Measure Theoretic Probability 1

Professor Suprio Bhar
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 23

Lebesgue–Stieltjes Measures

Welcome to this lecture. So, before proceeding to the topic of discussion in this lecture, let us first quickly recall what we have done in the past few lectures. So, in the last two lectures of this week, we have finished the proof of the correspondence between distribution functions on the real line, and probability measures on the real line, together with the borel σ- field. So, let us quickly recall what we mean by a distribution function.

So, it was defined as independent of any connection with any probability measure or a random variable. So, it is simply a function taking values between 0 and 1. And it is non-decreasing right continuous with limits at ∞ and $-\infty$ being 1 and 0, respectively. So, corresponding to such functions, so, we have constructed probability measures. And of course, when we originally defined the distribution functions, it was through probability measures.

So, that gave you a correspondence between these two classes of things. So, the first collection of things being the class of distribution functions, the second collection being the class of probability measures on the measurable space real line together with the borel σ- field. So, with that setup, we finished that discussion in the last lecture. So, that correspondence proof is now complete.

But now, we now move on to understanding that correspondence in a slightly more general setting. So, this is what we discuss in this lecture. And as we shall see, this is going to give us a larger class of examples of measures, beyond probability measures. Including probability measures, we will construct more types of measures, including infinite measures. So, we will exploit this connection between certain class of functions and measures. So, this is of course, extending the existing correspondence between distribution functions and probability measures. So, let us move ahead with the slides.

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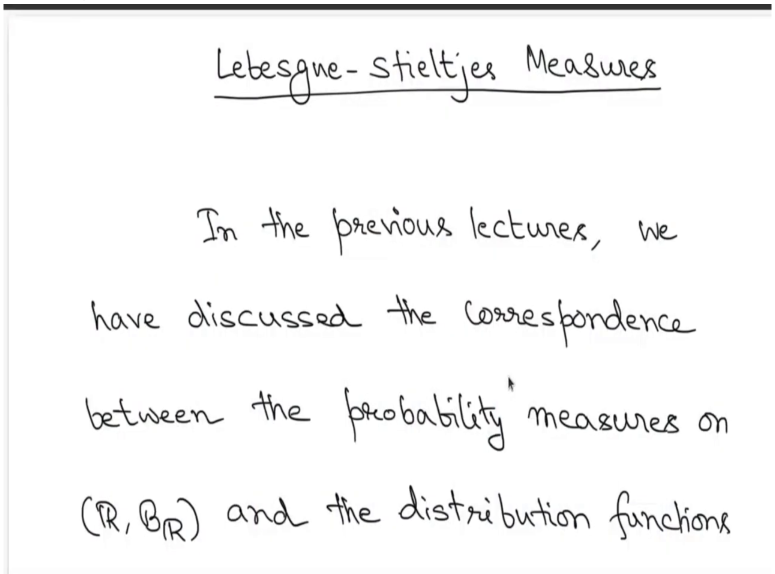

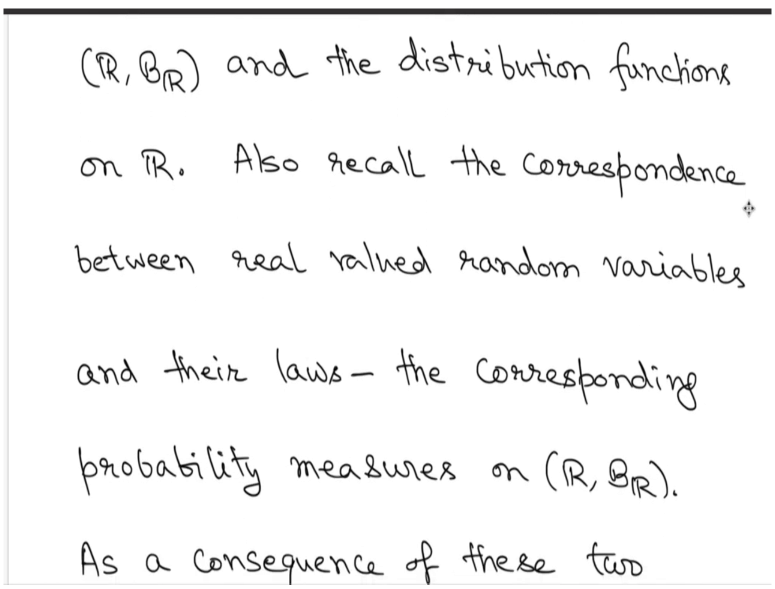

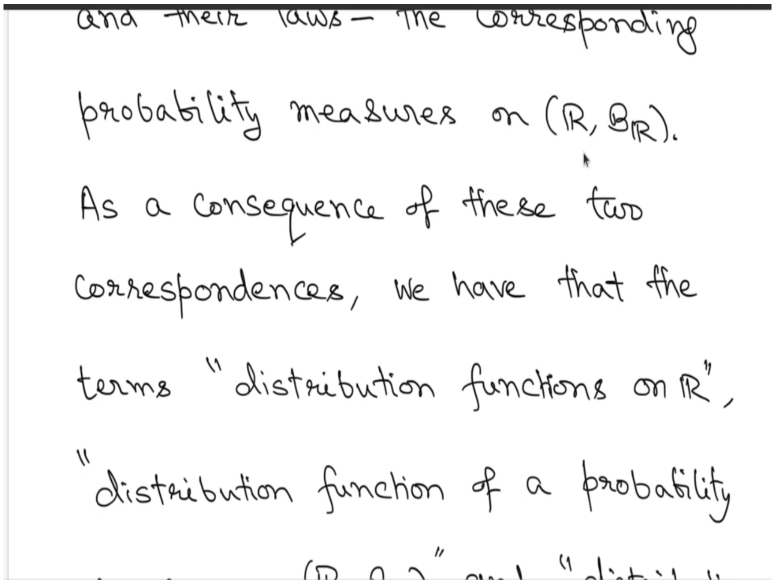

So, in the previous lectures, we have discussed this correspondence, between probability measures on this measurable space. And the distribution functions on ℝ, but recall in this setting, that the correspondence between real-valued random variables and their laws were also discussed in some earlier lectures. Then what do we have are two different correspondences, one between random variables and their laws, which are property measures on the real line. And you also have the correspondence between the laws and the corresponding distribution functions.

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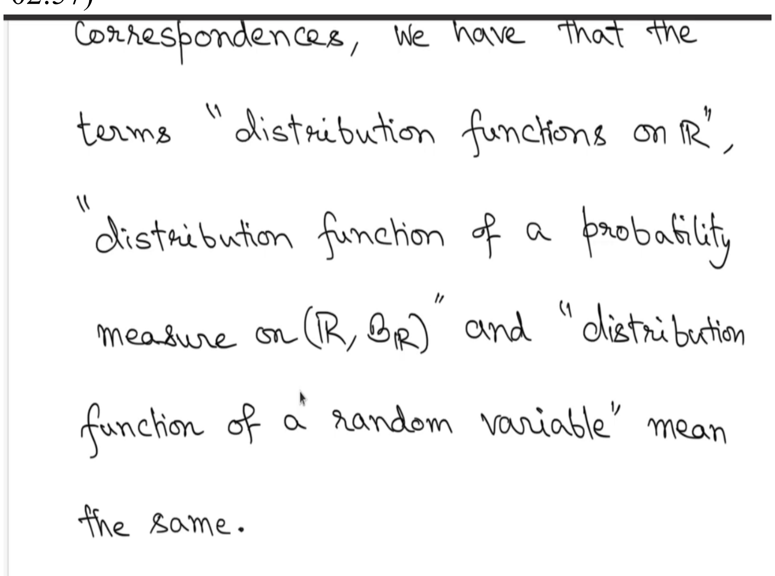

So, as a consequence of these two correspondences, we have that the three terms will mean the same things. The terms being distribution functions on ℝ, which are defined independent of any connection with probability measures and random variables. The second one is purely defined in terms of a probability measure.

So, given a probability measure, consider its corresponding distribution function. So, that is what this second term under quotes refers to. And the third type is the distribution function of a random variable. Again, given a random variable, you look at the corresponding law and corresponding to that you will look at the distribution function. So, you have these three types of objects, these three types of functions.

And according to our discussion, after all these correspondences has been assembled, these three things mean the same. So, even if the distribution functions originally were defined independent of the probability measures, we showed that all distribution functions arise as this form, that meaning that it is already a distribution function of some probability measure. So, that correspondence gives us this kind of very useful identification. But then, we are going to look into the correspondence between this class of functions and measures.

