negative measurable functions. So, again throughout you are assuming h to be measurable. In addition, if h is non-negative we might get some very nice interesting additional properties. So, what happens here, so, take h to be non-negative measurable, then we know that $\int h\, d\mu$ exists, we do not have to consider the $\int h^{-}\, d\mu$ you just have to consider $\int h^{+}\, d\mu$. So, here $\int h\, d\mu$ exists. So, this could be taking infinite values of course. So, this takes values between $[0.\ \infty]$.

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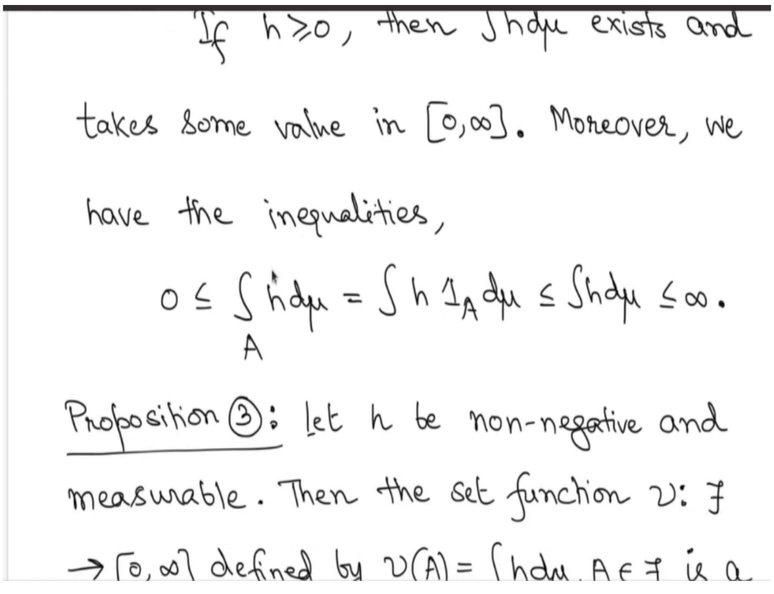

Here we also have the inequalities that $\int h\, 1_A\, d\mu$ that is nothing but $\int_A h\, d\mu$. So, this quantity, $h\, 1_A$ is dominated by h, because you are talking about a non-negative function h here and indicator is simply a non-negative measurable function. So, using this comparison type inequalities, you will claim that $\int_A h\, d\mu$ has this kind of structure, you have this upper bound given by $\int h$.

Now, this is what we had started off this lecture that we are going to fix h, we are going to fix the measure, we are going to vary the set A. And in the case when h is non-negative, the $\int h\, d\mu$ is also some non-negative quantity.

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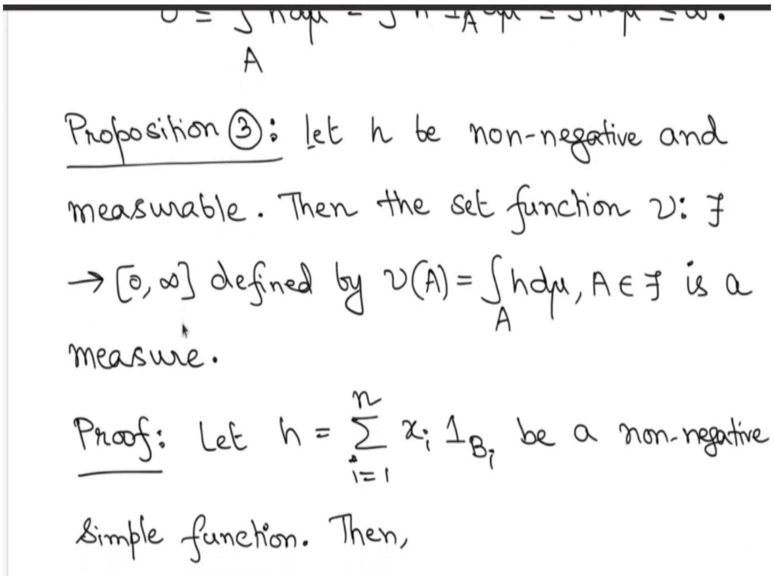

And it will give you one nice non-negative set function, how nice that is stated in this proposition. So, take h to be non-negative and measurable, consider this set function nu which takes any set to the $\int_A h\, d\mu$. And we are going to claim that this is a measure. So, if you integrate a non-negative measurable function and integrate over these sets A and vary the set A then you get a set function that turns out to be a very nice set function, which is non-negative and countable relative that is a measure. So, we are going to prove this.

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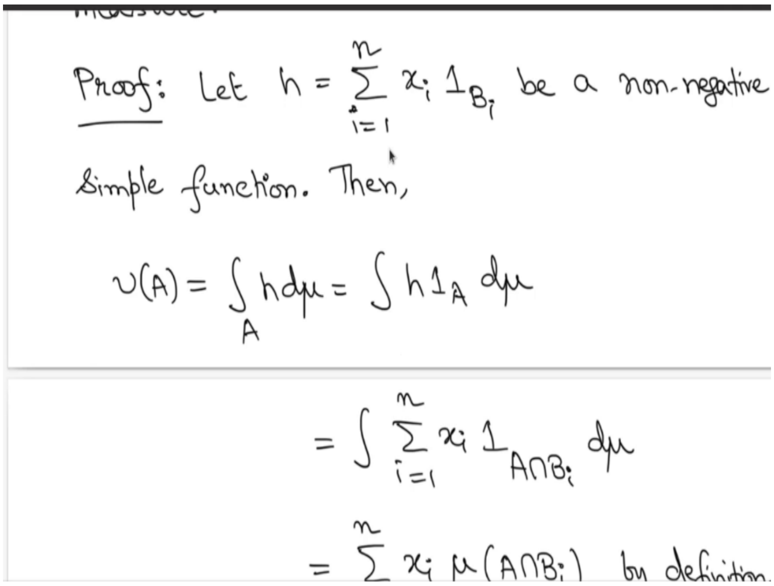

So, we start with the case when h is simple. So, again h is given to be non-negative, so you have to work with a non-negative simple function. So here look at this combination $x_i 1_{B_i}$. So, x_i are certain non-negative real numbers because h is non-negative. Now, what is happening here is that what is the integration now, $\int_A h\, d\mu$ is $\int h\, 1_A\, d\mu$.

