Measure Theoretic Probability 1

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Lecture - 14

Limiting Behavior of Measurable Functions

Welcome to this lecture. In this lecture we are going to talk about Limiting Behaviors of Measurable Functions. So, in the previous lectures of this week we have introduced the concepts of measurable functions and Borel measurable functions. And we have also seen many algebraic properties which were also discussed specifically in the previous lecture.

So, using such properties we can construct many examples from existing examples of constant functions, indicator functions, and continuous functions. So, with that at hand we are now going towards looking at the limiting behaviors of measurable functions.

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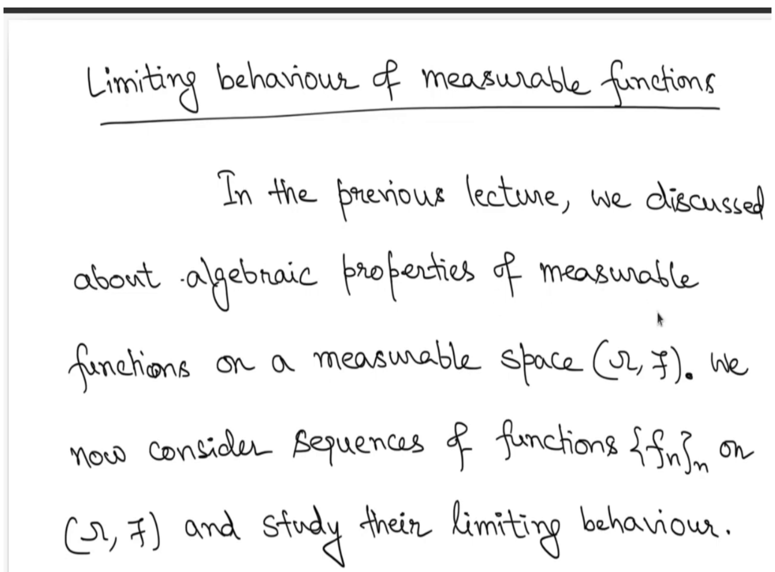

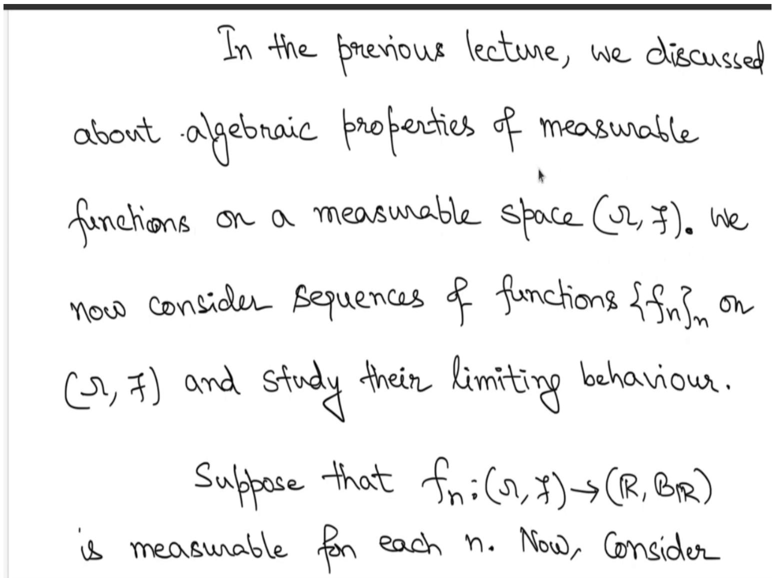

So, as usual, I will move on to the slides. So, when we have already discussed the algebraic properties. We have fixed these measurable functions on one specific measurable space. So, again we follow the same procedure we are going to fix the measurable space right at the beginning. We will be looking at real valued or extended real valued measurable functions Borel measurable functions and we will like to figure out certain limiting behavior for such functions.

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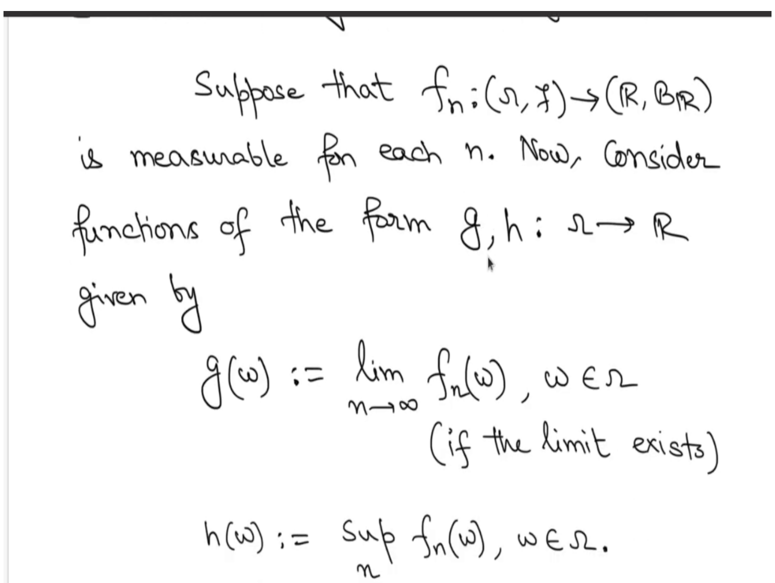

So, the setup is that we will be considering a sequence of such functions which we write it as f_n. And we want to look at we want to see whether their pointwise limit makes sense and if yes does it have some nice properties. So, suppose that each f_n is real valued and Borel measurable and then consider functions of these forms g and h.

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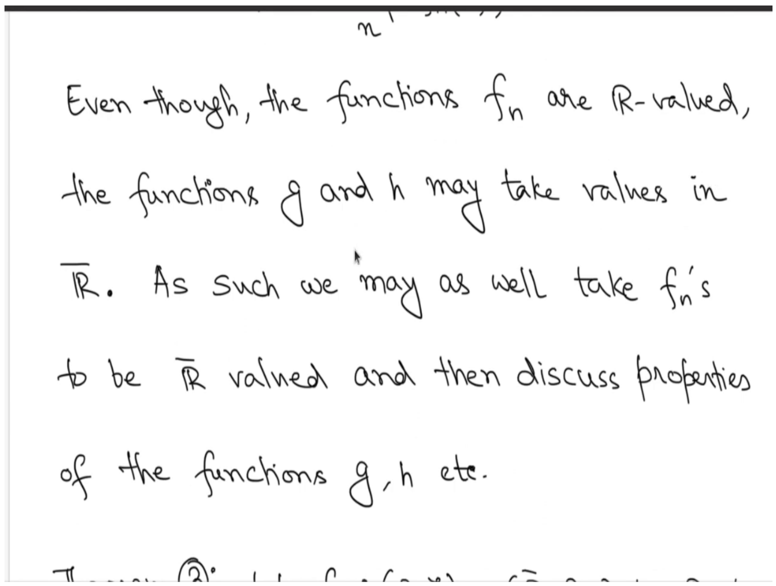

Which are given us pointwise limit. So, g is given as pointwise limit. So, for each point in the domain ω in the domain you look at the point wise limit. So, this is a sequence of real numbers and you are looking at the limit value of that and that is how the function g is getting defined. Of course, this can be defined only if the pointwise limit exists for all possible such ω.

