Measure Theoretic Probability 1
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Lecture 27
Properties of Measure Theoretic Integration (Part 1)

Welcome to this lecture. So, before proceeding to the discussion of this lecture, let us first quickly recall what we have mentioned in the previous lecture. So, in the previous lecture, we have defined the integration of measurable functions with respect to the domain side measure. Now, this definition went through several steps, the first that we considered was indicator functions and the motivation for defining integrals of indicator functions was simply the area under the curve which we simply took it as the area of the rectangle that is under the curve on it.

So, that was for the indicator functions. And then keeping the idea that for whatever integration procedure we managed to define it should be linear. So, with that idea at hand what we defined was that, since the simple functions are linear combinations of indicators, so we should be able to specify the integration values off simple functions, so we extended by linearity. But then we know that any non-negative measurable function can be approximated from below by simple functions, non-negative simple functions.

So, what we did was that the area under this non-negative simple functions that should increase to the area under the non-negative measurable function. So, anyway that is the idea. So, if you still continue to think of the area under the curve as the value of the integral then you must specify the value of the integral of a non-negative measurable function as the supremum of all possible integrals of simple functions which are below that given non-negative measurable function.

So, that was what we had used to define the integrals of non-negative measurable functions. And finally, for any general measurable function, which could take positive or negative values, what we did is that we split the function into positive parts and negative parts. So, we looked at the integrals of h^+ and h^- separately. If at least one of them is finite then what we said is that okay, so we can define the difference. So, in the case when exactly one of them is finite and the other is infinite, we call it as quasi integrable.

So, there is still the way of defining that integral which could be taking values plus or minus infinity. And the good case was when both h^+ and h^- have the integrals which are finite, and then what we can consider is the difference and then the integral of h will also be summed in finite quantity as per the description.

So, in this lecture, what we are going to do is to look at properties of such integration procedure. So, one of the important thing that we have to keep in mind that, we are yet to prove that the integration so defined is linear in structure, so, we will have to prove it as we go along. So, let us move on to the slides and continue the discussion.

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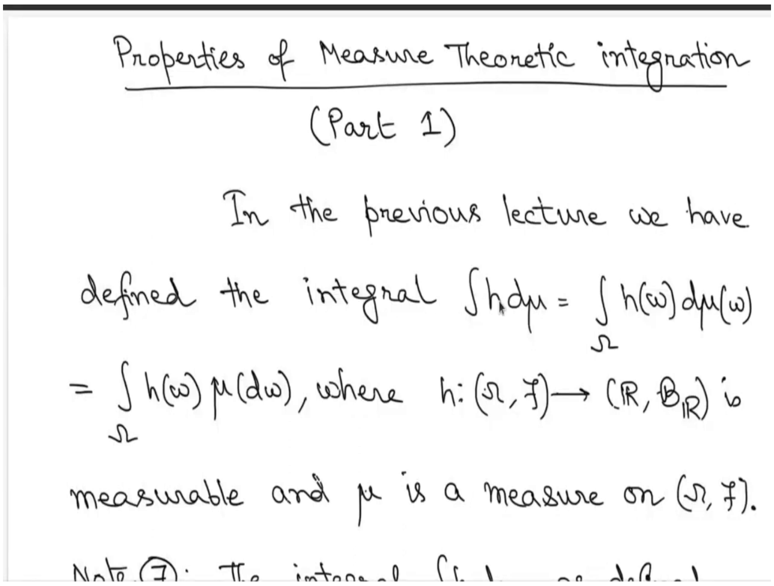

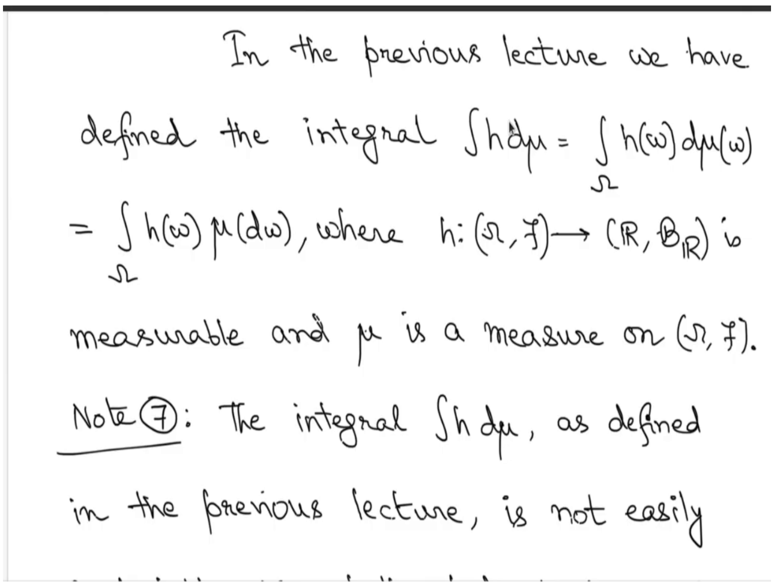

So, in the previous lecture, we have this define this integral, so, let us just formulate the notations for this lecture. So, h will stand for this measurable function defined on some appropriate measurable space. And suppose, you are given a measure on this domain side. So,

h is a real valued function measurable appropriately and then we want to talk about this kind of an integration procedure.

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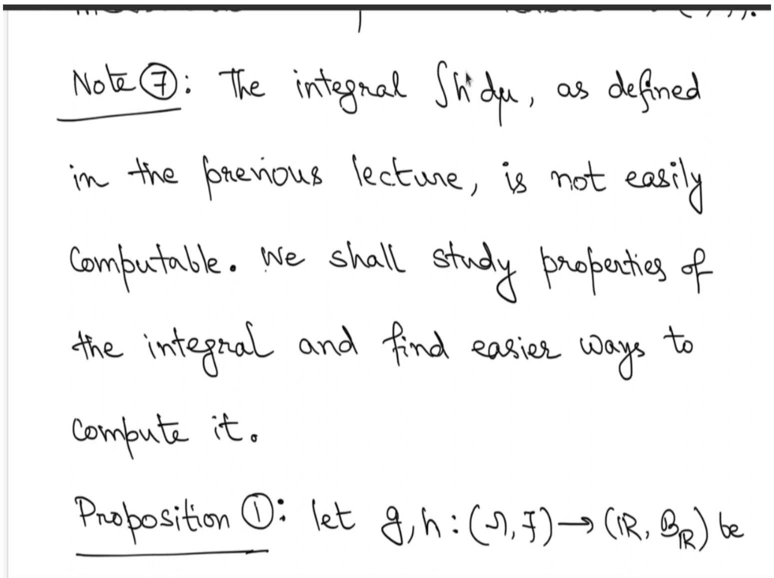

So, first you note that the integration, as you have defined in the previous lecture is not really easily computable. So again, our target now to is to understand the properties of this integration procedure, so that our job in computing these integrals becomes easier. So, you want to get hold of multiple nice properties, which should help us in computing these integrals. So, that is basically what we are planning to do now.