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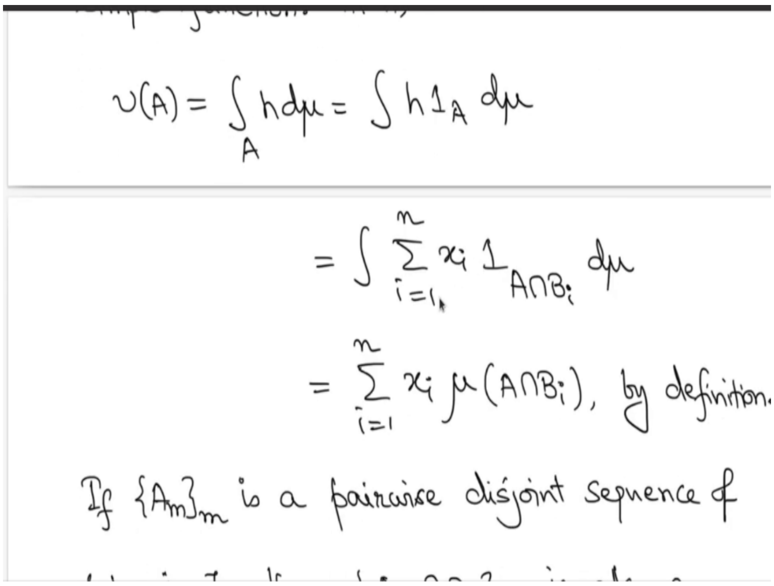

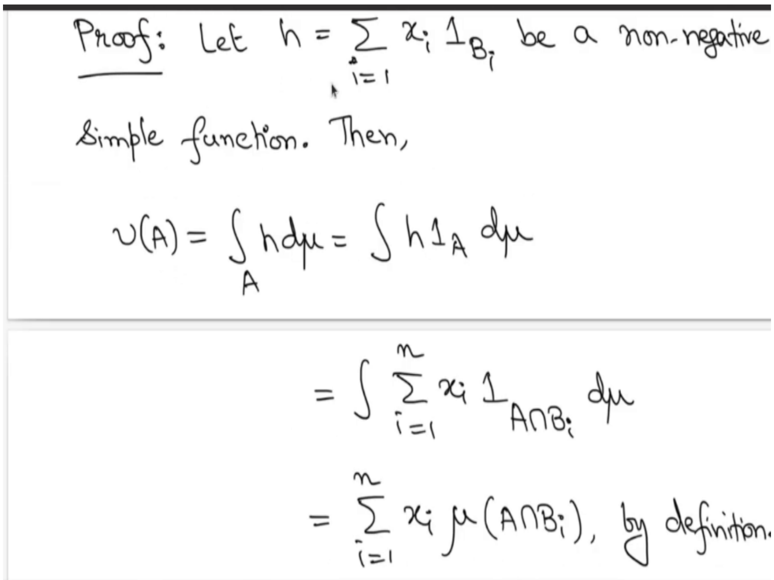

And that is as per our observation is nothing but this linear combination of x_i multiplied by this indicator of the intersections. So, that is something we had already mentioned right at the beginning. But now, as per definition, $\int \sum_{i=1}^{n} x_i 1_{A \cap B_i} \, d\mu$ is nothing but $\sum_{i=1}^{n} x_i \mu(A \cap B_i)$. So, that is by definition. So, again, here are all the terms x_i and measures are non-negative.

So, defining summation is not an issue $\infty - \infty$ situation does not arise, we are only dealing with non-negative quantities, it could be ∞ that is fine. So, this is our definition. So, integration of h gives me the set function and the set function here takes this explicit value. We are going to use this to prove that for the case when h is simple this set function is a measure.

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$$= \sum_{i=1}^{n} x_i \, \mu(A \cap B_i), \text{ by definition.}$$

If $\{A_m\}_m$ is a pairwise disjoint sequence of sets in $\mathcal{F}$, then $\{A_m \cap B_i\}_m$ is also a pairwise disjoint sequence of sets for every fixed i. Then,

$$\nu\left(\bigcup_{m=1}^{\infty} A_m\right) = \sum_{i=1}^{n} x_i \, \mu\left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cap B_i\right)$$

So, what, what is going to be proved is the countable additivity. So, start with a pairwise disjoint sequence of sets in the domain side σ-field, then observe that if A_m's are pairwise disjoint then, if you fix any i then $A_m \cap B_i$, these sets are also pairwise station, this is for any fixed i.

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pairwise disjoint sequence of sets for every fixed i. Then,

$$\nu\left(\bigcup_{m=1}^{\infty} A_m\right) = \sum_{i=1}^{n} x_i \, \mu\left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cap B_i\right)$$

$$= \sum_{i=1}^{n} \sum_{m=1}^{\infty} x_i \, \mu(A_m \cap B_i)$$

$$= \sum_{m=1}^{\infty} \sum_{i=1}^{n} x_i \, \mu(A_m \cap B_i)$$

Then use the fact that this set function nu has this form that it is this linear combination of the measures of this type of intersections. Now, use the fact that μ is given to be a measure and use the countability over the sets $A_m \cap B_i$. So, you will get this additional summation over m coming out, this is by countable additivity of the set function of the measure μ.

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$$\nu\Big(\bigcup_{m=1} A_m\Big) = \sum_{i=1} x_i \, \mu\Big(\Big(\bigcup_{m=1} A_m\Big) \cap B_i\Big)$$

$$= \sum_{i=1}^{n} \sum_{m=1}^{\infty} x_i \, \mu(A_m \cap B_i)$$

$$= \sum_{m=1}^{\infty} \sum_{i=1}^{n} x_i \, \mu(A_m \cap B_i)$$

$$= \sum_{m=1}^{\infty} \nu(A_m).$$

The exchange of sum over i & m above

measure.