But instead, you can also choose to look at this supremum function on each ω. So, for each sample point ω you will first evaluate the values f_n and then take the supremum. Now you have to be careful that while you are considering the supremum it might so happen that the supremum of this sequence of real numbers may be ∞. So, originally when you are trying to look at this function $h: \Omega \to \mathbb{R}$ you might have to consider the point $+\infty$ as a possible value there.

So, this you have to take care of so then even though the original functions are real valued g and h are limit point wise limits and the supremum functions. They might take values in the extended real line. So, as such for whatever discussions that we are going to perform or whatever discussions we are going to make we may as well take the functions f_n to start with to be extended real valued and then discuss the properties. And then the function g and h can automatically be thought of as functions fromΩ to the extended real line.

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Theorem ③: Let $f_n : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ be Borel measurable for all n and assume that

$\lim_{n\to\infty} f_n(\omega)$ exists in $\overline{\mathbb{R}}$ for all $\omega \in \Omega$. Then the function $g : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ defined by $g(\omega) := \lim_{n\to\infty} f_n(\omega)$, $\omega \in \Omega$ is Borel measurable.

So, with that set up at hand let us see the first result. So, it says that if you have this sequence of functions but now, we are taking extended real number valued functions. So, for such functions if you assume that the pointwise limit exists. So, this is an assumption because it needs not exist so in which case you cannot do anything there. But suppose it exists for all sample points ω.

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$g(\omega) := \lim_{n\to\infty} f_n(\omega)$, $\omega \in \Omega$ is Borel measurable.

Proof: Since the sets $(x, \infty]$, $x \in \mathbb{R}$ generate $\mathcal{B}_{\overline{\mathbb{R}}}$, using Note ⑬, we need to check if

$$g^{-1}((x,\infty]) = \{\omega \in \Omega \mid g(\omega) > x\} \in \mathcal{F} \quad \forall x \in \mathbb{R}.$$

Now,

$$\{\omega \in \Omega \mid g(\omega) > x\} = \{\omega \in \Omega \mid \lim_{n\to\infty} f_n(\omega) > x\}$$

Then you can choose to define this function g as done above in the motivation. So, you define it as pointwise limit value. So, that is how you get the function g and it turns out this result states that this also has to be Borel measurable provided each f_n that you have taken is Borel measurable. So, that is a nice property. So, how do you prove this? So the technique is now quite simple to go back to this note 13.

So, just to quickly recall what this result start suggested was that you choose a class of generating sets on the range side. So, on the range side we have the extended real line together with the Borel σ-field on the extended real line. So, here what you do? You choose a collection of generating sets there. So, for that we have chosen intervals of this type $(x, \infty]$. So, remember we are looking at the extended real line where $\pm\ \infty$ points are already there.

So, now we are looking at this type of sets my $(x, \infty]$ and these sets generate the Borel σ-field. So, now what you are interested in is to check whether the preimage of these sets will be in that domain side σ-field. If that happens then by the result mentioned in note 13 you will claim that g is measurable. So, that is basically the target.

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Now,

$$\{\omega \in \Omega \mid g(\omega) > x\} = \{\omega \in \Omega \mid \lim_{n\to\infty} f_n(\omega) > x\}$$

$$= \bigcup_{m=1}^{\infty} \{\omega \in \Omega \mid f_n(\omega) > x + \tfrac{1}{m} \text{ for all but finitely many } n\}$$

$$= \bigcup_{m=1}^{\infty} \liminf_{n\to\infty} \{\omega \in \Omega \mid f_n(\omega) > x + \tfrac{1}{m}\}$$

So, let us try to figure out what is the preimage of this interval under g. So, it means that you have to look at all points in the domain, all points ω in the domain such that $g(\omega) \in (x, \infty]$. And that can be written as this simple inequality that $g(\omega) > x$. So, that is a nice observation. So, now let us look at this set. So, all sample points ω such that $g(\omega) > x$.

So, you have fixed an x arbitrarily now and you are going to look at such sets. But then put in the definition of the function g. So, that is nothing but the pointwise limit of the f_n's let us put that in. But then if this limit value is strictly greater than x you have already assumed that the limit exists. So, that is a hypothesis in the statement. So, you have already assumed that so if this is strictly greater than x.

Then what you can say is that your actual value of the limit is something like some x plus ϵ. And then all the values of the f_n's the sequence of values that will not fall below the value $x + \epsilon$. So, it can happen for only finitely many such things. Otherwise, if it happens for an infinite such collection then you will get a limit which is smaller than x. So, you can just choose to you look at the inequality that is suggested by your limit.

So therefore, after a finite stage the values of f_n will cross the values $x + \epsilon$. So, now what you can rewrite these values ϵ as $\frac{1}{m}$ format. So, then you choose all possible such m from 1 to ∞ so all positive integers and choose this $x + \frac{1}{m}$ this kind of points. So, if it happens that $f_n(\omega)$ exceeds this value. Then of course the corresponding limit will exceed the value x.

And this is an if and only if condition. So, for finitely many n first few many n this inequality may not hold. But then afterwards after a certain stage if this holds then the limit is forced to be greater than x. So, that is the idea here. So, you have rewritten in terms of the individual f_n from the limit statement.

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$$= \bigcup_{m=1} \liminf_{n\to\infty} \{\omega \in \Omega \mid f_n(\omega) > x + \tfrac{1}{m}\}$$

$$= \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\omega \in \Omega \mid f_k(\omega) > x + \tfrac{1}{m}\}.$$

Since f_k's are Borel measurable, the sets

$$\{\omega \in \Omega \mid f_k(\omega) > x + \tfrac{1}{m}\} = f_k^{-1}\left(\left(x + \tfrac{1}{m}, \infty\right]\right) \in \mathcal{F}.$$

Hence, $g^{-1}((x,\infty]) \in \mathcal{F}$ $\forall x \in \mathbb{R}$. This completes

Now,

$$\{\omega \in \Omega \mid g(\omega) > x\} = \{\omega \in \Omega \mid \lim_{n\to\infty} f_n(\omega) > x\}$$

$$= \bigcup_{m=1}^{\infty} \{\omega \in \Omega \mid f_n(\omega) > x + \tfrac{1}{m} \text{ for all but finitely many } n\}$$

$$= \bigcup_{m=1}^{\infty} \liminf_{n\to\infty} \{\omega \in \Omega \mid f_n(\omega) > x + \tfrac{1}{m}\}$$

But then you have now written is in terms of a countable union. So, now our target is to write down or describe this set here in terms of explicitly the f_n's. So, we are going to rewrite this but then observe that for every fix m that you are looking at this set here can now be thought of as the limit infimum or certain sets what are these sets. That except few finitely many n your value of $f_n(\omega)$ that exceeds this value.