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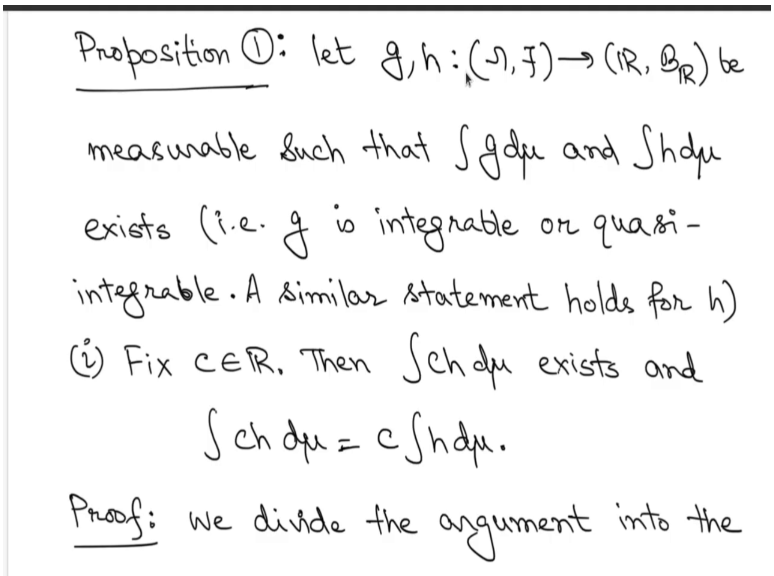

So, we start with this proposition that you take two measurable functions g and h defined on this measurable space and you assume that the integrals exist. So, what the, what do I mean by that. So, if you consider $\int g\,d\mu$, then there are two possible cases under which the integral exists, the first case is when g is integrable, in which case the integral value is certain finite number, some real number.

Otherwise, if g is quasi integrable even then the integral exists, but it takes certain values which is either, either $+\infty$ or $-\infty$. A similar statement you can make for the function h. So, with that at hand, so let us try to move on and try to prove certain nice properties.

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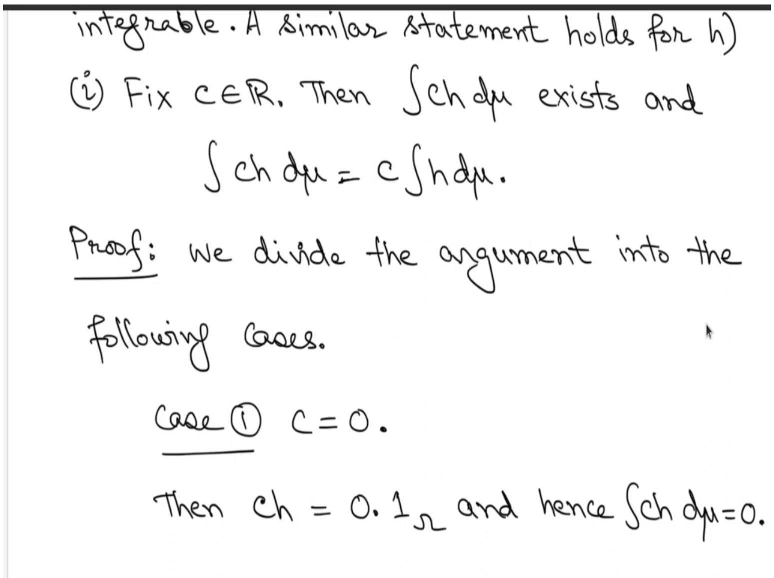

So, the first property is about scalar multiplication. What we are saying is that if you have a measurable function h, if you multiply it by some scalar quantity c, which is some real number, then you know that this is also a measurable function. So, what do you hope to prove is that $\int ch\,d\mu$ that should also exist with respect to the measure μ, and the integral value should be equal to $c\int h\,d\mu$. So that is basically the idea behind the scalar multiplication. So, but how do you prove this here.

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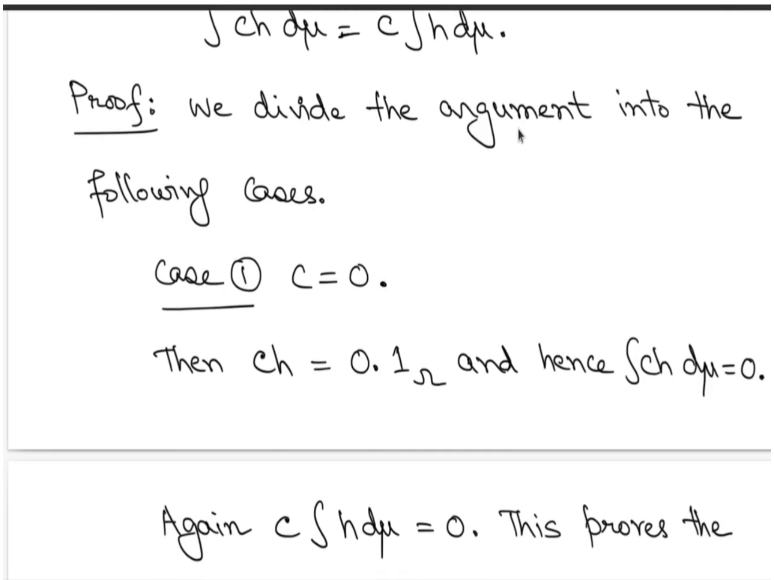

So, we again divide the argument into following cases to make the argument clearer, so let us see.