Proof: Let $h = \sum_{i=1}^{n} x_i 1_{B_i}$ be a non-negative simple function. Then,

$$\nu(A) = \int_A h \, d\mu = \int h 1_A \, d\mu$$

$$= \int \sum_{i=1}^{n} x_i 1_{A \cap B_i} \, d\mu$$

Now, exchange this order of the summations, so the summation over m goes outside here it is allowed, because you are dealing with non-negative quantities. Now, here inside you have this finite sum over these quantities. And here, what will happen is that this B_i's as per construction, this Bis appeared due to the decomposition of the simple function.

So, let us go back to the description. So, if you take h to be a simple function, we automatically assume that these B_i's are pairwise disjoint. So, just use the fact that measure of that set function ν has this form to write this inside combination as $\nu(A_m)$.

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fixed i. Then,

$$\nu\left(\bigcup_{m=1}^{\infty} A_m\right) = \sum_{i=1}^{n} x_i \, \mu\left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cap B_i\right)$$

$$= \sum_{i=1}^{n} \sum_{m=1}^{\infty} x_i \, \mu(A_m \cap B_i)$$

$$= \sum_{m=1}^{\infty} \sum_{i=1}^{n} x_i \, \mu(A_m \cap B_i)$$

$$= \sum_{m=1}^{\infty} \nu(A_m).$$

So, you have managed to show that when h is a non-negative simple function then this set function is countable additive and therefore it is a measure.

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when h is non-negative and simple.

Now, let h be non-negative and measurable. let $\{A_m\}_m$ be a sequence of pairwise disjoint sets in $\mathcal{F}$.

If s is a simple function with $0 \leq s \leq h$,

then $$s 1_{\bigcup_{m=1}^{\infty} A_m} = s\left(\sum_{m=1}^{\infty} 1_{A_m}\right) = \sum_{m=1}^{\infty} \left(s 1_{A_m}\right).$$

Now, how do you go from non-negative simple functions to non-negative measurable functions. So, this requires some effort. So, again start with a sequence of pairwise disjoint sets call them A_m once more. Our target is to prove that the corresponding set function is countable additive. So, you have to make be careful with certain inequalities here. Here, the main issue is that integration of h which is non-negative and measurable is defined through certain supremum over integrations of simple functions below h.

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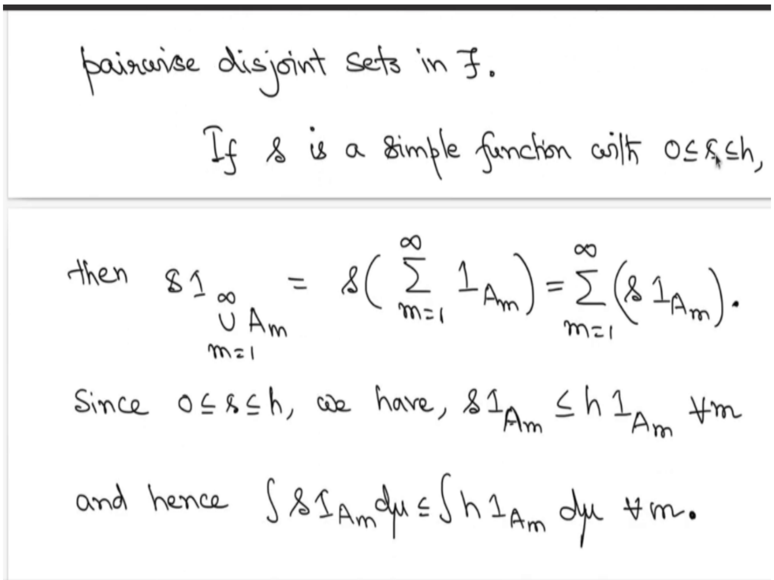

So, start with such a simple function which is below h and s is non-negative. Then look at this multiplication by indicator. So, $s\,1_{\cup_m^{\infty} A_m} = s \sum_{m=1}^{\infty} 1_{A_m} = \sum_{m=1}^{\infty} s\,1_{A_m}$. Why, this is easy to verify, put sample points ω on both sides and verify this equality. So, what is happening here is that indicator of this union of these pairwise disjoint sets A_m and can be simply written as the summation of the individual indicators. So, that is all we are using here.

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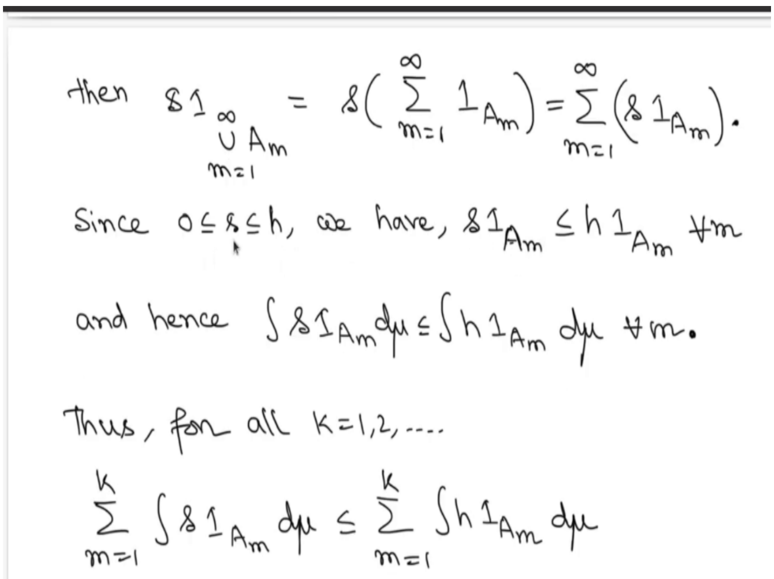

Now, for every fixed m, if you look at the inequality that $0 \leq s \leq h$, then by multiplying by 1_{A_m} you still have this inequality here. So, you have $s\,1_{A_m} \leq h\,1_{A_m}$. So, this is true for any fixed m. And hence, if you do the integration of this for every fixed m,

$\int s\,1_{A_m}\,d\mu \leq \int h\,1_{A_m}\,d\mu$. So, this is simply following that comparison type inequality that we discussed in the previous lecture.