So therefore, look back in the discussions in week one. So, now you can go back to this description of limit inferior of sets and you will immediately say that okay. So, that means that this is the limit inferior of these events or this kind of sets. So, what are these sets that $f_n(\omega)$ exceeds $x + \frac{1}{m}$. Remember limit inferior simply implies that $f_n(\omega)$ will exceed this value for all but finitely many n.

So, $f_n(\omega)$ will exceed $x + \frac{1}{m}$ eventually. So, x after a finite stage all $f_n(\omega)$ will be exceeding $x + \frac{1}{m}$. And that limit inferior now you put in the definition it comes in form of this additional unions and intersections. So, put that in and that turns in terms of the individual descriptions of the f case. But now these sets that you have at the end this $f_k(\omega)$ strictly greater than $x + \frac{1}{m}$ are nothing but preimages of certain type of intervals.

So, these sets are already in the σ-field domain sides σ-field because f_k's are given to the Borel measurable. Therefore, as soon as you are allowing these countable unions and intersections in the σ-field $\mathcal{F}$ you automatically get these countable unions and intersections to be in the σ-field.

So therefore, the set that you have started off with that was originally for the function g that was the point was limit function that preimage must fall inside the domain side domain side σ- field.

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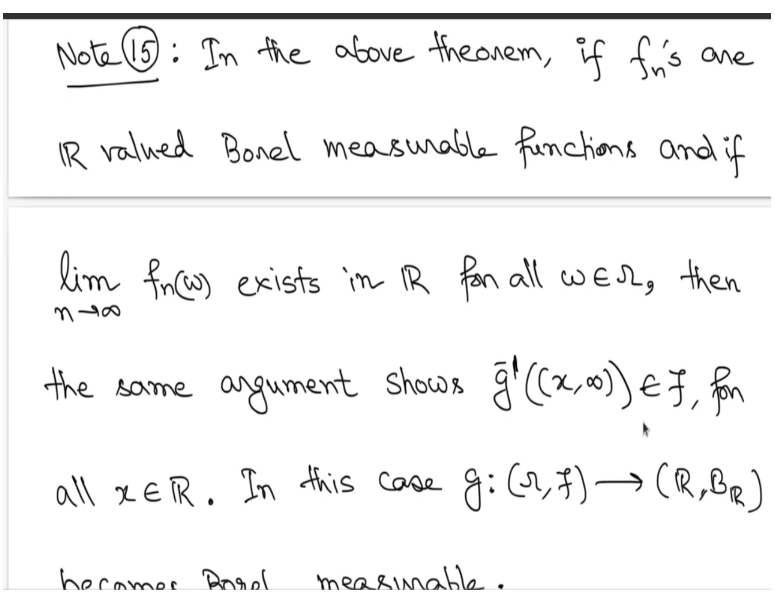

And therefore, this will imply that all the preimages for generating sets will be on the domain side σ-field and hence g must become measurable. So, pointwise limit function if it exists for all ω all points in the domain then g must be measurable. So, this is interesting result. So, in the above theorem if f_n's are real valued and if this limit exists for real line only. So, that means that if the f_n's are first of all start off with real valued functions.

And if it so happened that the pointwise limits these values for each fixed ω if this limit is a real number. So, you do not have to consider $+\infty$ or $-\infty$. Then the same argument will go through but then you will have to use this kind of preimages for (x, ∞). So, these sets are now in the real line. So, earlier it was including the ∞ so we had taken $(x, \infty]$.

But now you are taking (x, ∞). So, these sets are inside the real line. And as long as you go through this exact same structure of the arguments you will immediately be able to claim that the preimage are again on the domain side σ-field. So, there is not much difference in the arguments. So, if you can prove it for the extended real valued functions and provided the limit function, pointwise limit function is real valued the exact same argument will go through with minor modifications for the real valued case.

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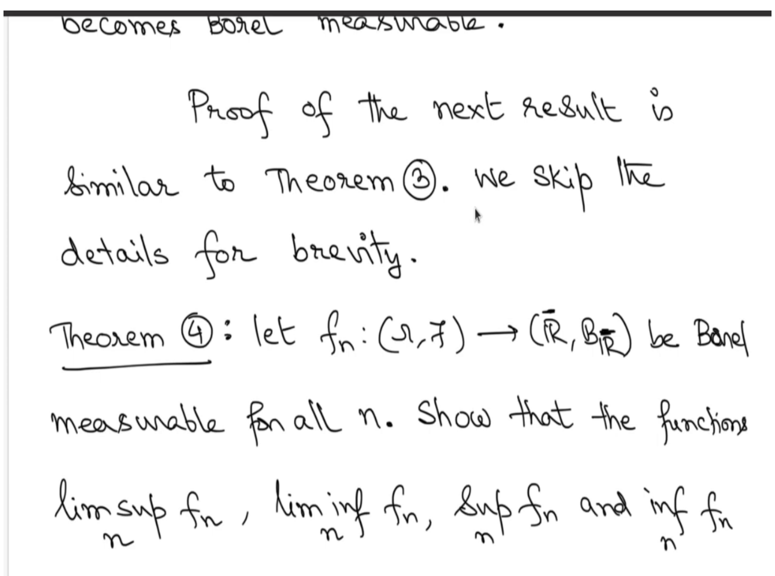

So, if you do that then you can show that g is actually Borel measurable real valued Borel measurable function. So, the limit function will be as long as the limit function is real valued you can show that it is actually real valued Borel measurable function. So, with that at hand you can actually extend these ideas. And I have put it in format of theorem 3. So, in a theorem 3 we had looked at the limit functions but then there are other such similar functions.

And the details are script for (())(12:15). So, what you do again you take this extended real valued setup. So, we will make a comment about them for the real valued case. So, first let us look at the real valued functions setup. So, given the sequence of functions look at the limit superior of the functions, limit inferior of the function, supremum of the functions and infimum of the functions. So, what are these limits superior?