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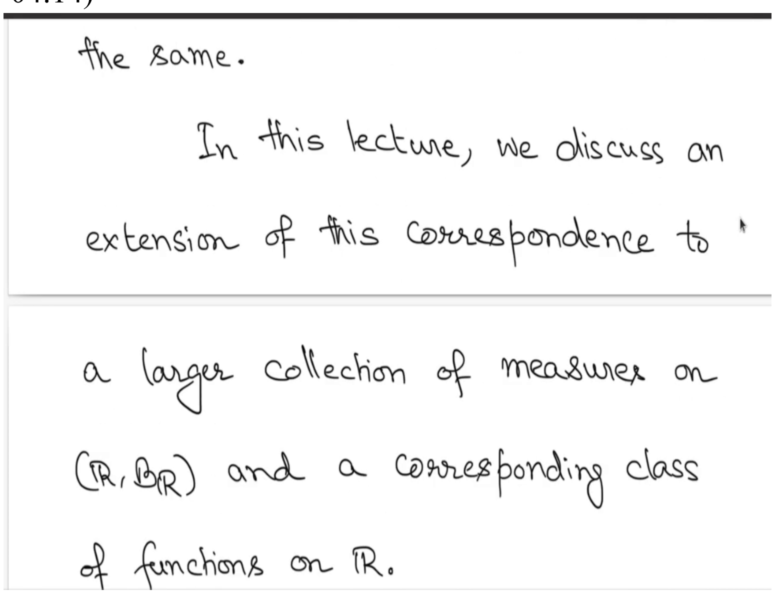

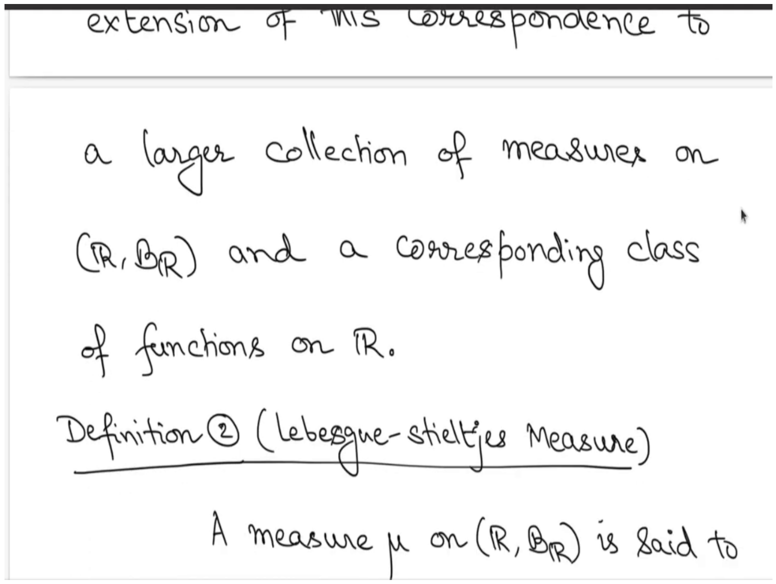

So, in this lecture, we are going to focus on a certain extension of this correspondence, to a larger class of measures. And of course, you will get a correspondingly a larger class of functions. So, before we go look into the corresponding class of functions, we again start off with whatever we did with probability measures. So, for probability measures, we first considered probability measures then define the corresponding distribution functions. And later on, we moved back from the class of functions to the measures. So, this is exactly what we are going to do now. We are again starting off with a slightly general class of functions with an appropriate definition.

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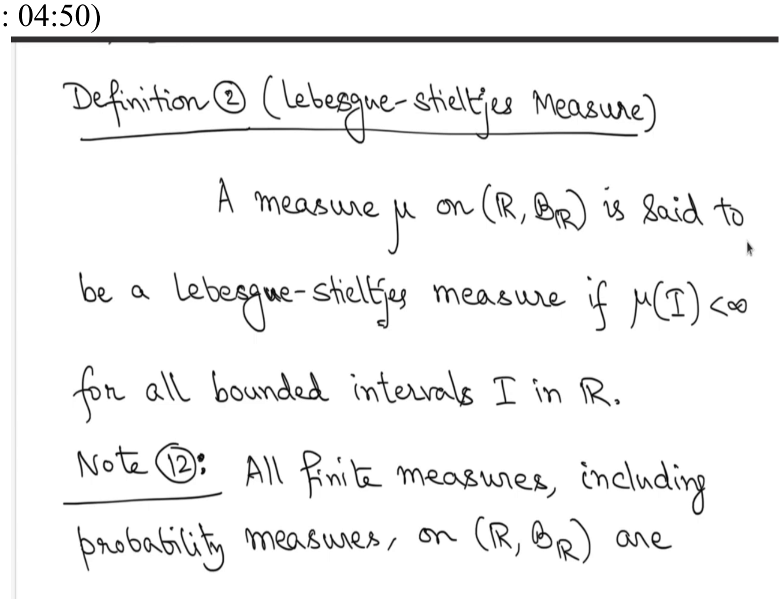

So, what is this class of measures? So, this, this is the class of measures called the Lebesgue Stieltje's measures? What do we do? We take measures on this now well-known measurable space, real line together with the borel σ- field. Take a measure, call it a Lebesgue stieltje's measure. So, this is more of an adjective. So, you will say that measure is Lebesgue stieltje's, if it associates finite mass to all bounded intervals.

So, take an interval of the form a to b, where a and b are real numbers. Then, any kind of (a, b) or $[a, b]$ or $(a, b]$ or, $[a, b)$, whichever combinations you take, as long as you are taking two real numbers a and b. And looking at the corresponding interval a b, so, any type of things as I mentioned so that is a bounded interval.

So, for all such bounded intervals, what we want is that the measure of that interval should be finite. So, we want this for all possible intervals in the real line. So, if this happens for a given measure mu, we will call this measure as a lebesgue stielje's measure. So, what are examples of this?

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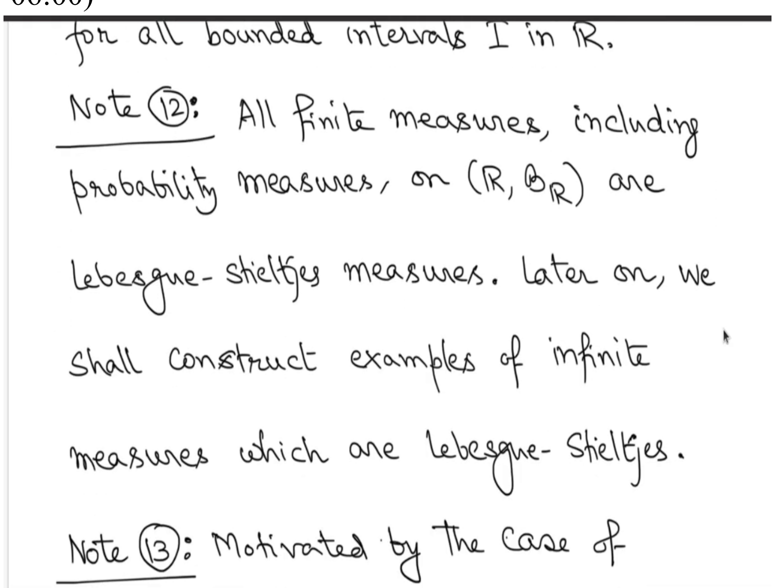

So, of course, any finite measure will satisfy this because the total mass anyway is finite. So, in particular, all bounded intervals will have finite mass. And finite measures, of course, includes probability measure. So, we already have quite a large class of measures, which are examples of Lebesgue Stieltje's measures. But now, the question is, are there any examples of infinite measures which are Lebesgue Stieltje's?

So, that is an interesting question now. So, all finite measures including probability measures are examples of this type, but what we would really would like to know is that, are there any infinite measures, which are Lebesgue Stieltje's. So, we will see such examples, which will turn out to be Lebesgue Stieltje's.

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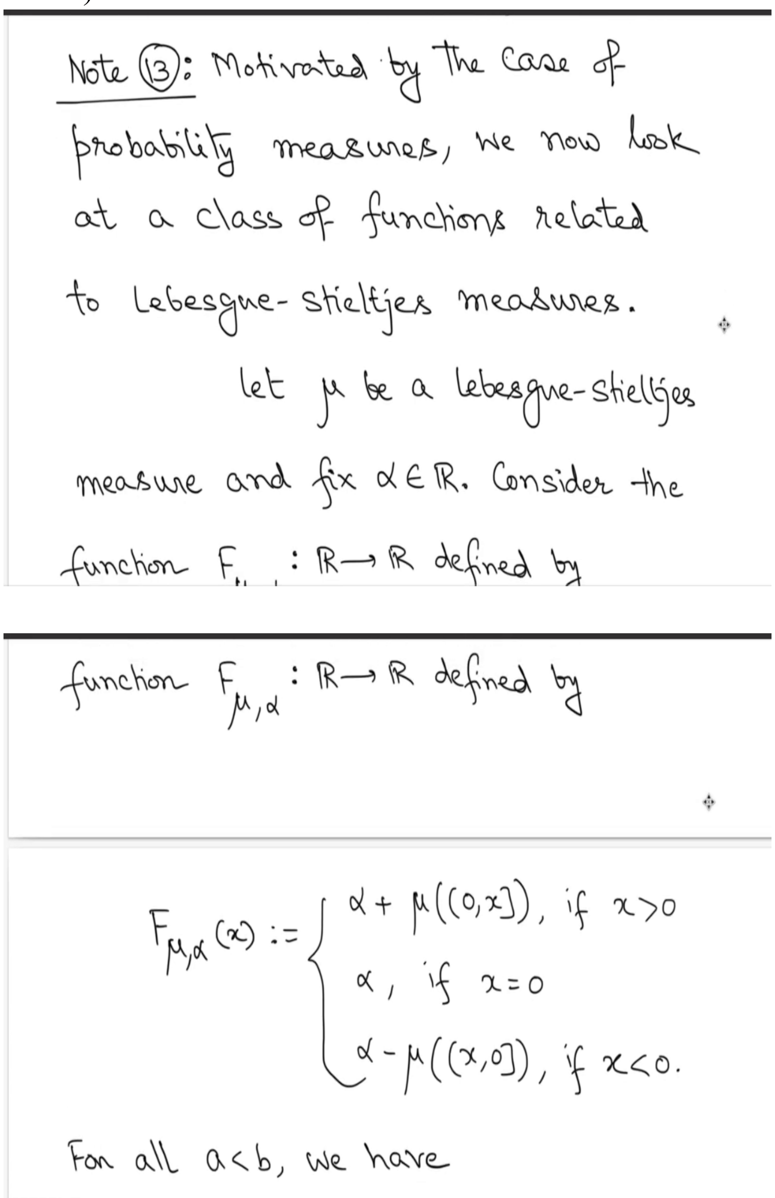

So, now, as I mentioned a few minutes back, so, we again start off with this class of measures and look at a corresponding class of functions. So, motivated by the case of this probability measures, we are now going to look at this class of functions associated or related to this lebesgue stieltje's measures. So, what do you do? Take a lebesgue stieltje's measure mu and fix a real number α.