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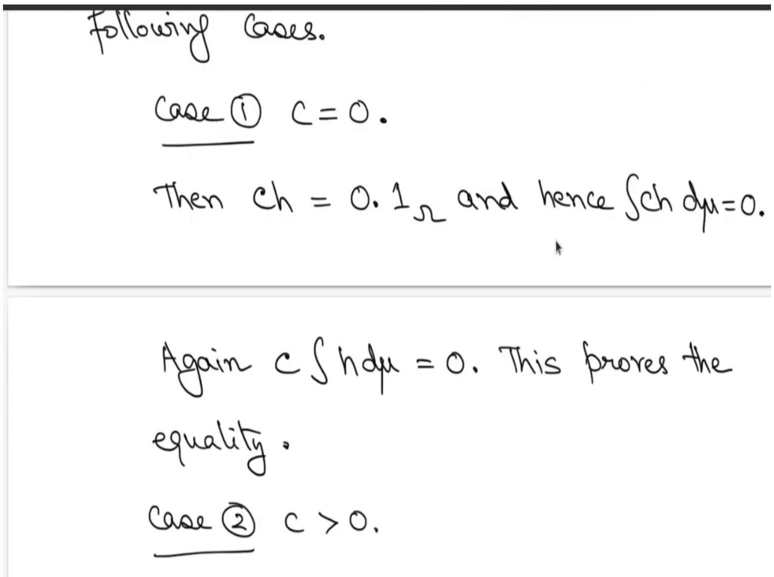

In the case that $c = 0$, if the scalar $c = 0$, then the function is identically the 0 function. So, you can think of it as a simple function, which is basically taking values 0 throughout the set Ω. So then as per the description, $\int ch\, d\mu = 0$, because you are just multiplying by the scalar 0 and multiplying it with the measure of the whole set Ω. But 0 times any real number, or 0 times ∞ is taken to be 0, so therefore, this integration is 0.

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Then $ch = 0 \cdot 1_{\Omega}$ and hence $\int ch \, d\mu = 0$.

Again $c \int h \, d\mu = 0$. This proves the equality.

Case (2) $c > 0$.

If $h = 1_A$ for some $A \in \mathcal{F}$, then

$$\int ch \, d\mu = \int c 1_A \, d\mu = c \mu(A) = c \int h \, d\mu.$$

But what happens to this $c \int h \, d\mu$? So again, whatever integral value h has it could be a real number, it could be $-\infty$, it could be $+\infty$, no matter what is the value, once you multiply by 0, it is 0. So therefore, you have that required equality. So, this equality holds when $c = 0$. Let us now consider the case when $c > 0$.

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Case (2) $c > 0$.

If $h = 1_A$ for some $A \in \mathcal{F}$, then

$$\int ch \, d\mu = \int c 1_A \, d\mu = c \mu(A) = c \int h \, d\mu.$$

If h is a simple function, then a computation similar to above yields the required equality.

So, what happens now, so again, now, we have to go through this several steps, which was used for the definition of the integral. So, start with the indicator function. So, take a set in your domain side σ field and look at the indicator of that set. So, if h is of that form, then what is the integral, so put in h as 1_A, so you are considering $c1_A$, and as per the description, this is nothing but $c\mu(A)$, but you can rewrite it as $c\int h\, d\mu$ because h is nothing but the indicator.

So remember, 1_A, if you integrate it, you just get back the measure. So that is what we are using them. So therefore, this required equality holds when h is the indicator function. But then, let us consider the case for simple functions.

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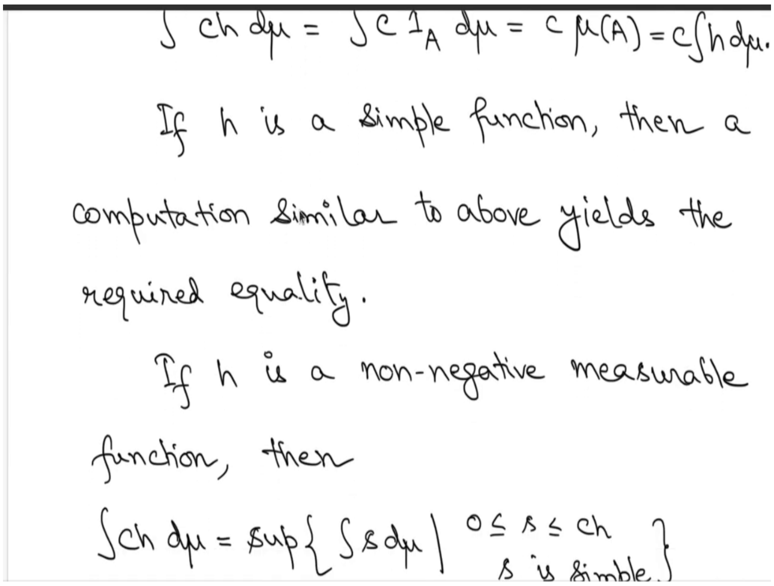

So, if you now do the similar computation, you will end exactly end up with a linear combination of these measures of the sets. And as long as the series or the summation that you are working with, as long as that makes sense, as long as $\infty - \infty$ does not appear together, you can immediately define the integral and verify this required equality. So, it just extends from the usual calculations that you have already seen for the indicators, you just have to deal with the appropriate linear combinations. And that is it. So, you get the result for simple functions.

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If h is a non-negative measurable function, then

$$\int ch \, d\mu = \sup\left\{ \int s \, d\mu \;\middle|\; \begin{array}{l} 0 \le s \le ch \\ s \text{ is simple} \end{array} \right\}$$

$$= \sup\left\{ c \int \frac{1}{c} \cdot s \, d\mu \;\middle|\; \begin{array}{l} 0 \le \frac{1}{c} s \le h \\ s \text{ is simple} \end{array} \right\}$$

$$= \sup\left\{ c \int \tilde{s} \, d\mu \;\middle|\; \begin{array}{l} 0 \le \tilde{s} \le h \\ \tilde{s} \text{ is simple} \end{array} \right\}$$

But then what do you do next, you go to non-negative and measurable functions, appeal to the definition. So, what do you do now, so you look at the definition, where you are looking at all simple functions below the function ch. And looking at those integral values, so you are looking at all simple functions, considering their integrals. So now, so that supremum quantity is the, this quantity. So here, remember, as long as h is non-negative, and $c > 0$, ch is a non-negative measurable function. So that is what we are using here.