So, for each sample point look at the real valued or extended real valued sequence $f_n(\omega)$. So, that is the extended real valued sequence and you can talk about the limit superior of that. Similarly, limit inferior is defined exactly in the same way you compute the pointwise limits of these functions. And as it happens that this limits superior, limit inferior supremum and infimum these functions always exists.

So, this is slightly different from the case of limit pointwise limit case. So, in the point was limit you have to assume that the sequence converges and then you are getting the values. Of course, in the convergence sense you were allowing the $\infty\ or\ -\infty$ as possible values but her, you do not have to worry about that. So, by definition supremum, infimum, limit inferior and limit superior of functions always exists. But they are no external real valued.

So, you can now show using a similar argument that as long as you take each f_n to be Borel measurable. Then d also will become Borel measurable functions. So, the modification of this as done for the limit functions case will be that as long as you assume that this limit superior limit inferior et cetera. This kind of functions are real valued then these functions are also real valued Borel measurable functions so you can just make the modification.

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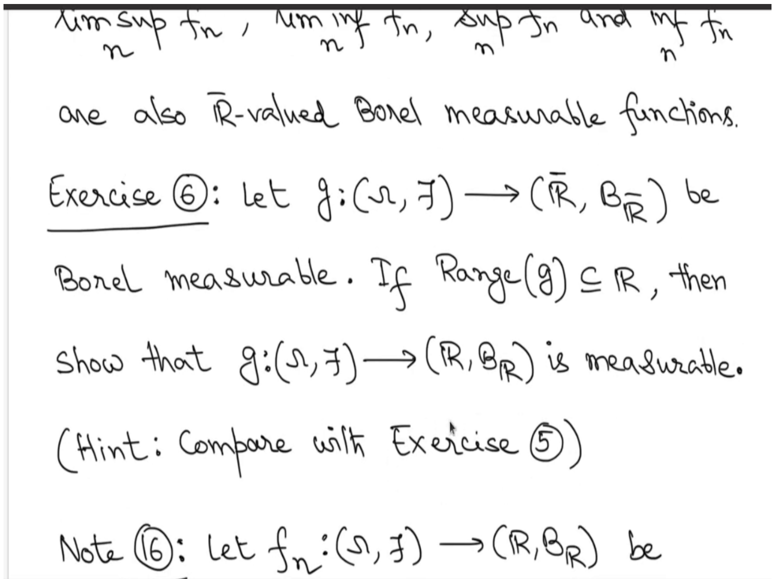

But then there was this exercise that we had mentioned in the previous lecture. So, this was exercise 5. So, there we had talked about a function g. So, we have been talking about a function g which we wanted to put in the denominator of a division operation. So, there we have taken this g which was not taking the value 0. So, this is a similar problem you go back to exercise 5 you will find the similarity.

So, what we are saying in exercise 6 here is that you will take g to be Borel measurable in that setup. But afterwards if you are given the information that range this actually within ℝ. Then can you say that g actually is Borel measurable in the as a real valued Borel measurable function and the exercise says yes it can do this. So, this was the exactly similar question that was asked in exercise 5. So, it was put in the form of a question there.

So, the answer to that question in exercise 5 is also, yes. So, the idea is this as long as you are restricting the domain. So, in the domain if there is this natural choice of Borel σ-fields you should expect that it will remain measurable. And that is what you are supposed to prove here. So, originally it was extended real valued Borel measurable function. But if you additionally give the information that the range of the function is contained within the real line. Then you will say that yes okay fine g is actually real valued Borel measurable function.

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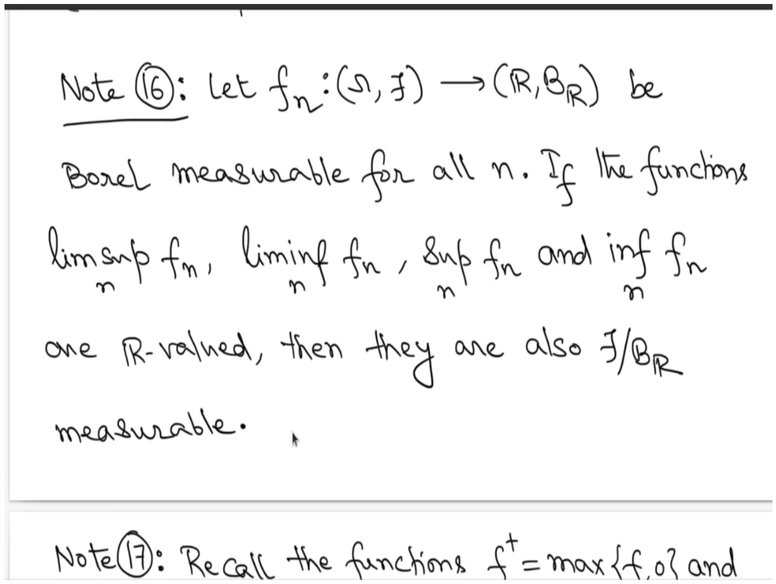

So now, this is the comment now we make. So, this is a different way of looking at the same thing that if you assume that the limit functions this $lim\, sup$ $lim\, inf$ or this supremum or infimum functions. If they are actually given to be real valued then they are also real valued Borel measurable functions. So, there are two ways of looking into it. So, one is if you are given the information before you start working.

That these functions limit superior for example is only taking real values then you directly work and try to prove the that they are real valued Borel measurable function. But if not if the information is not given beforehand what you can try to do is that you first work with extended real valued setup prove that there are extended real valued Borel measurable function. And afterwards whenever you are given this extra information that the function is actually taking values within the set of real numbers.

Then you improve upon that, you go from the extended real valued measurability to the real valued measurability question. So, there are two ways of going about it. So, only difference is when are you being given the information that the function is taking values within the set of real numbers.

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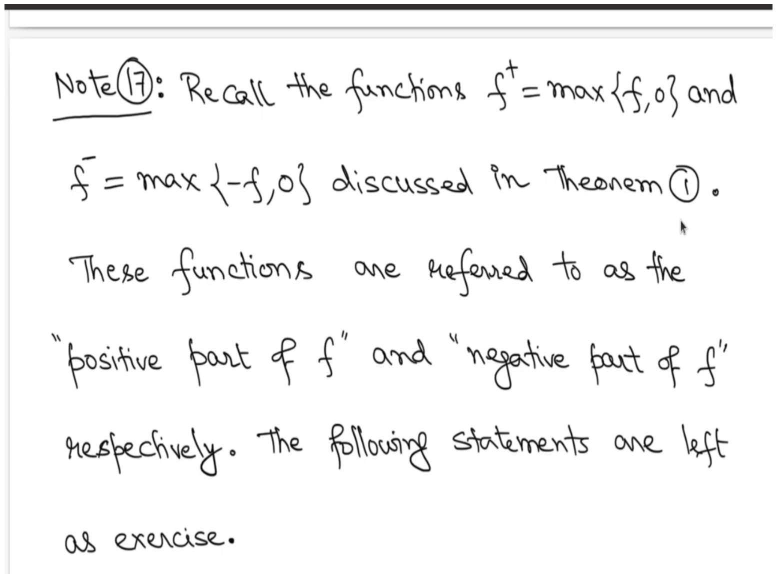

So now, we recall another two functions that were discussed in the previous lecture. So, we look at this positive part of f and negative part of f. So, again you take the maximum between f and 0. So, you just look at the values when $f(\omega)$ is strictly positive so there you will get the function value. Otherwise as long as the function f is taking negative values this maximum becomes 0.