So, you will see this α basically is playing the role of a parameter, you can fix it to be 0 if you wish. So, given this α, look at this type of functions, which we denote as $F_{\mu, \alpha}$. So, again, if you were to fix $\alpha = 0$, so, this function will purely depend on the measure chosen. So, once you have fixed a value of the parameter α, whatever function you are going to define is going to purely depend on the measure μ.

So, anyway, so, fix the real number α and look at this function defined on the real line and taking real values. So, how is this defined? So, for any real number x, you will look at its three possible cases. So, $x > 0$, $x = 0$ and $x < 0$, so three cases. So, if $x > 0$, you will look at this type of interval, $(0, x]$. So, this is a bounded interval. So, this value as given in the definition of our lebesgue stieltje's measured this value will be some finite quantity, so we add α to that.

So, that is what it is. So, when x is a positive real number, you assign this value, compute this value and assign this value to the function, for that point. If the real number $x = 0$, you assign the value α, to this function. And if $x < 0$, you assign this specific value, that you first compute the size of the interval, $(x, 0]$. So, remember, $x < o0$ here. So, you look at $(x, 0]$, that intervals or their sub bounded interval. So, these again are some real numbers, some non-negative real number, you subtract this value from α. So, whatever this value is, you assign this value to the function. So, this is the definition of the function.

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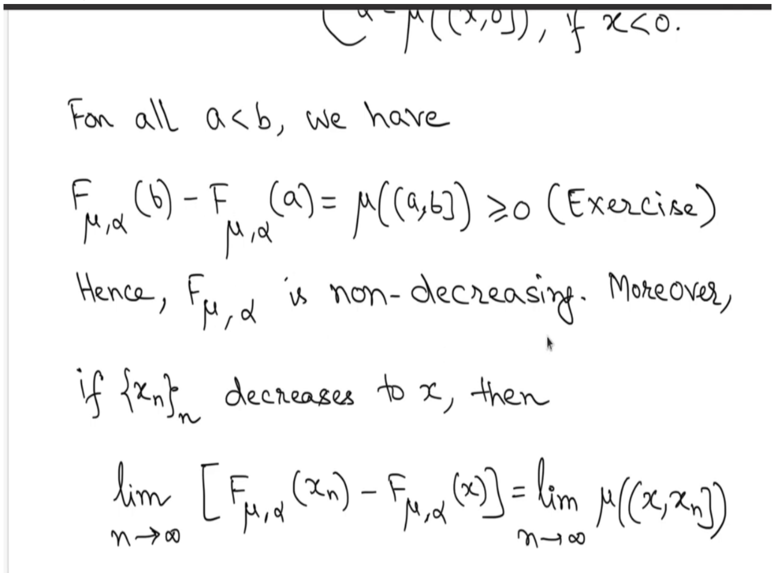

Now, what you can check is that for any real numbers, a and b, it does not matter whether they are in negative positive whatever, you choose any two real numbers a and b, then given this function, what you can verify is that the increment of the function values now will exactly be a measure of the interval, left open right closed interval, $(a, b]$. So, this you can check. So, just follow the definition of the function. Look at this increment.

Look at all possible cases where both a and b are positive, both a and b are negative, or a is negative, and b is positive. So, you have these kinds of cases. So, again, you could possibly include the case when $x = 0$. So, under all those possible situations, you try to look at the increment of this function, and then you compute this value. And it will turn out to be that it is exactly the measure in terms of the measure view of the interval $(a, b]$.

This is, you can take it as an exercise. But note that this value whatever this is, this is non-negative by the definition of a measure. So, this quantity, whatever it is, this size will be non-negative. So, therefore, as a consequence of this, we immediately claimed that $F_{\mu,\alpha}$, whatever function you have now defined, is non-decreasing. And moreover, you can also show this interesting property, that if you take a sequence of real numbers, $\{x_n\}$, decrease into a real number x.

Then you can consider this limit now, what is this limit? So, you look at the function value at x_n, x_n is a larger quantity so you subtract the smaller quantity here. So, this quantity whatever it is as per the observation above this is nothing but the length of the interval $(x, x_n]$ so, left open right closed interval. But now take the limit as $n \to \infty$ on both sides of this equality, great.

So, what happens to the right-hand side? So, remember x_n goes down to the point x. So, therefore, these intervals $(x, x_n]$, these will decrease and decrease to the empty sets. So, therefore, you simply use the continuity from above for this measure and therefore, you end up with a measure of the empty set, which is 0. So, that is a very important observation.

And remember, just to keep things complete, remember we are already given sets which has finite mass. So, therefore, for whatever type of measures you are considering, a measure of the empty set will be 0. So, just go back to the original discussion when we introduce measures. So, we said that if there is a set with finite mass, then you will get the emission of the empty set is 0, that is good. So, therefore, what you have actually managed to show is that if you have any sequence $\{x_n\}$ decreasing to x, then limit of this function values will approximate this quantity. So, the limit of these will be exactly this quantity.

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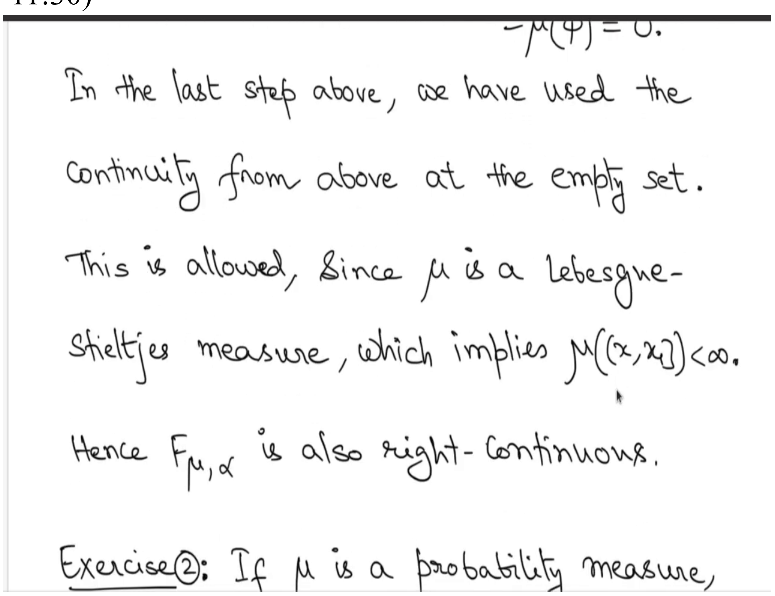

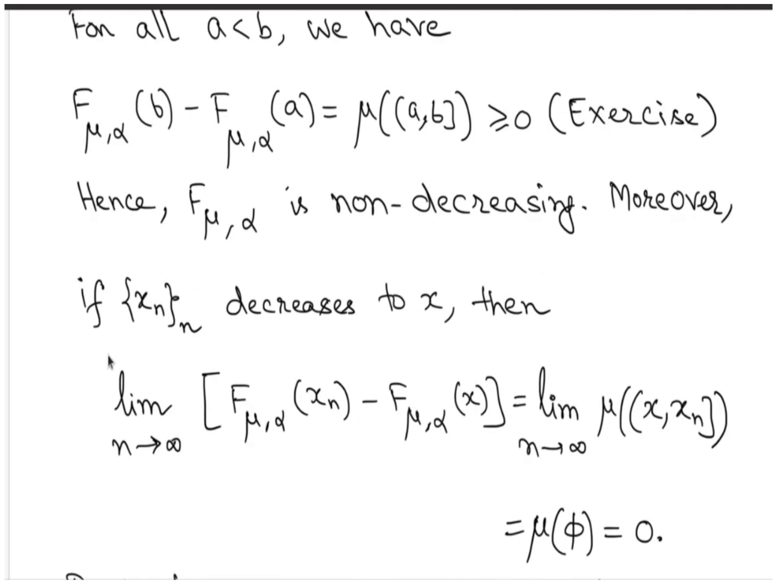

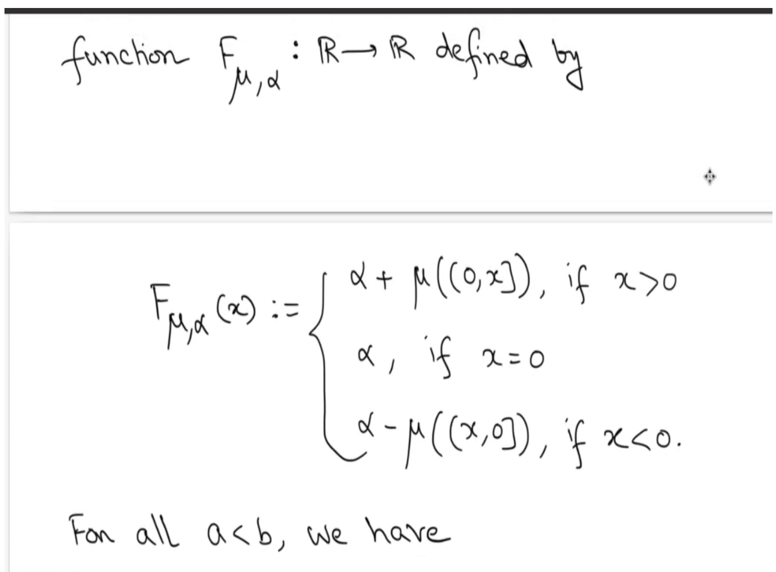

So, therefore, what do we get? We have used this continuity from at the empty set, and we get $F_{\mu,\alpha}$ is right continuous. So, this function whatever this is, this becomes right continuous and we have also shown that this is non-decreasing. So, this is simply generalizing the concept of a distribution function corresponding to a probability measure. So, there you have these limits at ∞ and $-\infty$ specified values.