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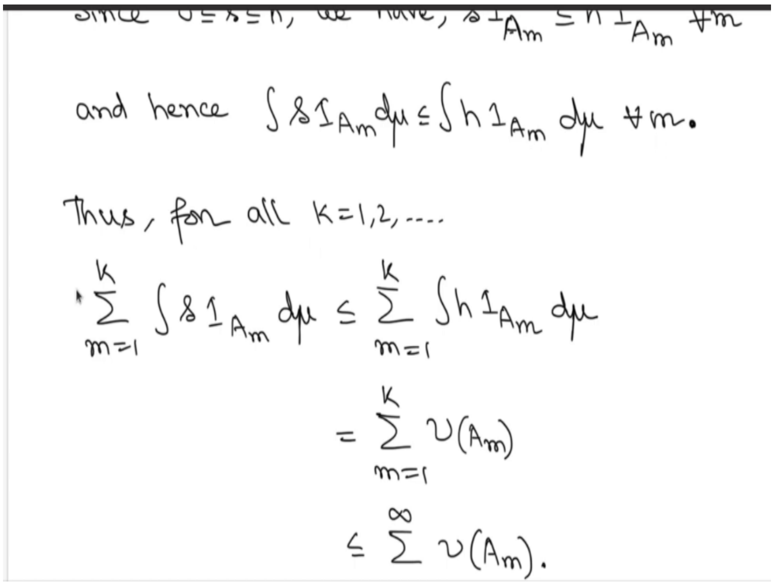

But then you vary your m, what do you do, you look at $\sum_{m=1}^{k} \int s\,1_{A_m}\,d\mu \leq \sum_{m=1}^{k} \int h\,1_{A_m}\,d\mu$, that is what these integrations are.

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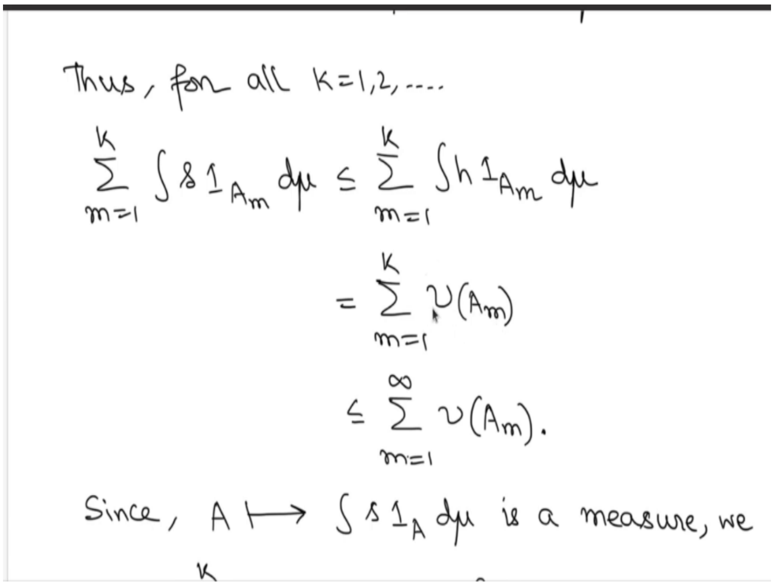

But as per definition, that is the set function $\nu(A_m)$ that is by definition, this is the summation here. But then $\nu(A_m)$ by definition, these are non-negative quantities. So, in particular you can dominate it by the full series. So, this full series need not converge, this full series can be plus infinity also it does not matter the inequality still holds. So, this finite sum is of course, dominated from above by this infinite sum. So, because these terms $\nu(A_m)$ are non-negative.

Therefore, for any fixed k, integer k, you have this $\sum_{m=1}^{k} \int s 1_{A_m} d\mu$ is dominated from above by this quantity which is now independent of k. So, the right hand side this final quantity that we ended up with is independent of k.

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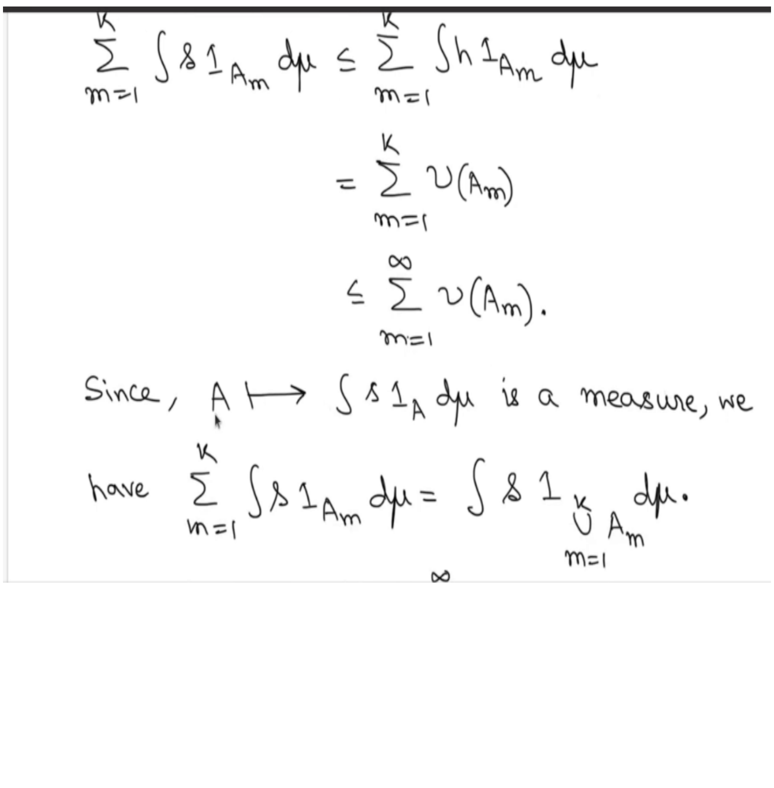

Therefore, you use the fact that the set function A mapping to integrations of non-negative simple functions is a measure to go from k to ∞ and you will use this bound here. So, that is all we are going to use.