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function, then

$$\int ch \, d\mu = \sup\left\{ \int s \, d\mu \;\middle|\; \begin{array}{l} 0 \le s \le ch \\ s \text{ is simple} \end{array} \right\}$$

$$= \sup\left\{ c \int \frac{1}{c} \cdot s \, d\mu \;\middle|\; \begin{array}{l} 0 \le \frac{1}{c} s \le h \\ s \text{ is simple} \end{array} \right\}$$

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$$= c \cdot \sup\left\{ \int \tilde{s} \, d\mu \;\middle|\; \begin{array}{l} 0 \le \tilde{s} \le h \\ \tilde{s} \text{ is simple} \end{array} \right\}$$

But then observe, that if you scale the simple functions that you are considering now, by the constant $1/c$. So, then what will happen, this will still remain a simple function, but these simple functions $\frac{1}{c}s$, so that will be smaller than h. But then you will look at this integration value. So, you divide and multiply by constant c. So, for simple functions, you have already allowed this multiplication by a scalar. So that is what we are using here. So, you can multiply and divide by the same scalar which is positive, then $\frac{1}{c}s$ is a simple function.

So, you can easily observe that that class of functions is nothing but in terms of this new notation that you look for all simple functions below h, call them as $\tilde{s}$. So, then we were just rewriting $\frac{1}{c}\tilde{s}$, that is all. So, you are just going from this supremum to this supremum, this is just a change in the indexing of the simple functions, but this supremum remains the same because the values remain the same.

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s is simple $\}$

$$= \sup\left\{ c \int \tilde{s}\, d\mu \;\middle|\; \begin{array}{l} 0 \le \tilde{s} \le h \\ \tilde{s} \text{ is simple} \end{array} \right\}$$

$$= c \cdot \sup\left\{ \int \tilde{s}\, d\mu \;\middle|\; \begin{array}{l} 0 \le \tilde{s} \le h \\ \tilde{s} \text{ is simple} \end{array} \right\}$$

$$= c \int h\, d\mu.$$

If h is $\mathbb{R}$-valued measurable, then observe that $(ch)^+ = ch^+$ and $(ch)^- = ch^-$

So therefore, let us look at this quantity now. So here you, you had multiplying all this terms here all the integration values here by this positive quantity c and you are considering supremum is there. So, you bring it out and that is nothing but the supremum of this set. But this is over this, this is the supremum is over all simple functions $\tilde{s}$ below h and that is nothing but the integration of the non-negative measurable function h. So, that is it.

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$$\int ch\, d\mu = \sup\left\{\int s\, d\mu \;\middle|\; \begin{array}{l} 0 \le s \le ch \\ s \text{ is simple} \end{array}\right\}$$

$$= \sup\left\{c\int \frac{1}{c}\cdot s\, d\mu \;\middle|\; \begin{array}{l} 0 \le \frac{1}{c} s \le h \\ s \text{ is simple} \end{array}\right\}$$

$$= \sup\left\{c\int \tilde{s}\, d\mu \;\middle|\; \begin{array}{l} 0 \le \tilde{s} \le h \\ \tilde{s} \text{ is simple} \end{array}\right\}$$

$$= c \cdot \sup\left\{\int \tilde{s}\, d\mu \;\middle|\; \begin{array}{l} 0 \le \tilde{s} \le h \\ \tilde{s} \text{ is simple} \end{array}\right\}$$

$$= c\int h\, d\mu.$$

So, therefore, you have managed to bring out that constant c from inside the integration to the outside, that is all you wanted to prove.

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$$= c\int h\, d\mu.$$

If h is $\mathbb{R}$-valued measurable, then observe that $(ch)^+ = ch^+$ and $(ch)^- = ch^-$

Therefore,

$$\int ch\, d\mu = \int (ch)^+\, d\mu - \int (ch)^-\, d\mu$$

$$= \int ch^+\, d\mu - \int ch^-\, d\mu$$

And therefore, the result is now true for non-negative measurable functions. How do you do it for real valued measurable functions, you just have to observe that the positive part of ch $(ch)^+$, that c times h is nothing but ch^+, because c is positive, this equality is true. Again, as c is positive, for the similar reason, the negative part of $ch(ch)^-$ is nothing but ch^-, so that is it. You identify it this way, and then for h^+ and h^-, which are non-negative measurable functions, use the result that was just proved.

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observe that $(ch)^+ = ch^+$ and $(ch)^- = ch^-$

Therefore,

$$\int ch \, d\mu = \int (ch)^+ d\mu - \int (ch)^- d\mu$$

$$= \int ch^+ d\mu - \int ch^- d\mu$$

$$= c\left(\int h^+ d\mu - \int h^- d\mu\right) = c\int h \, d\mu.$$

Case ③ $c < 0$.

So therefore, you start with $\int ch \, d\mu = \int (ch)^+ \, d\mu - \int (ch)^- \, d\mu$

$$= \int ch^+ \, d\mu - \int ch^- \, d\mu$$

$$= c\left(\int h^+ \, d\mu - \int h^- \, d\mu\right)$$

$$= c\int h \, d\mu$$

So, therefore, you get the required relation that the c can be brought out from your integration as long as $\int h \, d\mu$ exists.

So, therefore, you have managed to show that no matter what kind of measurable function you take, as long as $\int h \, d\mu$ exists, $\int ch \, d\mu$ also will exist, and it will just be that $c\int h \, d\mu$, so that is great. So, you have managed to provide for the cases when $c = 0$ and $c > 0$.

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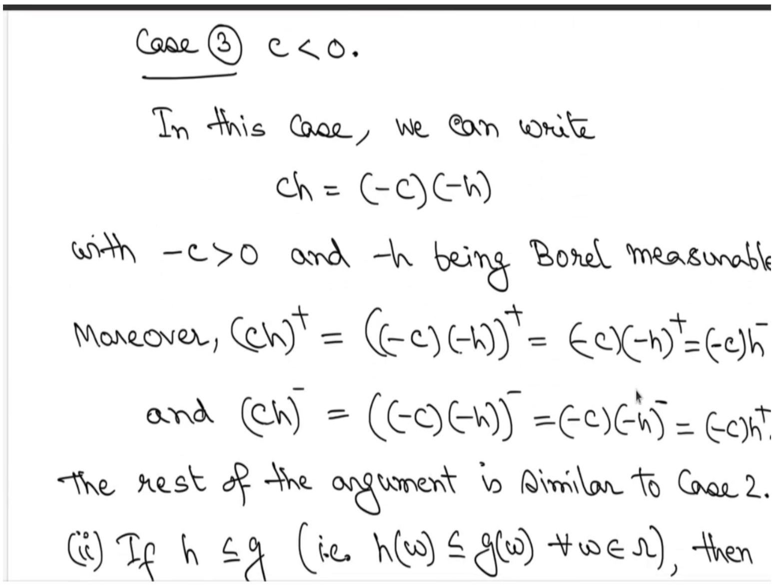