So, it just looks at the positive part of f and that is what we call the function f^+. Similarly, function f^- is the negative part of f. So, now there are some very interesting properties of these functions f^+ and f^-. So, some of these were discussed in theorem 1 earlier that if f is measurable then so are f^+ and f^-. So, this maximum operation preserves measurability and that is how we got this result that f^+ and f^- are also Borel measurable.

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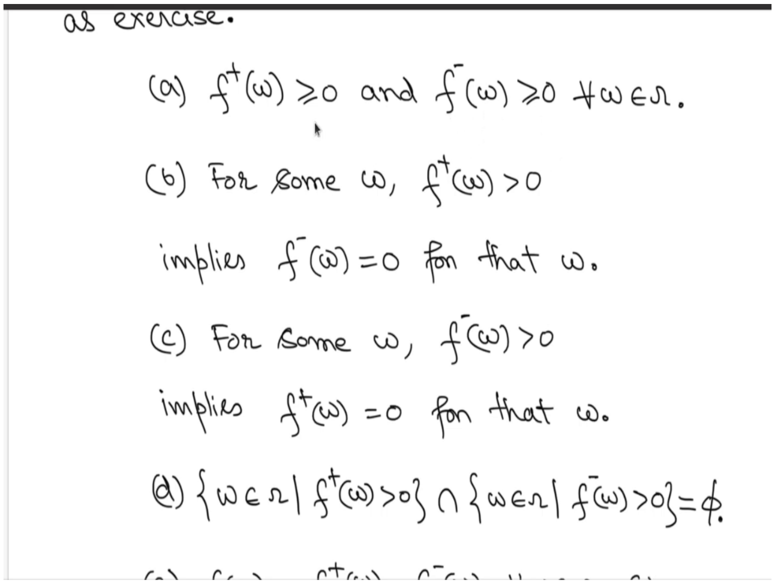

Now, you can actually show these nice properties and immediately this will follow. So, you just check the definitions you will immediately get these things. So, for example f^{+} is a max $\{f,\ 0\}$ so that maximum is always greater equals to the 0. Similarly, f^{-} is max $\{-\ f,\ 0\}$ and you will immediately get this.

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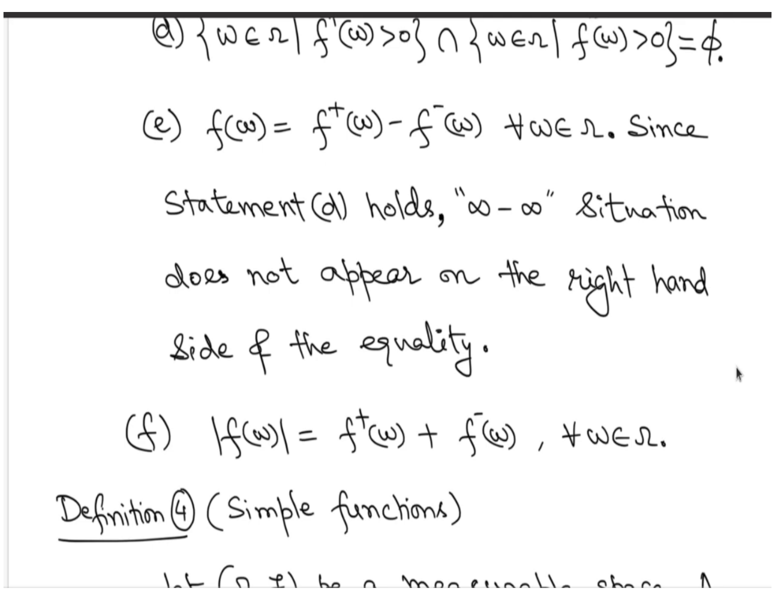

So, these are non-negative functions statement b says that if it happens that f^{+} is strictly positive. Then what it means is that the actual function is taking positive value. So, therefore, in the negative part of the function you will get 0 as the contribution. So, just check the definitions you will immediately get this and exactly the relevant statement when you exchange the role of f^{+} and f^{-} you will (())(18:23) this.

So, if the function f is taking negative values. So, if minus is giving you some positive quantity, then f^{+} the positive part of f will be 0. Again, following the same description that we have done in b and c if you put them together you can now try to prove this. That if you look at the set of points in the domain where f^{+} is positive and if you look at the set of points where f^{-} is positive, they will not have anything in common. So, the intersection is an empty set.

And then the very important identification that the function f, original function f can be written as a difference of these two things. So, here $f^{+} - f^{-}$ so original function f is written as $f^{+} - f^{-}$. So, here it is again important to note that by statement d you will not have an $\infty - \infty$ situation. So, if f^{+} becomes ∞, then f^{+} is positive then f^{-} must be 0 there.

Similarly, f^{-} becomes ∞ then f^{+} must be 0 at that point. And therefore, you will not have $\infty - \infty$ situation there. So, this is always defined. And this is a interesting observation that mod of f also can be written in terms of this positive part of f and negative part of f. This is simply the addition of these two functions please try to check this. So, these are easy verifications that are left as exercise for you.

And these all of these statements follow immediately as long as you follow the definitions of the positive part of f and negative part of f. So, with this at hand we are now going towards a very important collection of functions. So, now as mentioned earlier we have looked at three types of examples of measurable functions, constant functions indicator functions and in the restrictive fashion when you were restricting our domain to the real line or some Euclidean space, we had looked at continuous functions.

So, you have these three types of examples at hand. So, now with this putting together we want to construct more examples and with the algebraic properties at hand we can now do the following.