We have not talked about the values of the limits at ∞ or $-\infty$ for the functions $F_{\mu,\alpha}$, but you have already seen these properties that these are non-decreasing functions and they are right continuous. But, an i`mportant observation here is that depending on the value of α your function may take a negative value. So, for example, if you are working with $x = 0$, the function value is α.

So, if α you take to be something negative, then the function value is negative. This is allowed here. So, α is some real number as taken above. So, this value could be negative. So, whatever functions we are considering here, could take negative values even though their increments are now positive. Since simply because, using the properties of the measure μ. So, these functions now could be negative.

So, this is slightly the departure from the class of distribution functions that we have considered corresponding to probability measures. But with that at hand, we are now going towards talking about (())(12:55) properties of distribution functions, that we have already seen in connection with probability measures.

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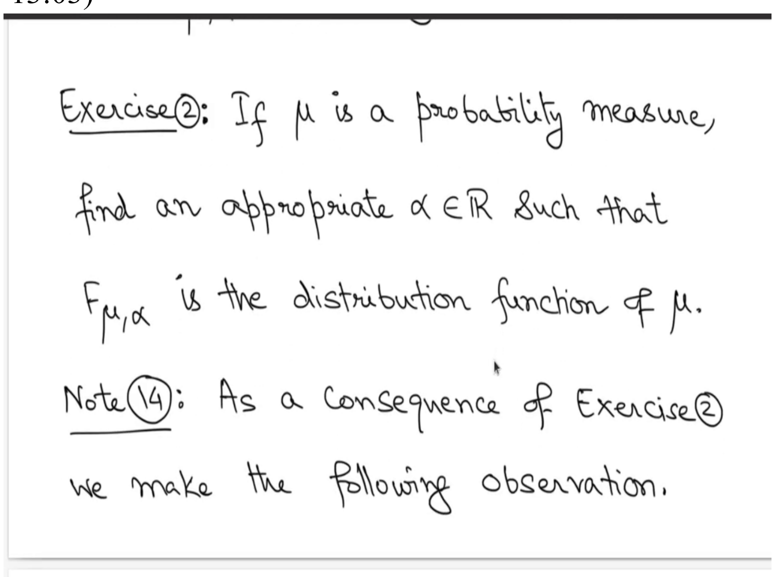

So, if μ is a probability measure, what do you really expect is that there should be an appropriate α. Such that, $F_{\mu,\alpha}$ is exactly the distribution function. So, this is exactly what we have been pointing towards. That since, this $F_{\mu,\alpha}$ is defined corresponding to any lebesgue stieltje's measure and probability measures are part of that and this $F_{\mu,\alpha}$ as seen above has this nine, non-decreasing and right continuous properties.

So, you expect to have some connection with the distribution functions corresponding to probability measures. So, what you can choose to do is to figure out this appropriate value of α. Such that, $F_{\mu,\alpha}$ exactly gives you the distribution function of the probability measure μ in the case when μ is a probability measure. So, please work this out.

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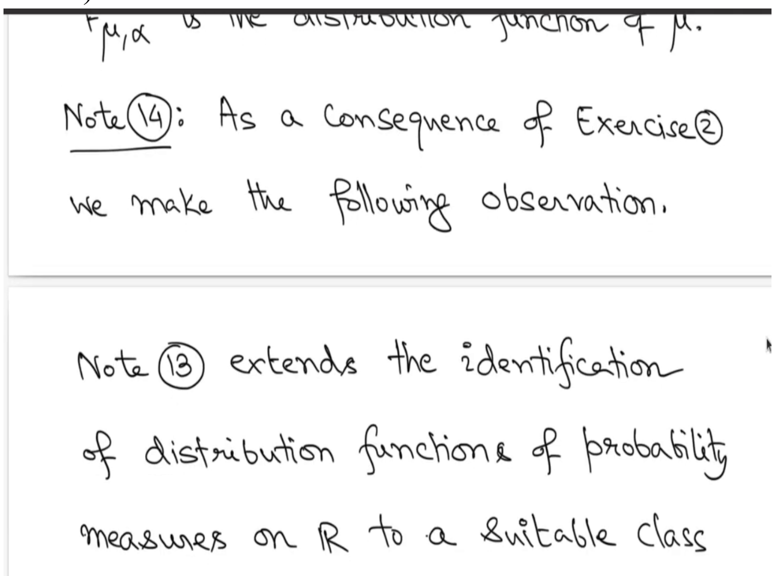

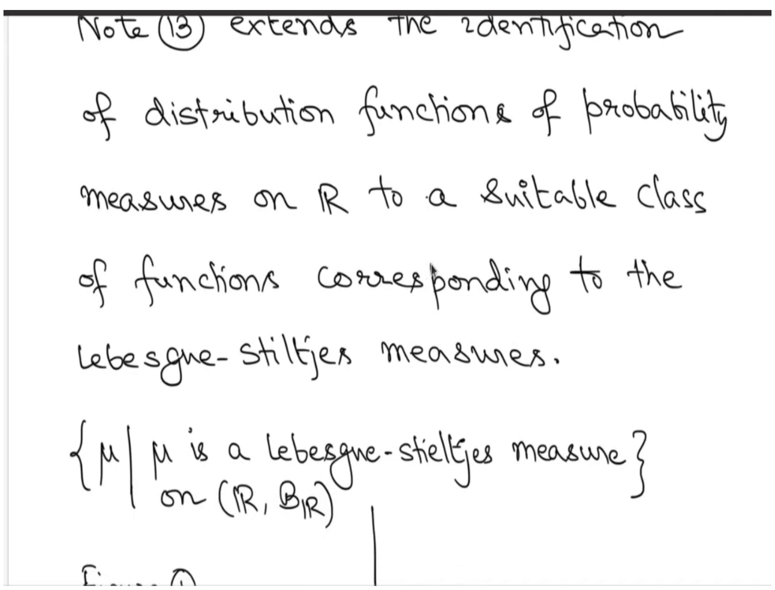

But now, as a consequence of this identification, that we have just mentioned, you can make the following observation, that in node 13 we have mentioned that these functions are non decreasing and right continuous. So, that is okay, put it together with this identification with the distribution function of a probability measure. But note 13 is basically saying is that we are just simply extending the identification of distribution functions of probability measures to the suitable class of functions, corresponding to lebesgue stieltje's measures.

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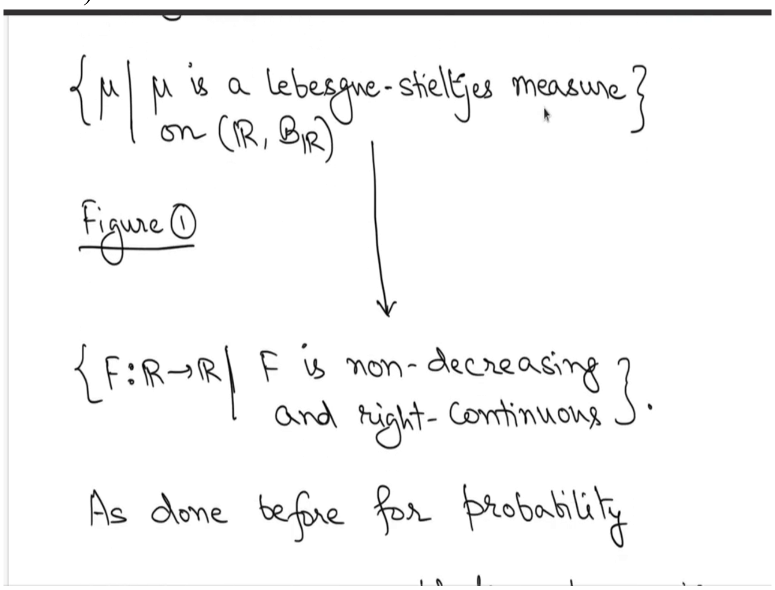

So, what is this suitable class of functions? So, as discussed above in note 13 we are taking any Lebesgue Stieltje's measure and looking at $F_{\mu,\alpha}$ and that is going to give us this class of functions which are, which are non-decreasing and right continuous. So, given any Lebesgue Stieltje's measure, you will construct this class of non-decreasing and right continuous functions.

So, to do that all you have to do is that basically fix the value of the parameter α and then you immediately get the function purely depending on the lebesgue stieltje's measure. So, corresponding to every lebesgue stieltje's measure, you then get a function non-decreasing and right continuous.