But what happens when $c < 0$, in that case, observe that ch, this function can be thought of as $(-c)(-h)$. So, $-c$ here is now a positive quantity. So, c was given to be negative. So therefore, $-c$ must be positive, but then h is given to be a measurable function. So $-h$ is also a measurable function, we just have to observe that now,

$(ch)^+ = ((-c)(-h))^+ = (-c)(-h)^+ = (-c)h^-$ (check the last step). So that will come out of that calculation. And similarly,

$(ch)^- = ((-c)(-h))^- = (-c)(-h)^- = (-c)h^+$. The rest of the argument follows similar to the case above when c was positive. So, just look at now, $\int ch\, d\mu$, split it into the positive part and negative part, use this identification here h^+ h^- are non-negative measurable functions and $-c$ is a positive constant, just use that and you will immediately get the required relation, that is it.

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Moreover, $(ch)^+ = ((-c)(-h))^+ = (-c)(-h)^+ = (-c)h^-$

and $(ch)^- = ((-c)(-h))^- = (-c)(-h)^- = (-c)h^+$.

The rest of the argument is similar to Case 2.

(ii) If $h \le g$ (i.e. $h(\omega) \le g(\omega)\ \forall \omega \in \Omega$), then

$\int h\, d\mu \le \int g\, d\mu$.

Proof: First consider the case $0 \le h \le g$.

For any simple function s satisfying

$0 \le s \le h$, we have $0 \le s \le g$. Hence

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So, you have managed to show that scalar multiplication works nicely with this integration procedure. But then, we now go to a very important comparison type inequalities involving this integration procedure, we are looking at two functions now h and g. Suppose you have that the function h is point wise below the function g.

So, therefore, pick any point $\omega \in \Omega$, look at the function values, if for all such points in the domain function value of h is below function value of g, i.e. $h(\omega) \le g(\omega)$ then what we are saying is that the corresponding integrals will also satisfy the same relation. So, therefore, if $h \le g$, then $\int h\, d\mu \le \int g\, d\mu$. So, how do you show this? So, here we consider a simple case we prove it in that in detail that simple case, but the rest of the case we refer to appropriate reference.

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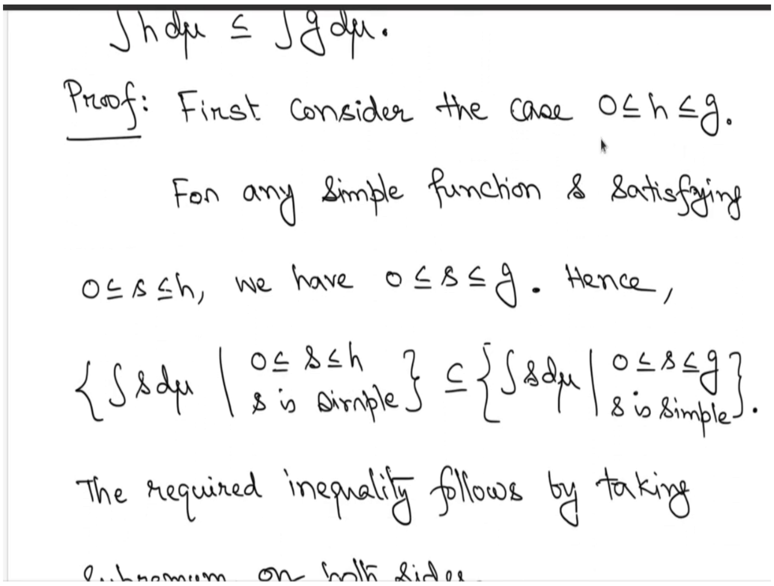

So, we first look at the simple case. So, here we consider the case when h is non-negative and since g is above the function h, g also becomes a non-negative measurable function. So, now consider the integrations for h and g separately. First look at the function h, what is the integration for this non-negative measurable function, what you have to do is to look at all simple functions below the function h. So, look at such simple functions s, but then since g is above h immediately for all such simple functions, we immediately have the inequality that the function values of this simple functions $0 \leq s \leq g$.

So, this is automatically satisfied. So, therefore, any simple functions that is below h is also below the function g, but then there might be some simple functions which are below g which is a bigger function, g is a bigger function. So, therefore, there might be some simple functions which are below g but need not be compatible with the function h. So, therefore, if you are looking at this class of simple functions that are below h that is contained inside the class of simple functions that are below g, but there might be more functions, more simple functions below g. So, that can happen. So, that is it.

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For any simple function s satisfying $0 \leq s \leq h$, we have $0 \leq s \leq g$. Hence,

$$\left\{ \int s \, d\mu \;\middle|\; \begin{array}{l} 0 \leq s \leq h \\ s \text{ is simple} \end{array} \right\} \subseteq \left\{ \int s \, d\mu \;\middle|\; \begin{array}{l} 0 \leq s \leq g \\ s \text{ is simple} \end{array} \right\}.$$

The required inequality follows by taking supremum on both sides.

For the other case, h may now take

So now, for all such simple functions, since this is a smaller collection of functions, then whatever values you get out of integration, so these values are already contained in the right-hand side set. So, for any simple function that is on the left-hand side, that simple function is already in the right-hand side. So, therefore, corresponding values of the integrations are already listed on the right-hand side.

So, this is a containment, but therefore, if you are going to look at the supremum of these two quantities, the left-hand side will be smaller. So, since you are looking at a bigger set on the right-hand side that quantity will be bigger, the supremum will be bigger. So, the required inequality simply follows by taking supremum on both sides.

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$$\int h\,d\mu \le \int g\,d\mu.$$

<u>Proof</u>: First consider the case $0 \le h \le g$.