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$$\sum_{m=1}^{k} \int s 1_{A_m} \, d\mu \le \sum_{m=1}^{k} \int h 1_{A_m} \, d\mu$$

$$= \sum_{m=1}^{k} \nu(A_m)$$

$$\le \sum_{m=1}^{\infty} \nu(A_m).$$

Since, $A \mapsto \int s 1_A \, d\mu$ is a measure, we have $\sum_{m=1}^{k} \int s 1_{A_m} \, d\mu = \int s 1_{\bigcup_{m=1}^{k} A_m} \, d\mu.$

Then, $\int s 1_{\bigcup_{m=1}^{k} A_m} \, d\mu \le \sum_{m=1}^{\infty} \nu(A_m)$ for $k = 1, 2, \ldots$

Since, $A \mapsto \int s 1_A \, d\mu$ is a measure,

$$\lim_{k \to \infty} \sum_{m=1}^{k} \int s 1_{A_m} \, d\mu = \lim_{k \to \infty} \int s 1_{\bigcup_{m=1}^{k} A_m} \, d\mu$$

$$= \int s 1_{\bigcup_{m=1}^{\infty} A_m} \, d\mu.$$

So, first use the fact that $A \to \int s \, 1_A \, d\mu$ is a measure which we have already proved, we have that we can push this integration inside. So, for simple functions, this is already a measure. So, therefore, you have this finite additivity. So, therefore, you have this union happening here. So, that is all you are using, because this set function becomes a measure for non-negative simple functions. So, this is just rewriting the equality for finite additivity of this measure in question.

So, therefore, what you have now just proved is that $\int s\, 1_{\bigcup_{m=1}^{\infty} A_m} d\mu$ is dominated from above by some quantity which is independent of k. Again, this quantity that is happening on the right hand side, this is positive real number or $+\infty$. Now, again use the fact that for non-negative simple functions, this set function is a measure to g from this limit, this is basically limit of finite additivity going to countably additivity.

So, take limit as $k \to \infty$ of this finite sum. Now, this finite sum has already observed is nothing but this but then using the countably additivity you get this limiting value this implies continuity from below. So, therefore, the since these sets increase to the whole union you will get this limit will give you this integration.

So, here you are using the fact that this set function is a measure and in particular it is continuous from below. So, since this finite union increases to the whole union, that is what you will get as the limit of this integrals. So, that is nothing but this $\int_{\cup A_m} s\, d\mu$. So, here you have used multiple times the fact that for non-negative simple functions, this is a measure.

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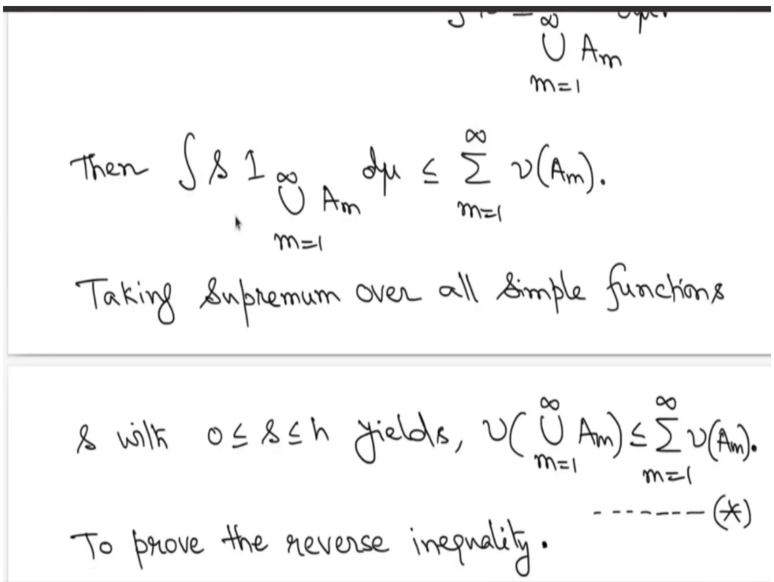

Then, what you have managed to prove at the end is that s times indicator of all these unions is dominated from above by this summation. Here the new function is defined in terms of the general function h which is non-negative and measurable. And s is a non-negative simple function, which is below the function h. So, this is true for any such simple function. So, we have this upper bound appearing here.

So, if you take supremum over all such simple functions, you get $\nu(A_m)$ on the left hand side and summation of the individual $\nu(A_m)$s on the right hand side. So, you have some kind of a countable sub-additivities for the set function ν. You want to prove the other side of the inequality. And that will give you the appropriate equality that we are after it will imply that nu will be countably additive. So, how do you show this?

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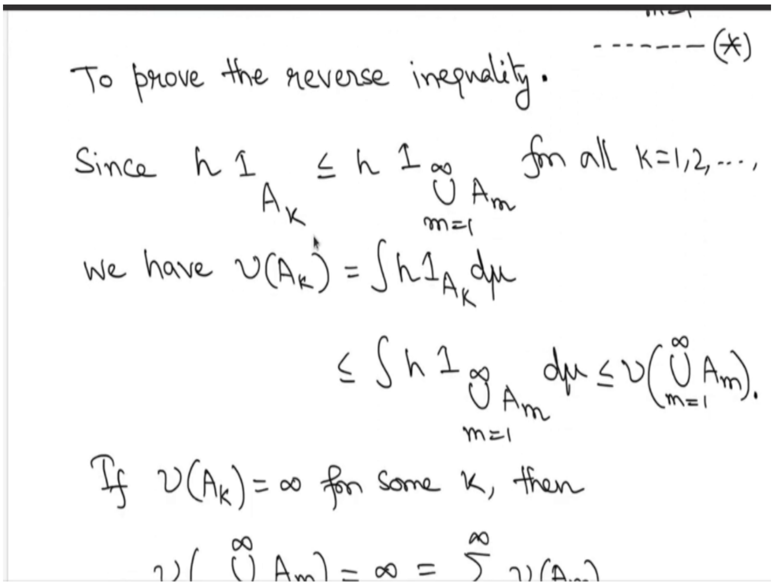