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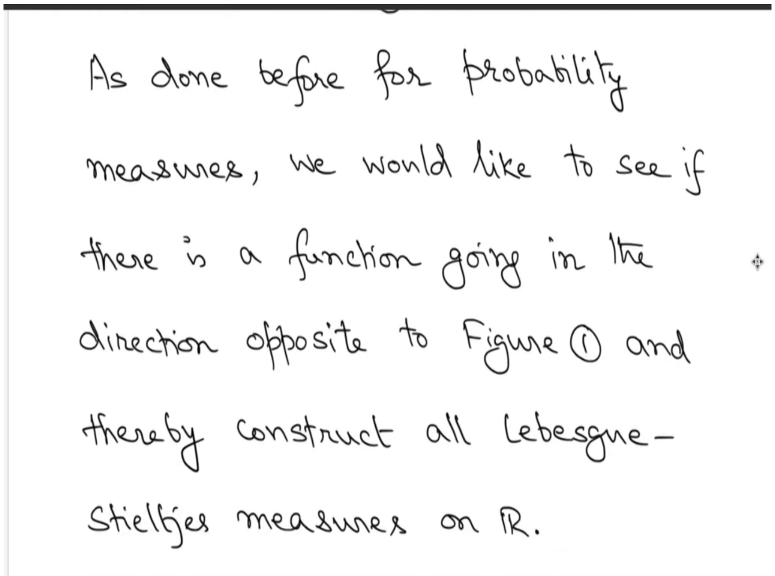

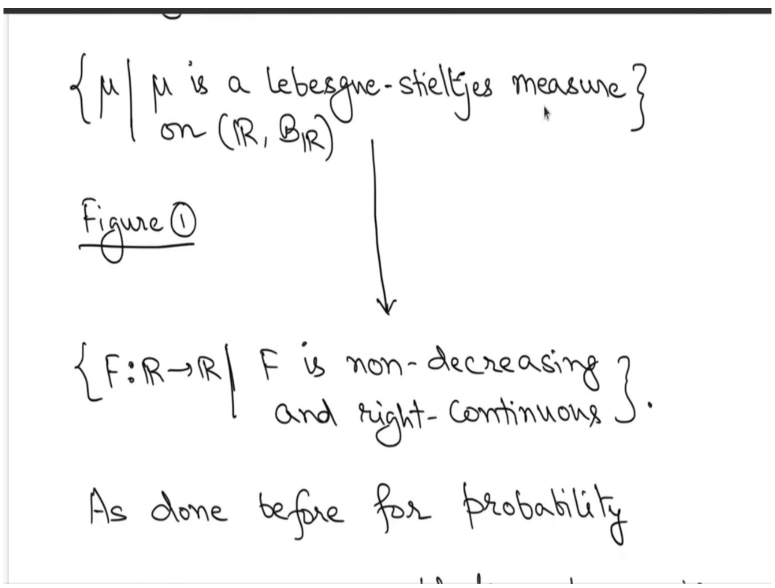

So, as done before for probability measures, we are now going to ask this question. That can we go back? So, given a non-decreasing and right continuous function, can we construct a lebesgue stieltje's measure, such that that will be the function corresponding to that measure. So, that is the identification that we are now after.

So, again our motivation is simply following the construction as done in the case of probability measures and the corresponding class of distribution functions. So, we are simply following that idea. We would like to extend that correspondence to Lebesgue Stieltje's measures and this class of functions. So, let us try to look at this class of functions and consider other properties of this class of functions.

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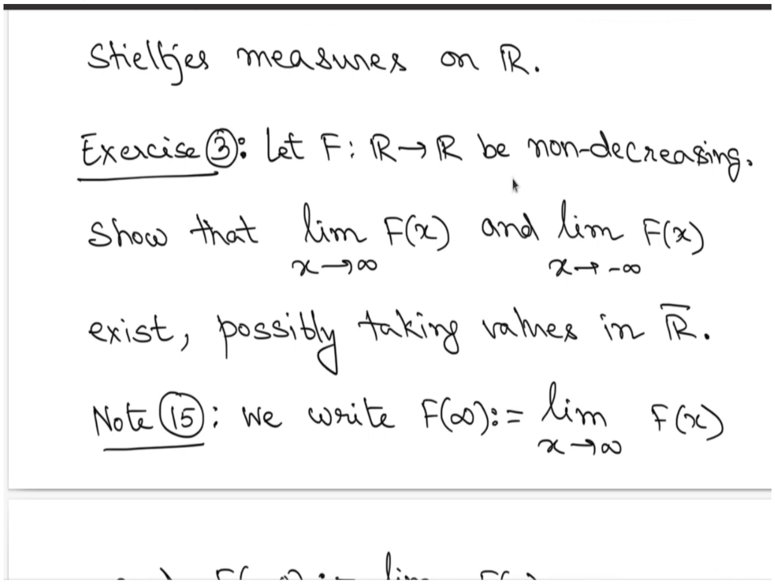

So, here is an interesting exercise. So, take our non-decreasing function, you do not need it to be right continuous. You can now try to show, that limits at ∞ and $-\infty$ will exist. Of course, they exist in the sense that these could be taking values in real numbers and also could take values in $\pm\infty$. So, essentially these limits whatever they are, they can be ∞ or $-\infty$ or some real number. So, basically that is why we are writing it is that these limits exist but possibly taking values in the extended real line. So, take any non-decreasing function, then you get the limits. So, this limit could be now ∞. So, or $-\infty$, so, be careful with this case.

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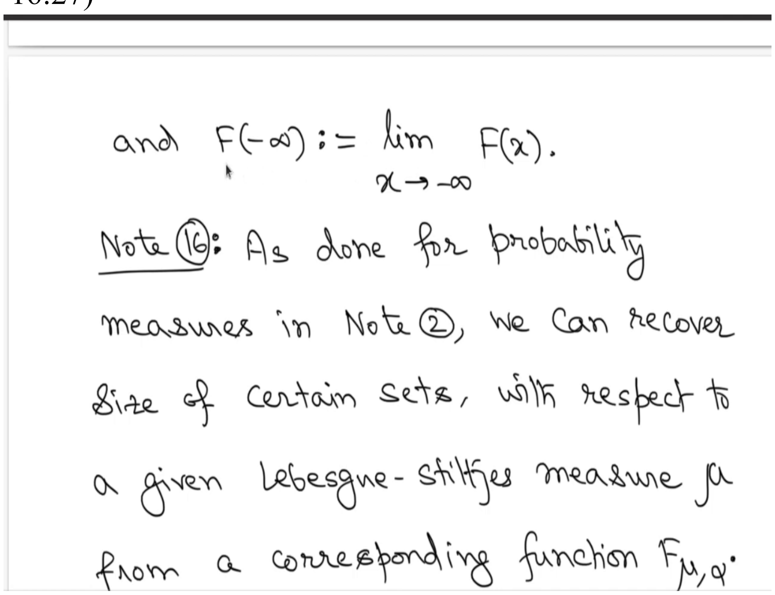

For any fixed $\alpha \in \mathbb{R}$, we have

$$F_{\mu,\alpha}(b) - F_{\mu,\alpha}(a) = \mu((a,b]) \quad \forall a<b.$$

If $(a_1,b_1], (a_2,b_2], \ldots, (a_n,b_n]$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{n}(a_i,b_i]\right) = \sum_{i=1}^{n}\left[F_{\mu,\alpha}(b_i) - F_{\mu,\alpha}(a_i)\right]$$

Furthermore,

$$\mu(\mathbb{R}) = \lim_{n\to\infty}\mu((-n,n]) = \lim_{n\to\infty}\left[F_{\mu,\alpha}(n) - F_{\mu,\alpha}(-n)\right]$$

Show that $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$ exist, possibly taking values in $\overline{\mathbb{R}}$.

Note (15): We write $F(\infty) := \lim_{x\to\infty} F(x)$

and $F(-\infty) := \lim_{x\to-\infty} F(x)$.

Note (16): As done for probability measures in Note (2), we can recover

But then, as done for the case of distribution functions corresponding probability measures, so, we follow the same description. So, limit at ∞ exists so, it could be something infinite. But whatever that quantity is we assign that value to $F(\infty)$. So, that means that, we say that the value associated to the point ∞ under the function F is that limit value. Similarly, consider of the limit at $-\infty$ and assign it like this.

So, as done for probability measures, we are doing this same argument. And therefore, you can now think of this function F, this non-decreasing function that you can think of it as a function from the extended real line, defined on the external real line and taking values in the extended real line. So, we will clarify this concept in a minute. But as done for probability measures, earlier in note two, we can now recover values of, sorry, sizes of certain sets under the corresponding measure.

And we can write down that values or the sizes of the sets in terms of the corresponding distribution function. So, this is what we had seen for probability measures. But we want to do the same thing for lebesgue stieltje's measures. But whatever you want to do, we want to write it in terms of the corresponding functions that we have just seen. So, given the lebesgue stieltje's measure μ, you look at the corresponding function $F_{\mu,\alpha}$.

And what you do is that, you look at this increment. As we have already discussed, this increment simply gives you the size of the set $(a, b]$. But then you can extend this observation, you can extend this observation by saying that since μ is finitely additive, it is a measure. So, you can in particular to finance the additive. So, therefore, if you are given this pairwise disjoint, this left open right closed intervals. Then the value associated to or the size associated to this finite digit union will be simply the summation of these increments of the function values.

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$$\mu\left(\bigcup_{i=1} (a_i, b_i]\right) = \sum_{i=1} \left[F_{\mu,\alpha}(b_i) - F_{\mu,\alpha}(a_i)\right]$$

Furthermore,

$$\mu(\mathbb{R}) = \lim_{n\to\infty} \mu((-n,n]) = \lim_{n\to\infty} \left[F_{\mu,\alpha}(n) - F_{\mu,\alpha}(-n)\right]$$

$$= \lim_{x\to\infty} F_{\mu,\alpha}(x) - \lim_{x\to-\infty} F_{\mu,\alpha}(x).$$

$$= F_{\mu,\alpha}(\infty) - F_{\mu,\alpha}(-\infty).$$

Note (17): If μ is a Lebesgue-Stieltjes

If $(a_1, b_1], (a_2, b_2], \ldots, (a_n, b_n]$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) = \sum_{i=1}^{n} \left[F_{\mu,\alpha}(b_i) - F_{\mu,\alpha}(a_i)\right]$$

Furthermore,

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$$= \lim_{x\to\infty} F_{\mu,\alpha}(x) - \lim_{x\to-\infty} F_{\mu,\alpha}(x).$$

And more generally, you can also get back the size of the whole real line, under the measure μ by computing these limits. So, again we are simply following the conclusions as done for the case of probability measures. So, you look at these intervals of $(-n, n]$. So, these sets increase and increase to the whole real line. So, therefore, you can use continuity from below and get this quantities.