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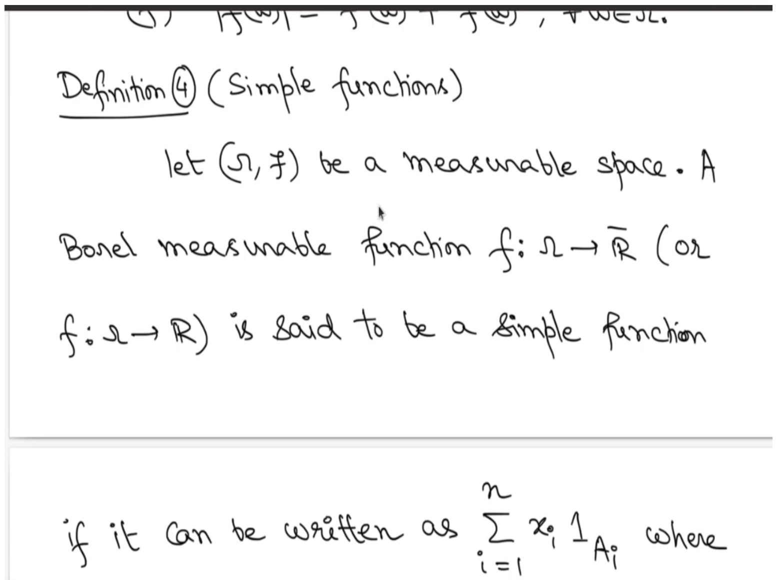

We look at something called simple functions. So, what are these? So again, we are fixing that measurable space right at the beginning and look at functions of this type Ω to $\bar{\mathbb{R}}$. So, either you look at extended real valued functions or the real valued functions in either case you say that a function is called simple function.

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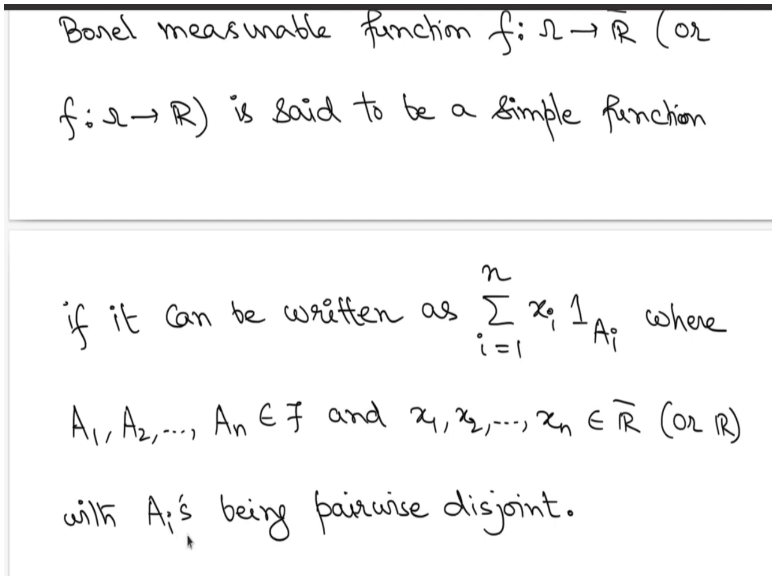

If it can be written as a linear combinations of indicators. So, here you want this finite number of sets you wanted be in they should be pairwise disjoint. They are in the domain and they are they should be pairwise disjoint. So, that is how the indicators come up. And then we want this scalers $x_1, x_2, \ldots, x_n$ extended real numbers or real numbers something then you look at this linear combination finite linear combinations of indicators.

You look at such things. So, if your function can be represented like this, then you say that it is a simple function. And it is important that this A_i's this sets that you are using they should be pairwise disjoint.

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<u>Note</u> (18): If $A, B \in \mathcal{F}$, with $A \cap B \neq \phi$, then observe that $1_A + 1_B = 1_{A\setminus B} + 2.1_{A\cap B} + 1_{B\setminus A}$, with the sets appearing on the right hand side being pairwise disjoint. More generally, given a finite linear combination of indicator functions, we

So now, you will immediately look at certain nice properties of these indicators just to understand how these simple functions (())(21:43) and we would also like to talk about this measurability questions. So, in the definition itself we have mentioned that these functions are Borel measurable but we will explain that in a minute. So, take any two sets such that their intersection is not empty. So, in the definition we wanted these intersections all these sets to be pairwise disjoint.

But if it does not happen what kind of situations might occur? So, look at any two sets A, B on the domain side look at $1_A + 1_B$. So, this is a linear combination only difference with the type of things that we have seen in the definition of a simple function is that A and B are not disjoint so let us see. But then what you see is that this addition of these two indicator functions $1_A + 1_B$ can be written as a linear combinations of certain other type of sets with sets being pairwise disjoint.

So, the sets on the right hand side are $A\setminus B$ meaning $A \cap B^c$. Then $A \cap B$ and $B\setminus A$ meaning $B \cap A^c$ all of these sets are also in the domain side σ-field. Therefore, these indicators make sense and then these indicators, using these indicators you can write down this combination of functions. You just check that this equality is fine. So, put any point in the domain and check that this equality is fine.

But what is happening at the end once you verify this equality what is happening is that the sets on the right hand side are pairwise disjoint. And therefore, the function that appears on the right hand side that is a simple function. But then originally, I did not start with sets which were pairwise disjoint. So, what does it tell you?

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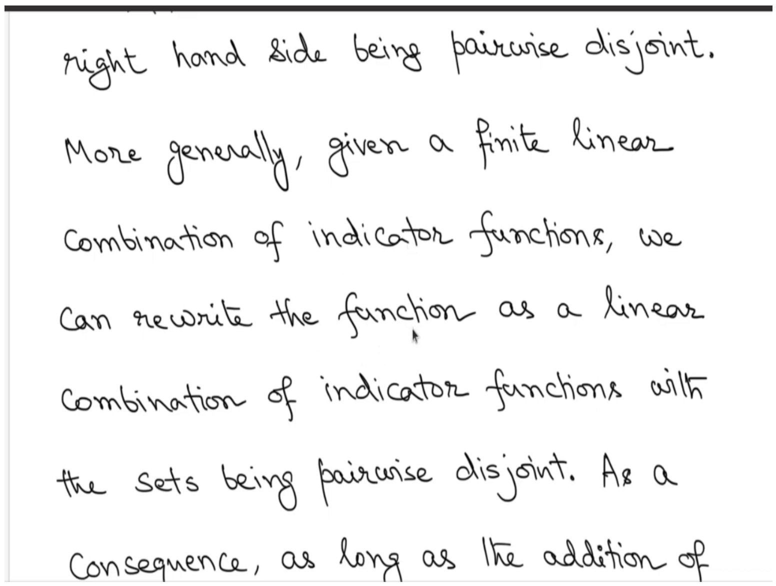

It says that given finite linear combinations of indicator functions where the sets need not be pairwise disjoint. You can rewrite this same function as a linear combinations of indicators such that the sets are pairwise disjoint.