To show this again start off with this observation that for every fixed k from 1 to 3 onwards $h\, 1_{A_k}$ is dominated from above by $h\, 1_{\bigcup_{m=1}^{\infty} A_m}$. And therefore, if you consider the integrations of these functions now, that is nothing but that $\nu(A_k)$ is dominated from above by $\nu(\bigcup_{m=1}^{\infty} A_m)$. So, that is all. So, since these functions have this relation, their corresponding integrations also have this relation and therefore, $\nu(A_k)$ is dominated from above by this quantity.

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Since $h \mathbb{1}_{A_k} \le h \mathbb{1}_{\bigcup_{m=1}^{\infty} A_m}$ for all $k=1,2,\dots$,

we have $\nu(A_k) = \int h \mathbb{1}_{A_k} \, d\mu$

$$\le \int h \mathbb{1}_{\bigcup_{m=1}^{\infty} A_m} \, d\mu \le \nu\left(\bigcup_{m=1}^{\infty} A_m\right).$$

If, $\nu(A_k) = \infty$ for some k, then

$$\nu\left(\bigcup_{m=1}^{\infty} A_m\right) = \infty = \sum_{m=1}^{\infty} \nu(A_m).$$

Suppose $\nu(A_k) < \infty$ for all k. Then for any

But if it happens that $\nu(A_k)$ is infinite, then of course, that ν of this countable union will also have infinite mass. Therefore, you will get the countable additivity in this case. When, $\nu(A_k)$ is infinite for some k.

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$$\nu\left(\bigcup_{m=1}^{\infty} A_m\right) = \infty = \sum_{m=1}^{\infty} \nu(A_m).$$

Suppose $\nu(A_k) < \infty$ for all k. Then for any

fixed k and $\varepsilon > 0$, we can find a simple function s with $0 \le s \le h$ and

$$\int_{A_m} s \, d\mu \ge \int_{A_m} h \, d\mu - \frac{\varepsilon}{k} \quad \forall m = 1, 2, \dots, k$$

(Exercise)

But now, let us consider the finiteness condition that $\nu(A_k)$ is finite for all things. So, this is the interesting condition. And here you have to separately prove the countable additivity of the set function ν. Here what is happening, if you fix any k and fix $\epsilon > 0$.

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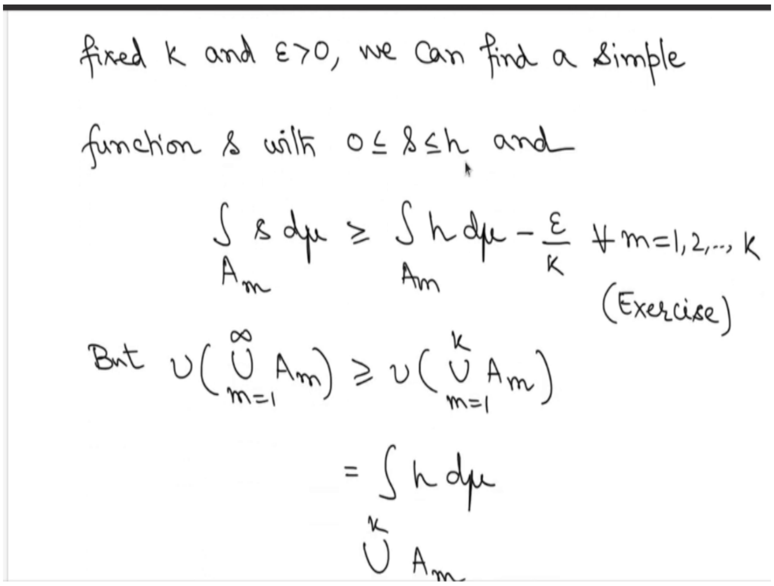

You can now find a simple function $0 \le s \le h$ such that $\int_{A_m} h \, d\mu$ is approximated by these simple functions. So, you can choose such a simple function with this error $\frac{\epsilon}{k}$. So, you can choose this this is given as an exercise, the hint is that the supremum over all such simple functions that you consider is exactly going to give you $\int_{A_m} h \, d\mu$. So, that is all we are using here, but we are saying you can choose a common simple function s here. So, try to work this out this is left as an exercise for you.

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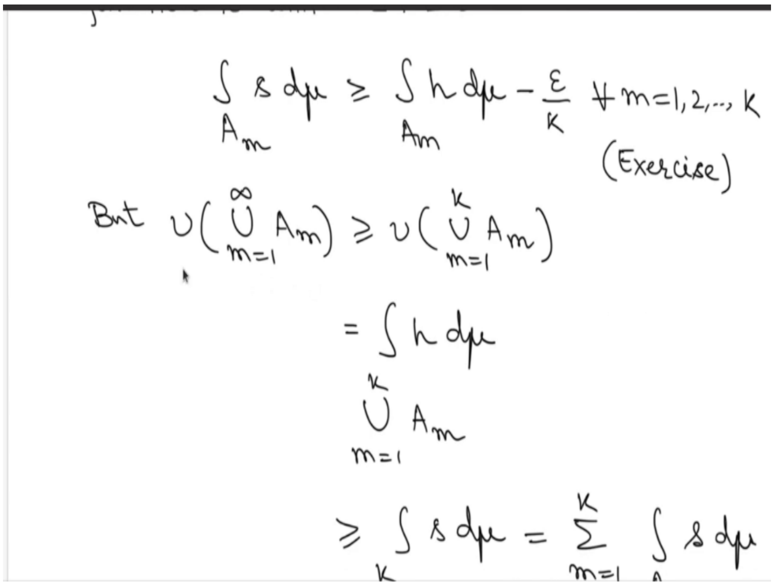