That, this is the difference of the function values or the increment of the function values. But then you have just said that these are non-decreasing functions. So, therefore, limits at ∞ and $-\infty$ will exist. Therefore, this difference if you take the limits, it is exactly the difference of the two limits. So, that is nothing but as per our earlier notation, these are the values assigned to the points ∞ and $-\infty$

So, this is simply the shorthand notation for the limits, as defined above. So, therefore, the value associated to the whole real line under the measure μ can be computed as the difference of the function values at ∞ and $-\infty$. This is a very important observation and all of these results, that we are writing down here corresponding to the size of sets, which we are writing in terms of the corresponding functions is simply extending whatever we did for the case of probability measures.

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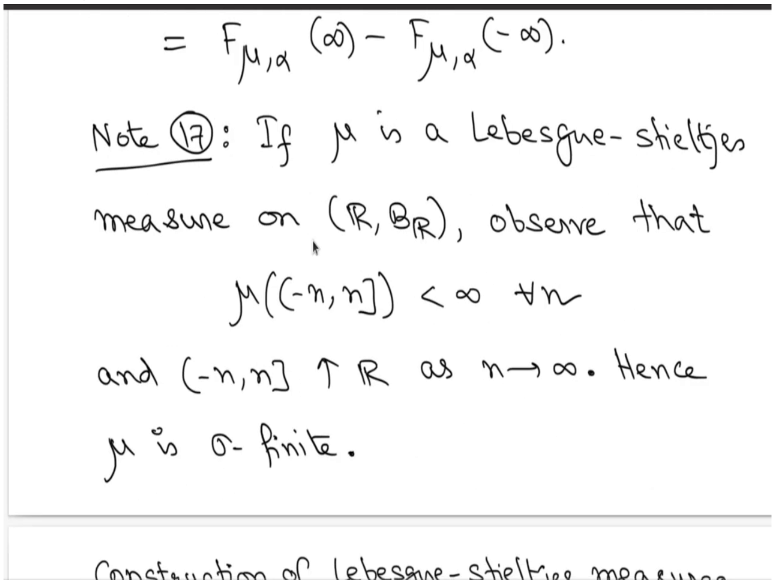

So, with that at hand, we would like to go back, if we start with this class of non-decreasing functions, which are right continuous, we will have to go back and construct the corresponding lebesgue stieltje's measure. But here, before starting the construction, we now want to go back to a important clarification. What is this? That start with a lebesgue stieltje's measure, as per the definition the size associated to any bounded interval is finite.

In particular, if you go back and look at now, this familiar interval $(-\ n,\ n]$. So, if you look at such a measure, then what you happen to notice is that this size is finite as for the definition of lebesgue stieltje's measure. And you have the fact that the sets, $(-\ n,\ n]$, these sets increase and increase to the whole real line.

Because this sets increase in the union if you compute, then it will turn out to be the holy real line. Because you are simply looking at sets within the real line. So, you will only get back the whole real line as you take the unions. So, what you get at the end is that each of these sets has finite mass. And these sets increase to the whole real line. Therefore, any Lebesgue Stieltje's measure turns out to be a σ- finite measure.

So, this is exactly falling in the set of the Caratheodory's extension theorem, that was applied as the last step in the construction of probability measures corresponding to the (())(21:10) functions. So, this is a very important clarification and we are going to use this in our construction, when we want to start the construction of a Lebesgue Stieltje's measure corresponding to a given non decreasing and continuous function.

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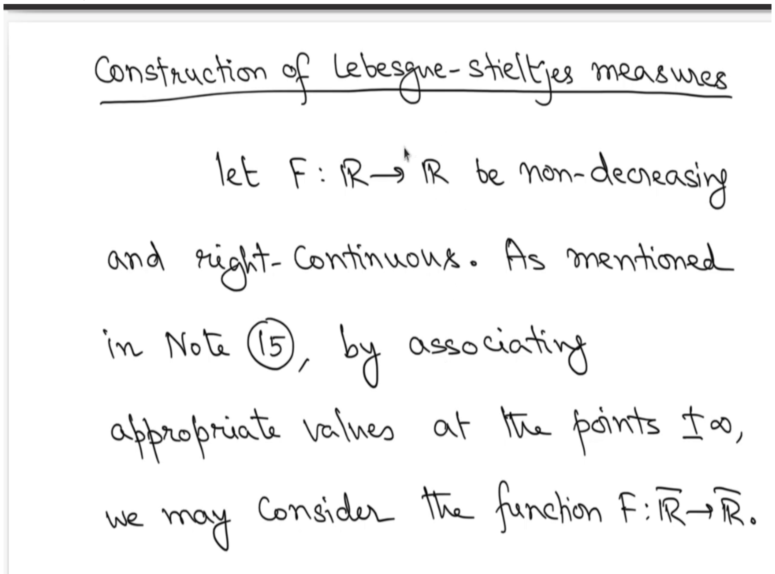

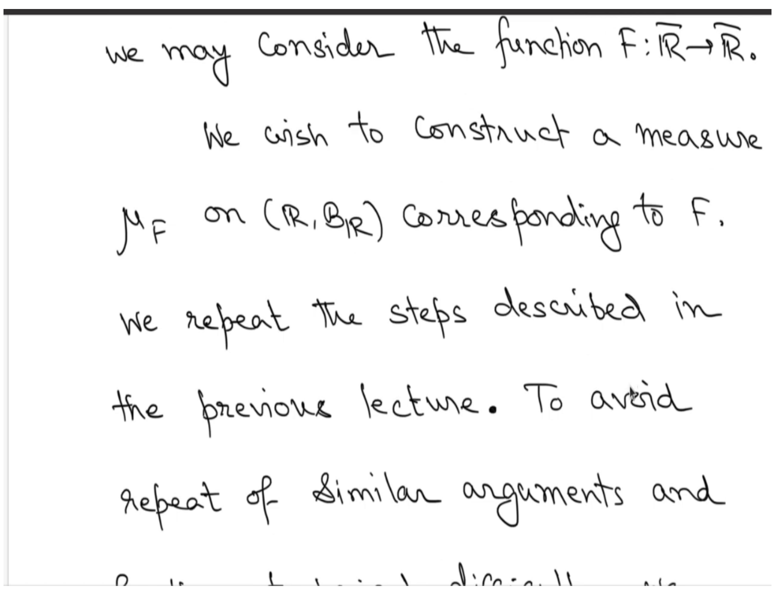

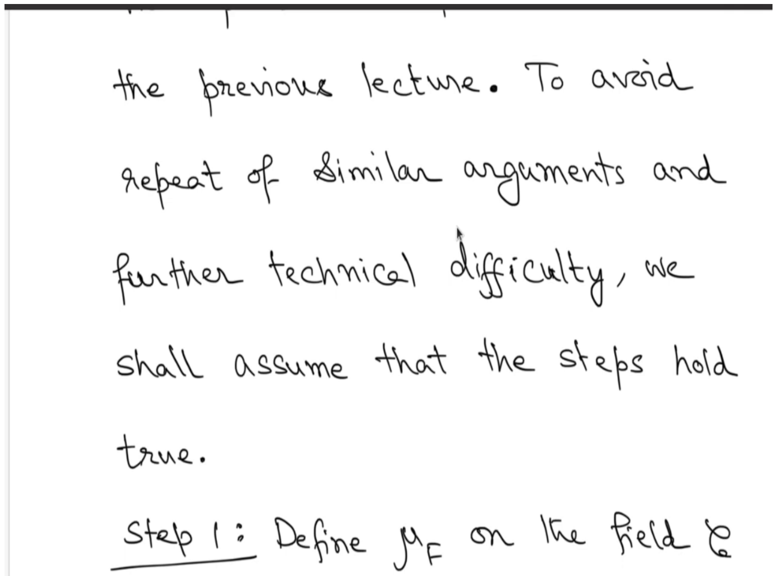

So, we are going to use this fact, great. So, now, what do you do? You start with a non-decreasing right continuous function defined on the real line and taking values in the real line. So, this is a real valued function defined on the real line. But, as mentioned above these non-decreasing functions will have limits. And you can associate the, these values of the limits at this points, this $\pm\infty$.

So, you can see that $F(\infty)$ is the limit at ∞, $F(-\infty)$ is nothing but the limit at $-\infty$. So, therefore, you can as well consider the given function F from the extended real line to the extended real line. So, the values at ∞ and $-\infty$ could now be taking ∞ or $-\infty$ as their values.

So, with that at hand now, we wish to construct a measure. And as done for the case of probability measures, we will again denote the set function by μ_F just to denote that it depends on the given function F. So, again what do you need to do? We just follow the same steps as described for the case of probability measures. And we want to avoid a repeat of similar arguments.

But in this case, in the general case, there are certainly more technical difficulties that will be arising and what we do is that we take that these steps will hold true. So, we will assume that the steps are true, many of the steps are very easy to verify whatever you have done for probability measure the same arguments we will go through. But one difficult step will be the case where you want to verify the continuity properties, of the set functions. As we see, we will mention them. But many of the steps that you have already seen, the same arguments will immediately give you the results and we are just following the same steps. So, we avoid the complete details, write down the main steps.

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Step 1: Define μ_F on the field $\mathcal{C}$ finite disjoint unions of left-open right-closed intervals as follows:

$$\mu_F(\phi) := 0,$$

$$\mu_F(\mathbb{R}) := \lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x),$$

$$\mu_F((a, b]) := F(b) - F(a) \quad \forall a < b$$

$$\mu_F(\mathbb{R}) := \lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x),$$

$$\mu_F((a, b]) := F(b) - F(a) \quad \forall a < b$$

$$\mu_F\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) := \sum_{i=1}^{n} \left[F(b_i) - F(a_i)\right],$$

if $(a_i, b_i]$'s are pairwise disjoint.