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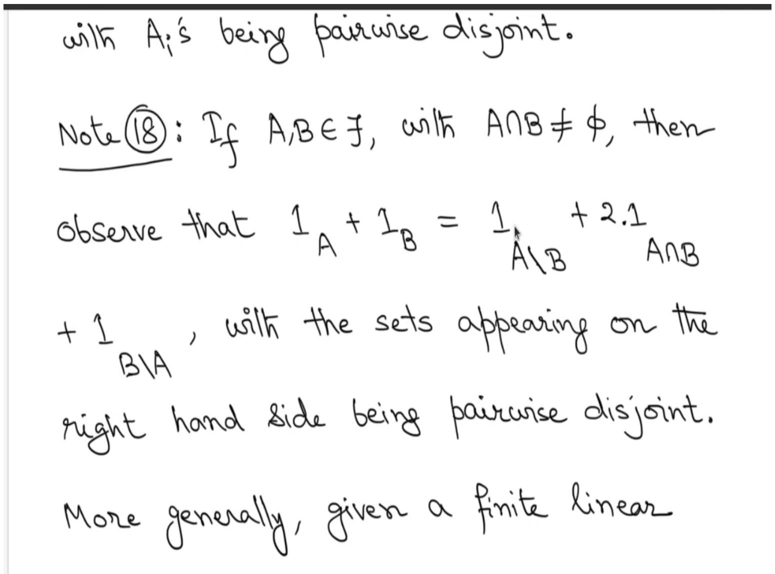

So, start with some linear combinations where the sets are not pairwise disjoint. But split them up into the appropriate intersections. So, intersections of A and A^c, B and B^c and so on. If you split them up if you start with such finitely many sets you can go to these intersections of all these many sets and including their complements.

You can write down this equality of this addition that linear combination in terms of another linear combinition where the sets appear with these intersections indicators of intersections and they turn out to be pairwise disjoint.

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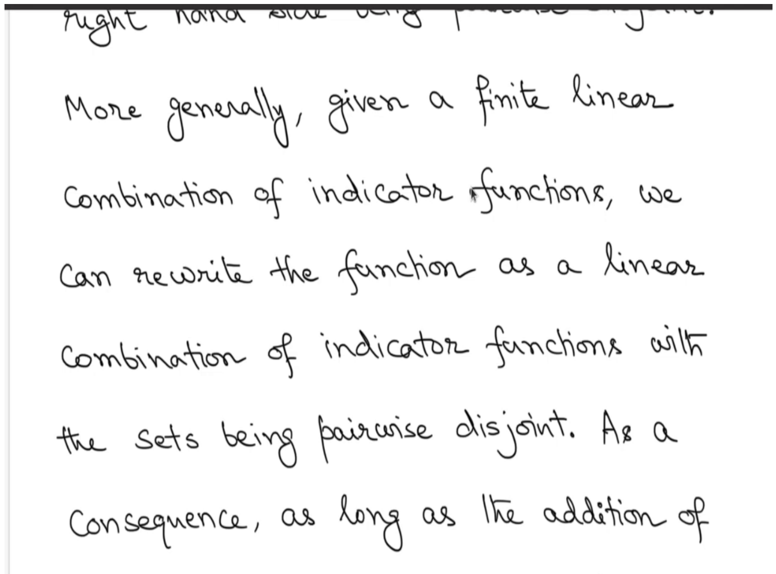

So, even if you start off with some linear combinations of indicators where the sets are not necessarily pairwise disjoint. You can rewrite it and in terms of certain other sets where they also appear as a linear combination of indicators where the sets are pairwise disjoint. So, therefore the restriction of pairwise disjointness is not that severe you can simply add up indicators and as long as the summation makes sense you can get a nice function.

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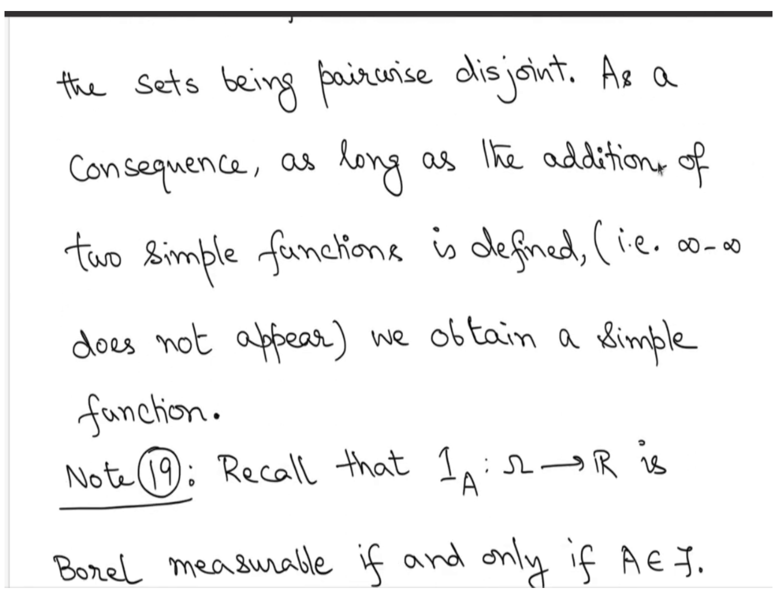

So, as long as this addition of two simple functions is defined you can rewrite things in this form. So, what do I mean by this addition making sense? $\infty - \infty$ situation should not appear. So, we want as long as the $\infty - \infty$ situation does not appear you can talk about addition of functions.

This is a new pointwise addition. So, pointwise addition will make sense as long as $\infty - \infty$ situation does not appear. So, as long as that happens you can obtain the simple functions.

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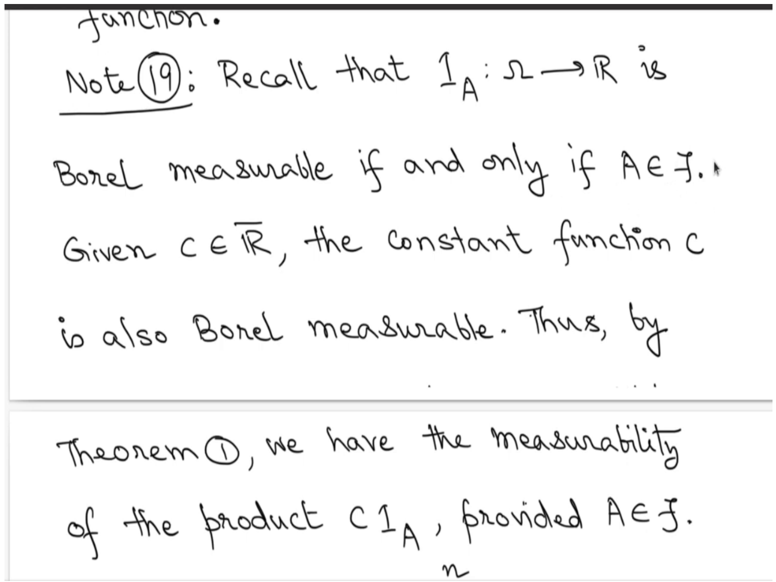

Now, another important observation now about the comment about measurability of such functions is that this 1_A will be Borel measurable if and only if the set is in the domain side σ-field.