But then what is happening here is that look at these inequalities, you have already observed that for this countable union, whatever this set is, this contains this finite union and therefore, you get this inequality. So, that is already proved. But now, this finite union is nothing but

$$\int_{\cup A_m} h\, d\mu$$

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$$= \int_{\bigcup_{m=1}^{k} A_m} h\, d\mu$$

$$\geq \int_{\bigcup_{m=1}^{k} A_m} s\, d\mu = \sum_{m=1}^{k} \int_{A_m} s\, d\mu$$

$$\geq \sum_{m=1}^{k} \int_{A_m} h\, d\mu - \varepsilon$$

On this finite union of sets use the lower bound given by the simple functions s. So, since, $s \leq h$ you have this inequality. So, that is simple to observe. But then, here this sets A_m are pairwise disjoint. And $\int s\, d\mu$ over such sets is going to give you a measure. So, use finite additivity to write it as a $\sum \int_{A_m} s\, d\mu$.

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$$\bigcup_{m=1} A_m$$

$$\geq \int_{\bigcup_{m=1}^{k} A_m} s\,d\mu = \sum_{m=1}^{k} \int_{A_m} s\,d\mu$$

$$\geq \sum_{m=1}^{k} \int_{A_m} h\,d\mu - \varepsilon$$

$$= \sum^{k} \nu(A_m) - \varepsilon ..$$

fixed k and $\varepsilon > 0$, we can find a simple function s with $0 \leq s \leq h$ and

$$\int_{A_m} s\,d\mu \geq \int_{A_m} h\,d\mu - \frac{\varepsilon}{k} \quad \forall\, m = 1, 2, \ldots, k$$

(Exercise)

But $$\nu\left(\bigcup_{m=1}^{\infty} A_m\right) \geq \nu\left(\bigcup_{m=1}^{k} A_m\right)$$

$$= \int_{\bigcup^{k} A_m} h\,d\mu$$

And then you use the inequality that is left as an exercise here to claim that this is also further equal to $\sum_{m=1}^{k} \int_{A_m} h\,d\mu - \epsilon$.

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$$\bigcup_{m=1} A_m \qquad\qquad A_m$$

$$\geq \sum_{m=1}^{K} \int_{A_m} h\,d\mu - \varepsilon$$

$$= \sum_{m=1}^{K} \nu(A_m) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\int_{A_m} s\,d\mu \geq \int_{A_m} h\,d\mu - \frac{\varepsilon}{K} \quad \forall\, m = 1, 2, \ldots, K$$

(Exercise)

But $\nu\left(\bigcup_{m=1}^{\infty} A_m\right) \geq \nu\left(\bigcup_{m=1}^{K} A_m\right)$

$$= \int_{\bigcup_{m=1}^{K} A_m} h\,d\mu$$

$$\geq \int_{\bigcup^{K}} s\,d\mu = \sum_{m=1}^{K} \int_{A_m} s\,d\mu$$

$$= \sum_{m=1}^{k} \nu(A_m) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\nu\Big(\bigcup_{m=1}^{\infty} A_m\Big) \geq \sum_{m=1}^{k} \nu(A_m) \quad \forall k = 1, 2, \dots$$

Hence $\nu\Big(\bigcup_{m=1}^{\infty} A_m\Big) \geq \sum_{m=1}^{\infty} \nu(A_m)$. This

proves the other inequality. Combining with

Put them together, once you get this, you will get that rewriting it in terms of ν is nothing but $\nu(A_m)$. So, that is nothing but the definition of $\nu(A_m)$ being used here.

So, what did we start off with, we started off with the ν of the complete countable union, and that is greater equals to $\sum_{m=1}^{k} \nu(A_m) - \epsilon$. But since this is true for every fixed k, and every fixed $\epsilon > 0$, you get that this inequality is true, is true for any k now.

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$m=1$

Since $\varepsilon > 0$ is arbitrary, we have

$$\nu\Big(\bigcup_{m=1}^{\infty} A_m\Big) \geq \sum_{m=1}^{k} \nu(A_m) \quad \forall k = 1, 2, \dots$$

Hence $\nu\Big(\bigcup_{m=1}^{\infty} A_m\Big) \geq \sum_{m=1}^{\infty} \nu(A_m)$. This

proves the other inequality. Combining with

(*) above, we have the result.

Note (14): If h is non-negative and integrable

And hence, if you let $k \to \infty$ on the right hand side, the left hand side is independent of k and therefore, this will give you an upper bound and that is the upper bound we were after we had proved the other side of the inequality and hence the set function ν is countably additive and therefore, it becomes a measure.

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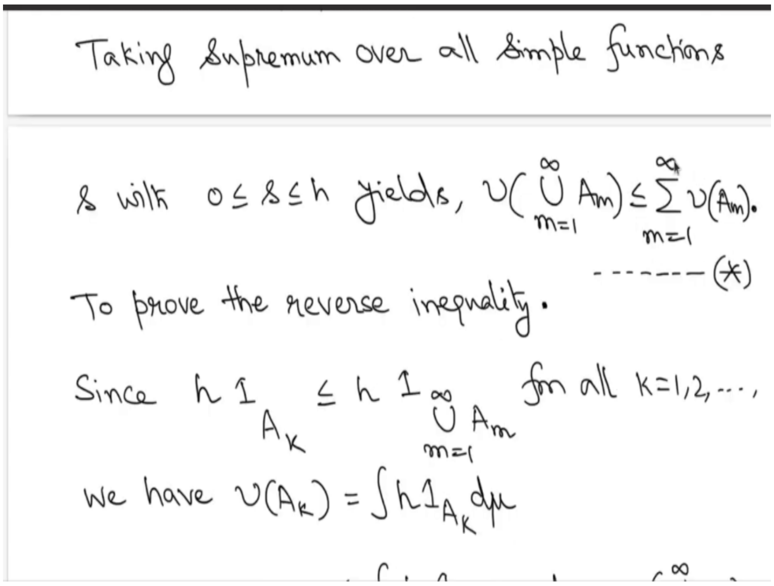

This series is upper bound for ν of this countable union.