Step 2: μ_F is non-negative and

So, the first step is that, we want to take this set function defined on the field of finite disjoint unions of left open right closed intervals. Here is our familiar field once more, $\mathcal{C}$ we want to define the set function first on the field $\mathcal{C}$. So, what do we do? You first start with the measure of the empty set, which you assign the value 0. Then look at the difference in the limits. So, this is nothing but $F(\infty) - F(-\infty)$.

So, look at that quantity and assign that value as the size of the real line. But then, we are now familiar with how to give the size for this left open right closed interval, this is simply the increment of the function handles down. And again, given the idea at hand, we want to describe measure at the end.

So, therefore, μ_F whatever it is, it should be finitely additive. So, therefore, for finite disjoint unions of such left open right closed intervals, you will look at the submission values at the submissions of the increments of the function levels, great. So, you have now managed to define the set function on the fields C of finite disjoint union of left open right closed intervals on the real line.

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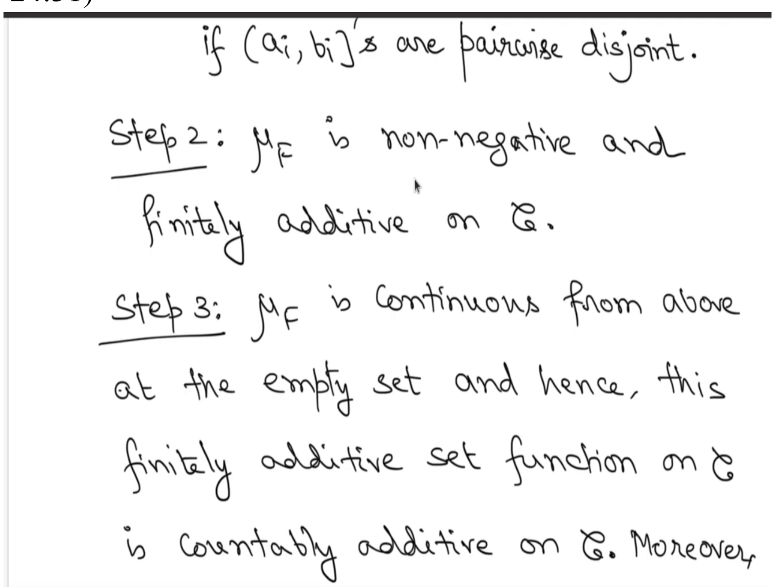
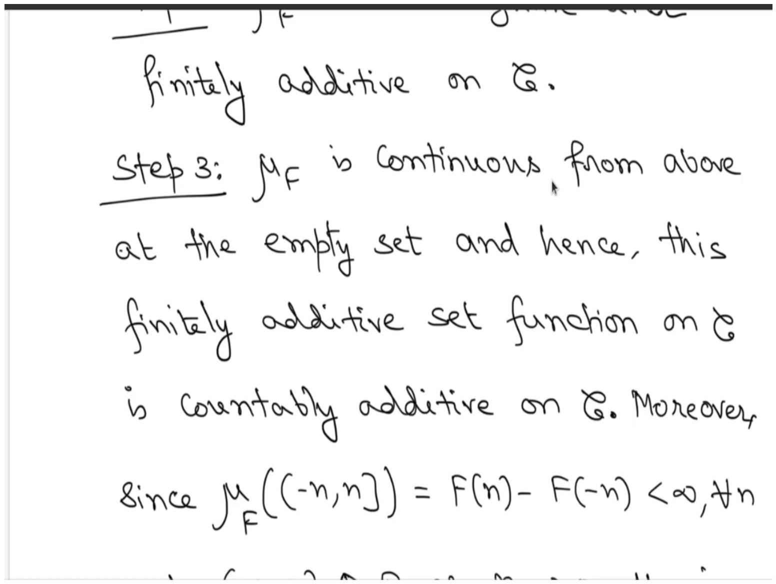
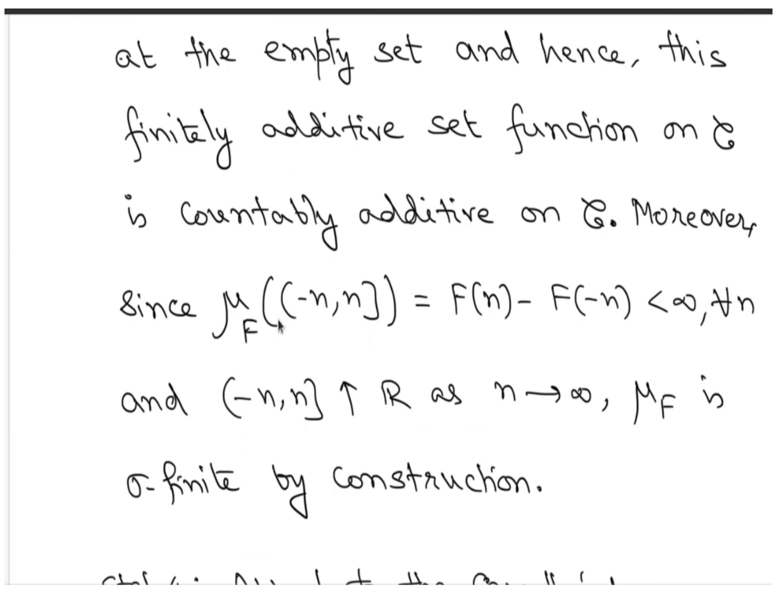




finitely additive on $\mathcal{C}$.

Step 3: μ_F is continuous from above at the empty set and hence, this finitely additive set function on $\mathcal{C}$ is countably additive on $\mathcal{C}$. Moreover, since $\mu_F((-n,n]) = F(n) - F(-n) < \infty, \forall n$



at the empty set and hence, this finitely additive set function on $\mathcal{C}$ is countably additive on $\mathcal{C}$. Moreover, since $\mu_F((-n,n]) = F(n) - F(-n) < \infty, \forall n$
and $(-n,n] \uparrow \mathbb{R}$ as $n \to \infty$, μ_F is σ-finite by construction.

Once you have that, the next steps are no pretty clear. You will now claim that μ_F is non-negative. So again, you have already started off with a non-decreasing function. So, whatever increments of the function values that you consider are non-negative. So, therefore, the values associated with all these sets that we have already considered in the field, they will get associated non-negative values.

So, therefore, the set function is non-negative. Moreover, by the structure as defined above, for the set function μ_F, it is easy to see again as in the case of probability measures that this set function that we have just defined becomes finitely additive on the field. So, here comes that technical step which we are avoiding now, is that you can again verify that μ_F as defined above is continuous from above at the empty set. So, this is very important, we again verify the same condition that μ_F is continuous from above at the empty set.

And hence, these finitely additive set function that you look at here on the field is countably additive on the field. So, you have this continuity property put it together with the finitely additivity, you will end up with the fact that this set function is countaby additive on the field. Moreover, since, you already have observed that the size of the sets, this left open right close intervals is nothing but the increment of the function values.

This quantity is now finite because original function is defined on the real line taking real numbers as their values. So, in particular, the values associated to the points n and $-n$ are also finite, finite real numbers. So, therefore, their difference is also finite. So, therefore, the size of these sets are finite. And this, this sets as observed above, these increase to the whole real line. So, this μ_F which you have now constructed, this is σ- finite by construction. This is very very important. So, you now have a non-negative set function on the field $\mathcal{C}$, which is countably additive therefore, this set function is a measure on the field $\mathcal{C}$. Moreover, this is σ- finite.

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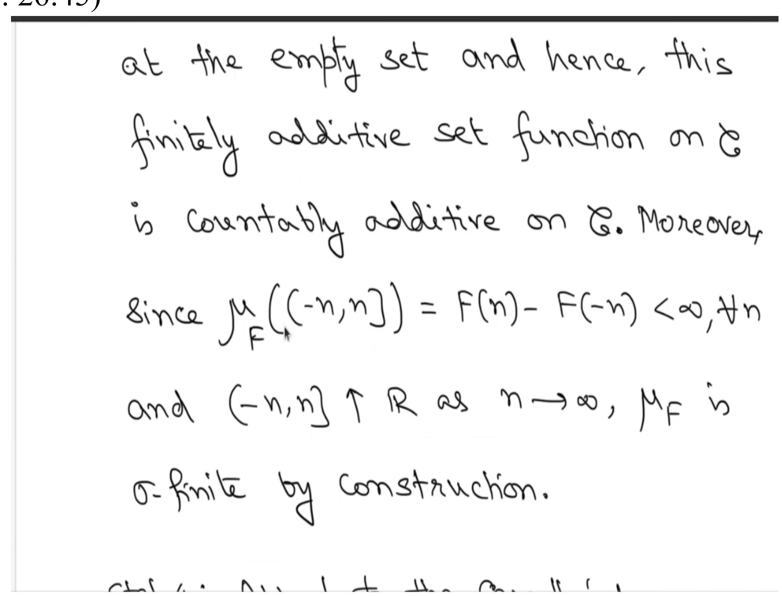

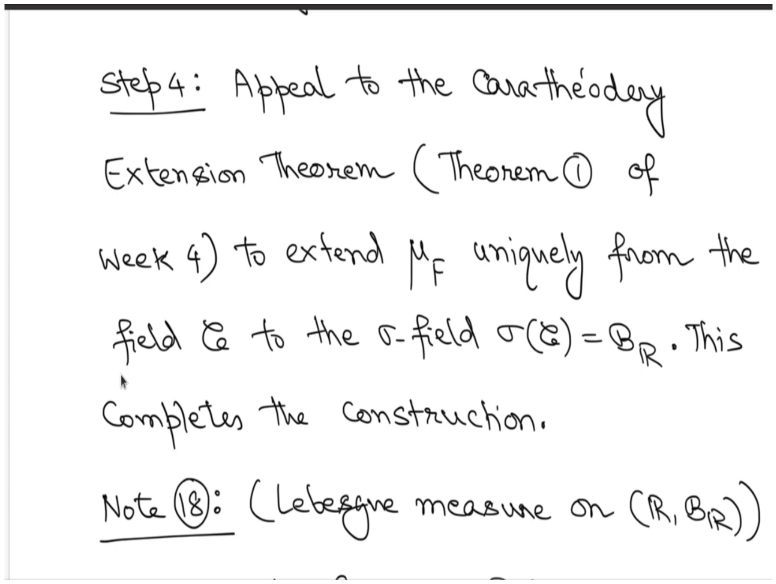

Once you have all these quantities, you are now ready to appeal to the Caratheodory's extension theorem. And what do you get? You can now extend this set function that you have taken on the field $\mathcal{C}$, you can extend it to the σ- field generated by the field which is nothing but the borel σ-field on the real line. So, therefore, you can extend this measure that you have obtained on the field $\mathcal{C}$, which is a σ- finite measure.