For any simple function s satisfying $0 \le s \le h$, we have $0 \le s \le g$. Hence,

$$\left\{ \int s\,d\mu \;\middle|\; \begin{matrix} 0 \le s \le h \\ s \text{ is simple} \end{matrix} \right\} \subseteq \left\{ \int s\,d\mu \;\middle|\; \begin{matrix} 0 \le s \le g \\ s \text{ is simple} \end{matrix} \right\}.$$

The required inequality follows by taking

Now, if you allow the function h to take negative values

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For the other case, h may now take both positive and negative values. The proof is similar to the case above, by carefully accounting for h^+ and h^-. The

explicit calculation is being skipped to reduce technicalities in the notes. The

Then that that argument becomes a slightly more complicated *h* might take positive or negative values.

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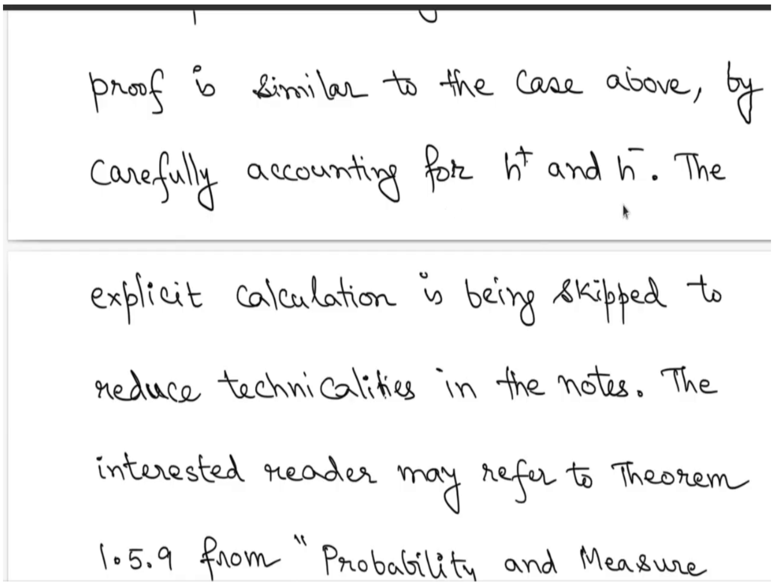

And then what do you have to do is to split the cases into h^+ and h^- and compare with the appropriate parts for g, appropriate g^+ and g^-. So, that explicit calculation here is being skipped this, this argument will go through, but the only problem is that you have to divide into several possible cases, you have to compare h plus with g, g^+ and g^- and so on. So, we are simply avoiding that case but the argument will go through by comparing all these simple functions.

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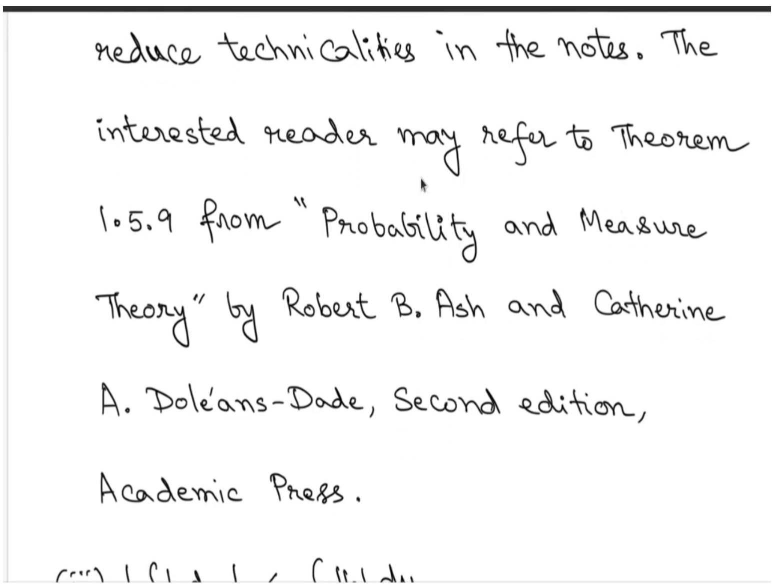

If you are interested, please go take a look at the book by Robert B. Ash and Catherine A. Dole'ans-Dade, the book title is Probability and Measure Theory, the theorem number is

1.5.9. So, this is the second edition of the book published by Academic Press, you might choose to take a look at the detailed proof there. But we are going ahead assuming that the result is true. So, we can compare the integrals provided the original functions already satisfied that required relation.

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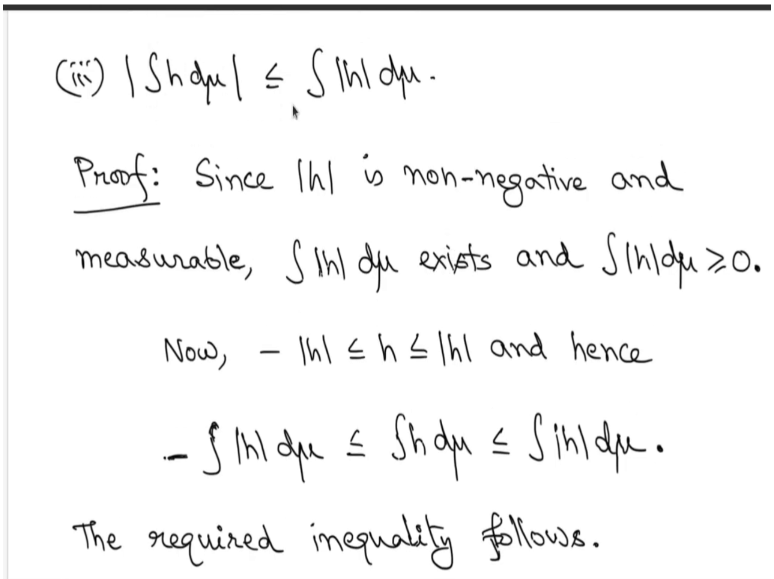