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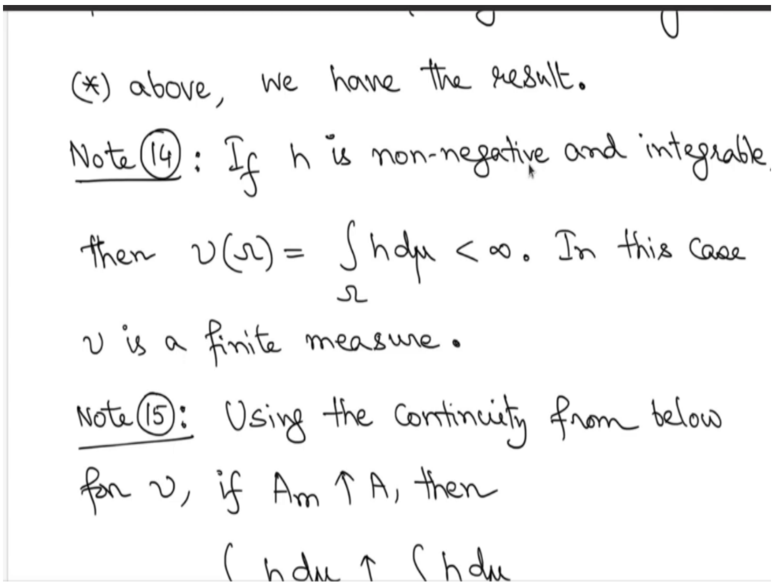

So, we have this result and using this observation, we now make some interesting comments that if h is non-negative and integrable, then $\nu(\Omega)$ whatever that is, that is $\int_{\Omega} h\, d\mu$ is finite provided h is integrable. So, therefore, you get a finite measure provided h is integrable.

Otherwise, if it is just non-negative and measurable, then the integration of h will give you a measure.

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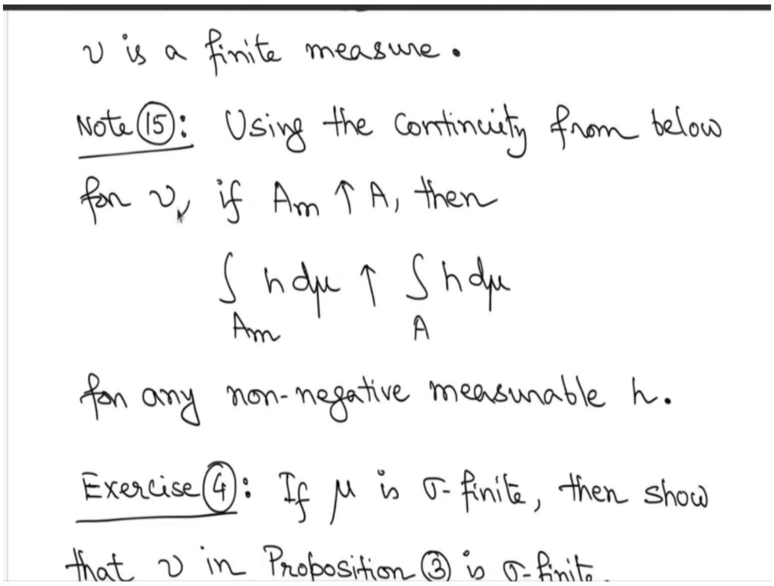

In particular, you can now have these observations about continuity from below that will imply this limiting behaviors for the corresponding integrals, that if $A_m \uparrow A$ then $\int_{A_m} h\, d\mu \uparrow \int_A h\, d\mu$. So, this is for non-negative measurable h. This is simply using continuity from below of the set function ν.

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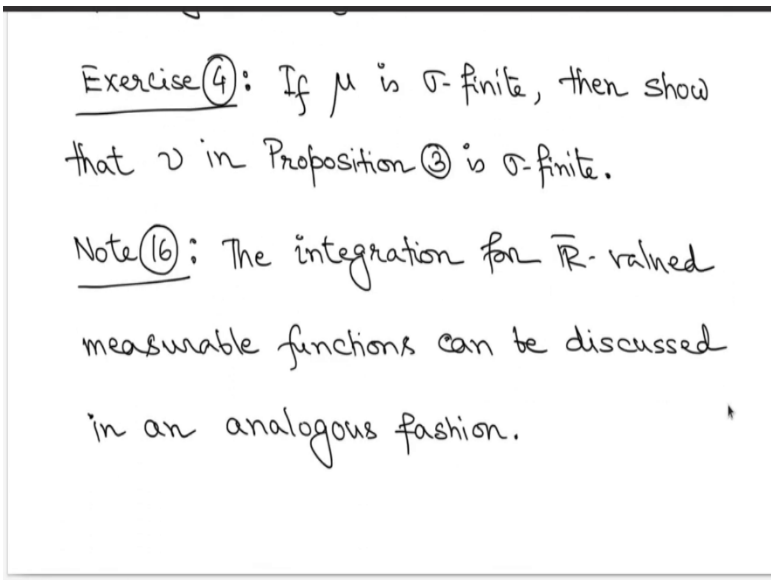

And we end with short discussion on other interesting properties of this set function. So, you can try to show that if μ is σ $-$finite, then the measure that we have constructed out of integrating this function h will also be $\sigma -$ finite, please try to check this. Here we have discussed the results for non-negative measurable functions taking values in the real line, but you can actually do all these arguments for extended real valued functions, but non-negative functions.

So, you allow $+\ \infty$ for possible values of h. So, again the similar results can be discussed in an analogous fashion. We will continue the discussion of properties of integrations in the next lecture. We stop here.