You can extend this to a σ- finite measure on the real line, together with the Borel σ- field. So, on that measurable space, you can extend this measure. So, therefore, corresponding to this given non-decreasing and right continuous function, you have managed to construct this measure. And as done earlier for probability measures, if you now try to look at that corresponding function $F_{\mu, \alpha}$ for these measures, you can now try to connect it with the given function F.

So, that connection I will leave it to you, please try to check this. But what we are most interested in, are examples of infinite measures that fall under this collection. Of course, by this construction, you can also construct finite measures, but these are essentially scaling of probability measures. Given a finite measure, if you divide the measure by the scalar, which is the size of the whole real line, you get back a probability measure.

So, that identification already implies that given an extension of a probability measure you should be able to construct extensions of finite measures. So, this construction should go through. But then the idea is this, that this construction is not really restricted to the class of finite measures, this will also include these collections of Lebesgue Stieltje's measures, which are in particular σ- finite.

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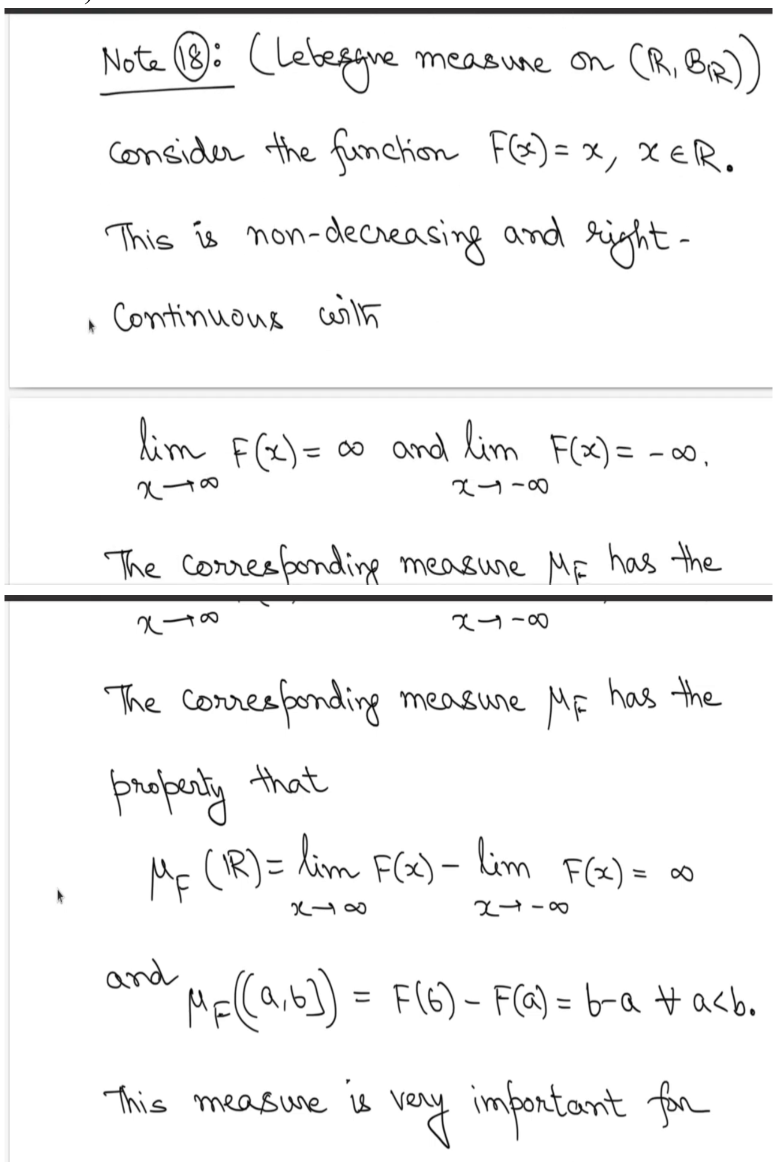

So, as a special case of this we are going to construct a very important infinite measure on the real line. This is called the lebesgue measure on this measurable space real line together with the borel σ- field. So, what is this? So, consider this function $F(x) = x$. So, this is the identity function. So, this is non-decreasing and right continuous, compute the limits at ∞ and $-\infty$.

So, for this function it turns out to be ∞ and $-$ ∞ respectively. So, for if x equals to x you can immediately compute this limit values. But once you have this, you can now appeal to the construction above and look at the corresponding measure. So, what is the corresponding measure appearing here? So, again you have the first observation that, what is the measure associated with the whole real line?

That is there is a difference in the function values, and difference of the function values which are limits at ∞ and $-\infty$. So, limit at ∞ is ∞, you subtract $-\infty$ from that. So, you are just adding $\infty + \infty$ which is nothing but ∞ So, here what you are ending up with is $\infty - (-\infty)$, which is $\infty + \infty$ which is ∞. So, therefore, the measure which you have just constructed is an infinite measure. By construction, of course, it is a σ- finite measure and it is also a lebesgue stieltje's measure.

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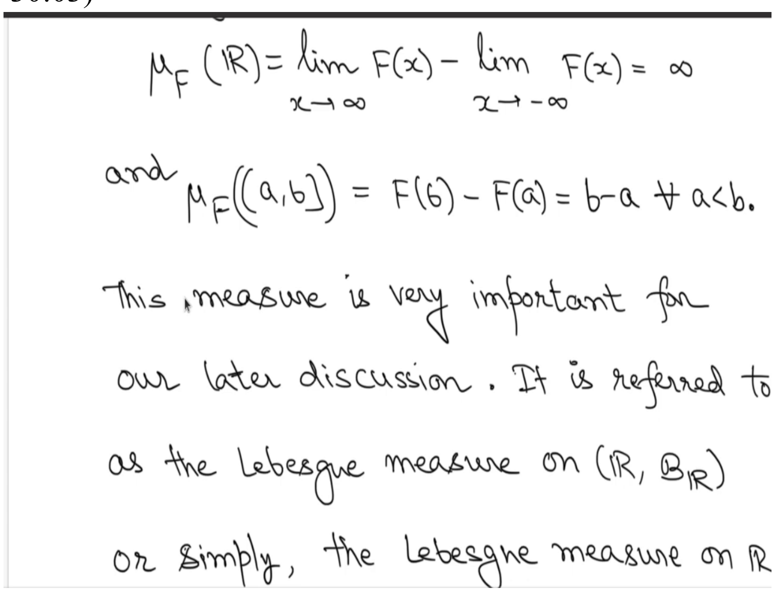

But, let us look at what happens to special subsets like left open right closed intervals. So, that is as per the description above this is the increment of the function values at the endpoints, but that is nothing but $b - a$, if you take the interval $(a, b]$. So, basically, what you end up with is that you get back the length of the interval $(a, b]$. So, this measure that we have just now constructed is very important for our later discussion. You will see this measure useful for the description of absolutely continuous random variables and that description or that discussion will do in week (())(30:41). But this measure will be very important for us and in the next lecture, we are going to focus on the properties of this measure.

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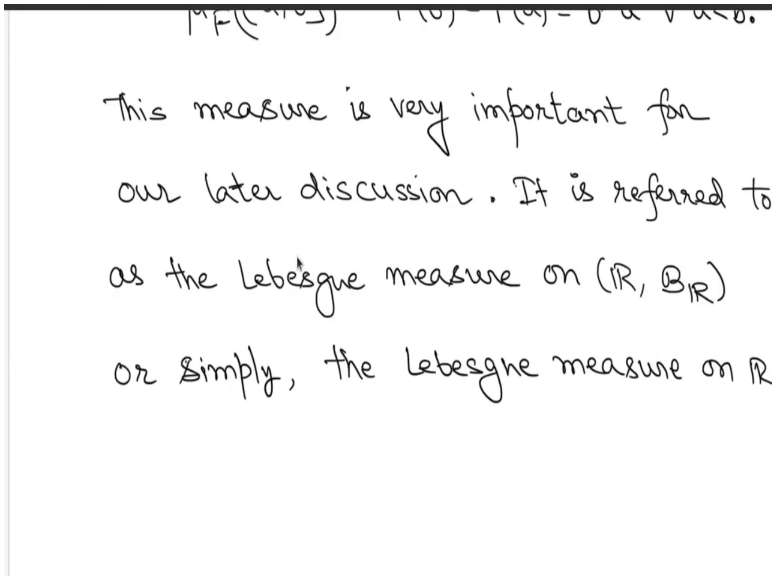

We are going to refer to this measure as a lebesgue measure on the, this measurable space real line together with the borel σ- field. But this measure is very important. So, just to simplify the terminology, we will refer to it as a lebesgue measure on the real line. So, lebesgue measure on the real line is an infinite measure. It is a σ- finite measure. It is a (()(31:08) finite measure, which associates length of intervals for the case of left open right closed intervals. But then we would like to look at further properties of this lebesgue measure in the next lecture. So, we will continue this discussion in the next lecture. So, we stop here.