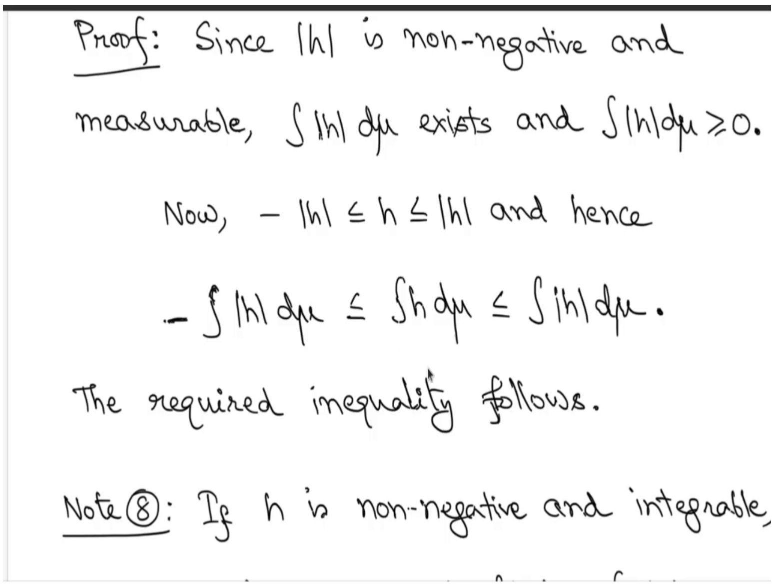

So, then we now get very nice interesting inequalities following that inequality. So, what do we do is that we look at $\int h \, d\mu$ and $\int |h| \, d\mu$. So, observed that given any measurable function h, the function $|h|$ is always non-negative and measurable, this we have already observed earlier, but then you can consider $\int |h| \, d\mu$.

So, this is a integration of the non-negative and measurable function mod h. So, in particular, $\int |h|\, d\mu$ will be something non-negative, so that, that is the observation we can make very easily. But now, compare these functions.

So, since h falls between these functions $-\ |h|$ to $+\ |h|$, then you can automatically compare the integral values. So, what is $\int -\ |h|\, d\mu$ it is nothing but $-\int |h|\, d\mu$. And that is dominated from above by $\int h\, d\mu$simply by following that inequality.

Again, since $h\ \leq |h|$, $\int h\, d\mu \leq \int |h|\, d\mu$, that is it, you get both sided inequalities. And hence, to get that $\left|\int h\, d\mu\right|$, you get $\left|\int h\, d\mu\right| \leq \int |h|\, d\mu$. So, this is a very simple inequality to prove, as long as we have managed to compare any two arbitrary measurable functions and their integrations.

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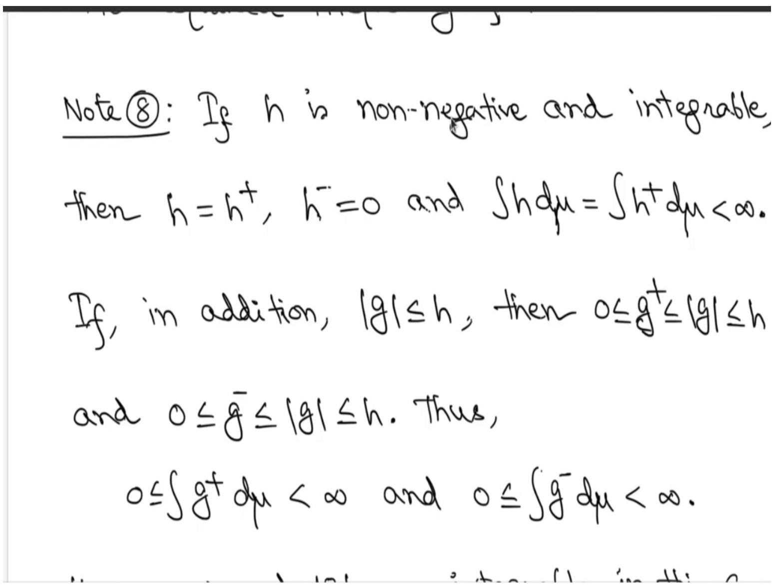

So, with this relation at hand, we can now make some very interesting comments. So, here start with a h, which is non-negative and integrable. So, if h is already non-negative, then $h^- = 0$. So, you do not have to worry about h^-. So, $h = h^+$ here. So, if h is non-negative, this is what is going to happen. Then integration of h is nothing but $\int h^+\ d\mu$ and that is a finite quantity provided that h is given to be integrable. Then, what is going to happen is that provided such h, if you can get arbitrary function g, such that $|g| \leq h$.

So, you are starting off with the arbitrary function g now, and if $|g|$ gets dominated from above by this function, this non-negative function h, then you will immediately say that g^+ is dominated by h simply following this inequality that g^+ is a non-negative function and it is dominated from above by $|g|$ but $|g|$ is dominated by h. So, therefore, g^+ is also dominated by h from above.

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If, in addition, $|g| \leq h$, then $0 \leq g^+ \leq |g| \leq h$

and $0 \leq g^- \leq |g| \leq h$. Thus,

$$0 \leq \int g^+ d\mu < \infty \text{ and } 0 \leq \int g^- d\mu < \infty.$$

Hence, g and $|g|$ are integrable in this case

Note 9: If $|g|$ is integrable, then by

A similar inequality holds for g^-, so $g^- \leq |g|$ is also less equals to mod g, and if $|g| \leq h$, then $g^- \leq h$. So, therefore, you will immediately say that integration of $\int g^+ \, d\mu \leq \int h \, d\mu$.

And since $\int h \, d\mu$ is finite, you will immediately say that $\int g^+ \, d\mu$ is finite.

Similarly, $\int g^- \, d\mu$ is finite, because it is dominated from above by $\int h \, d\mu$. So, both is g^+ and g^- have finite integrations. So, therefore, how do you immediately say is that the function g is integrable. And you immediately also observed that g and $|g|$ both are integrable here.

So, this is something interesting that as long as you can dominate $|g|$ by an integrable function h, you can immediately say that the function g is integrable, you do not have to separately verify the integrability of g. So, if you can get a bound in terms of some known nice function h, which is known to be integrable, then g has to be integrable. So, this is some explicit condition that will allow you to check integrability in a slightly easier fashion.

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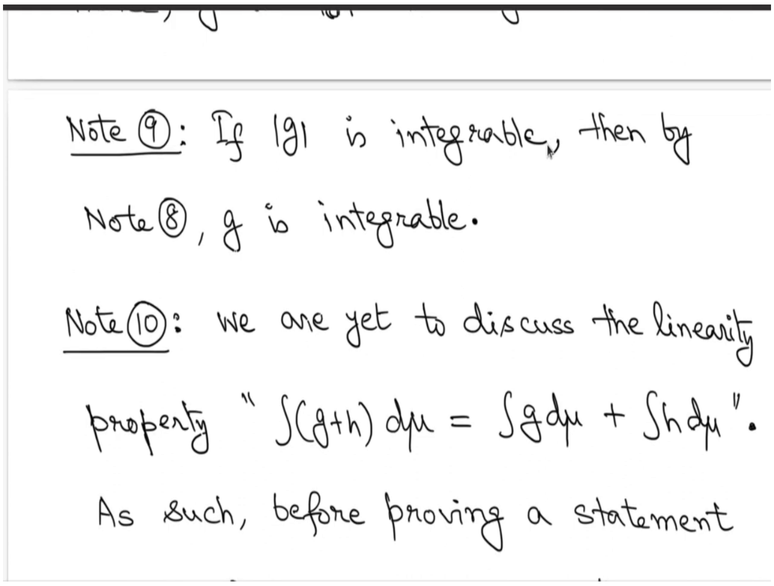

But then what is the relationship between integrability of g and $|g|$. So, you make a partial statement here, we will come back to this statement later on, once more in a later lecture. So, what we are observing here is that if $|g|$ is integrable, then we claim that g has to be integrable. Why?

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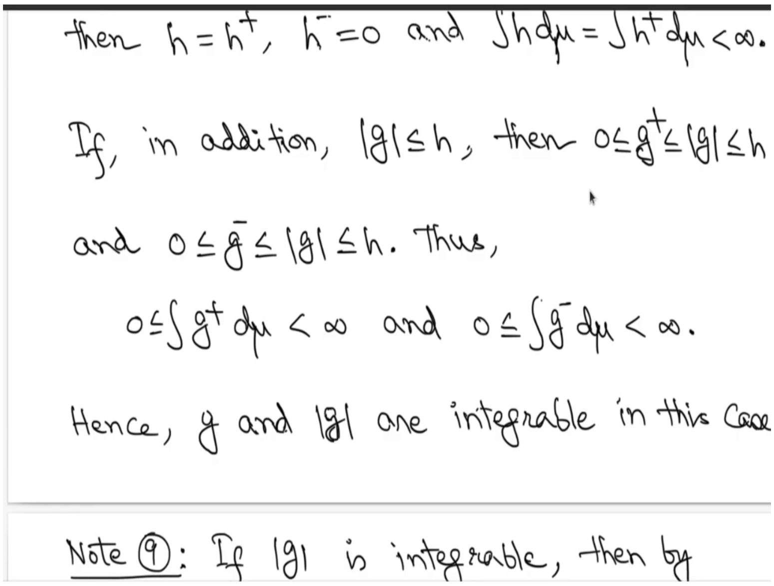

If mod g is already integrable observe these inequalities that we have stated earlier, forget about h g^{+} is dominated by $|g|$. So, therefore, $\int g^{+} d\mu$ will be finite provided $|g|$ is integrable. So, this is a non-negative function $|g|$ is non-negative, and if it is integrable, that means integration of $|g|$ is finite.

So, g^{+} has a finite integral, as long as g^{+} gets dominate from above by $|g|$, so that is it. So therefore, g^{+} has a finite integral. Similarly, g^{-} is also a finite integral. So, putting them together, you get the integrability of g.

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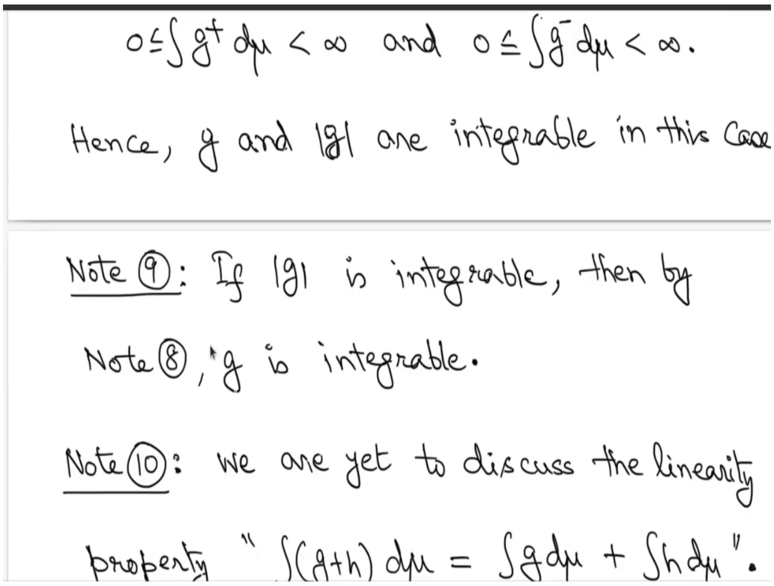

So therefore, to check that g is integrable, one sufficient condition is that $|g|$ is integrable. Another sufficient condition is that you find some other nice function h which is integrable and check that the inequality $|h| \le h$ holds, that is it. So, in all of these results, that we have discussed, so far, we have discussed this scalar multiplication and certain comparison inequalities, we are yet to discuss the linearity property of the integration.

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Note (10): we are yet to discuss the linearity property "$\int (g+h)\, d\mu = \int g\, d\mu + \int h\, d\mu$". As such, before proving a statement of this form, we cannot use this in our argument.

Note (11): Once we prove the statement in

So that is by that I mean that we would like to say that $\int (g + h)\, d\mu$ is the addition of $\int g\, d\mu$ and $\int h\, d\mu$. But the problem is as long as you do not prove this, you cannot use it in your argument. So, in all these arguments till we prove linearity, we have to be very careful, we cannot use linearity. So, this is caution that you should be taking care of.

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of this form, we cannot use this in our argument.

Note (11): Once we prove the statement in Note (10), we shall prove the converse of the statement in Note (9).

Note 9: If $|g|$ is integrable, then by Note 8, g is integrable.

Note 10: We are yet to discuss the linearity property "$\int (g+h)\, d\mu = \int g\, d\mu + \int h\, d\mu$". As such, before proving a statement

But once you prove the required linearity, then we can prove the converse of the statement in note 9. So, what is note 9 again, so, in note 9, we said that $|g|$ is integrable implies g is integrable. We are going to show that g is integrable implies $|g|$ is integrable. So, this will follow once we prove the linearity of the integration property. So, we will discuss these issues in later lectures. And we are going to continue looking at properties of measure theoretic integration in the next lecture. We stop here.