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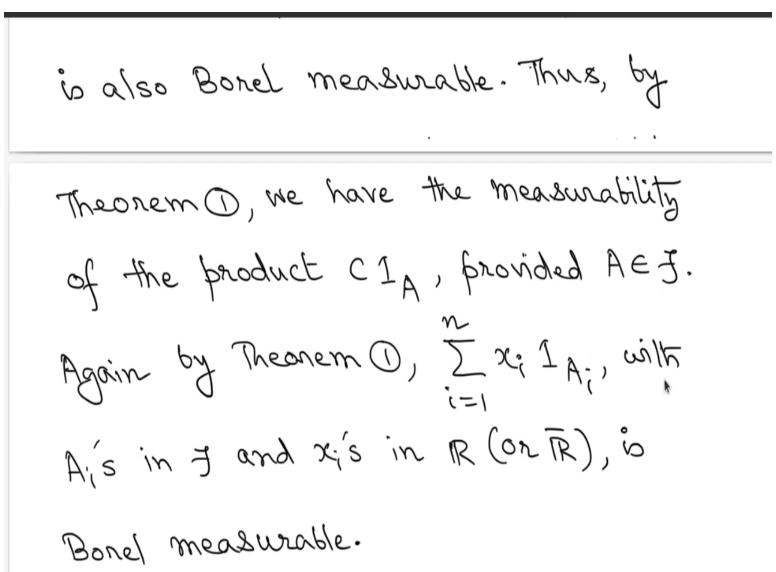

But then you recall that constant functions are also Borel measurable. And therefore, if you look at the multiplication of measurable functions it is also measurable. So, here we are multiplying the constant function C with the indicator 1_A. So, therefore again by theorem 1 linear combinations of those also will be Borel measurable addition of functions, addition of measurable functions will be Borel measurable. So, that is the idea that is being used here.

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Note 20: without loss of generality, we may assume $\bigcup_{i=1}^{n} A_i = \Omega$. If $\bigcup_{i=1}^{n} A_i \subsetneq \Omega$, then taking $A_{n+1} = \left(\bigcup_{i=1}^{n} A_i\right)^c$, we can rewrite the simple function as $\sum_{i=1}^{n} x_i 1_{A_i} + 0 \cdot 1_{A_{n+1}}$. By definition, a simple function only takes the finitely many values $x_1, x_2, \dots, x_n$.

And another interesting comment is that whenever you are looking at the simple functions you can without loss of generality assume this additional condition that the finite union of these sets that you are using in your definition of simple functions that covers the whole set. So, why? Because if you are looking at this finite union and if it is a proper subset of Ω then you can look at the complimentary part of that and think of it as the $(n + 1)$-th set.

So, originally you are given n many sets and I am saying just add one more set to your collection which is the complement of whatever the sets before was ignore all those things look at the complement. So, that will give you the $(n + 1)$-th set. So, the complement of the first n sets unions complement.

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$A_{n+1} = \left(\bigcup_{i=1}^{n} A_i\right)^c$, we can rewrite the simple function as $\sum_{i=1}^{n} x_i \mathbb{1}_{A_i} + 0 \cdot \mathbb{1}_{A_{n+1}}$. By definition, a simple function only takes the finitely many values $x_1, x_2, \ldots, x_n$.

Note (21): The values $\pm\infty$ are allowed for simple functions, provided $\sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$ makes sense.

So now, rewrite the linear combination by adding this extra set, adding this extra term in your collection. So, what is this? Originally the terms were n many so that you had n many terms, n many functions you were adding up. But then you can simply think of the term coming from the $(n+1)$-th set with the scalar 0. So, this will summation again will make sense because you are just adding 0 everywhere. So, this will be the exactly the same function but here the union of the sets will be the whole set Ω.

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Note (21): The values $\pm\infty$ are allowed for simple functions, provided $\sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$ makes sense.

However, in practice, we work with simple functions which take values in $\mathbb{R}$.

Theorem (4) (Approximation by simple functions)

Let $f : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ be Borel

So again, without loss of generality in a simple function description you can always assume that the sets you have they are pairwise disjoint and they are covered in the whole domain. So, again this is an important clarification once more. So, again simple functions as defined if you are talking about extended real valued simple functions then they are allowed to take the values $\pm\ \infty$.

As long as that summation makes sense, as long as $\infty\ -\ \infty$ situation does not appear you can define simple functions. But in practice we are going to work with simple functions which actually take values in the real line. So, you will see the reason why. So, you are going to look at finite linear combinations of indicator functions which are the simple functions those scalars that appear in the as the coefficients of the linear combination. They will be taken to be real numbers, you will see the reason why in the theorem 4.

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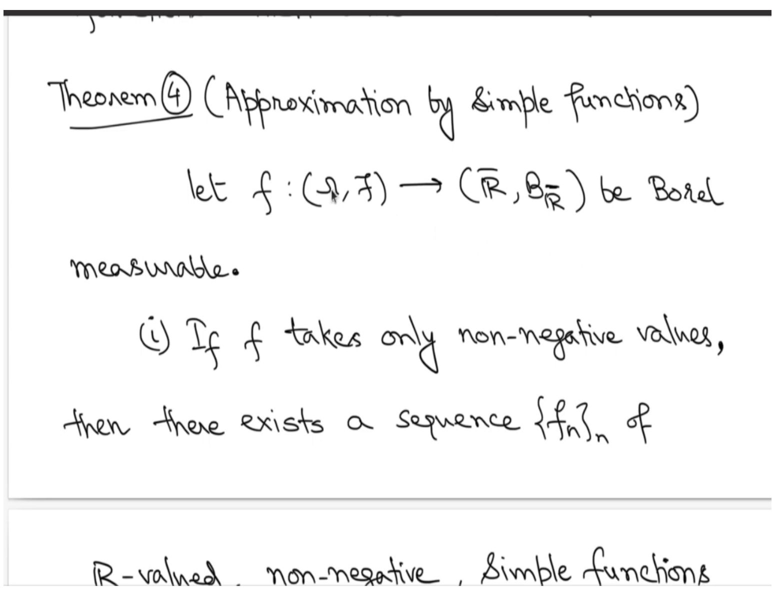

So, what is this? So, this is an approximation by simple functions what it says is that general functions by some appropriate mechanism can be approximated by simple functions. You start with extended real valued functions. But then as stated earlier you can also restate these results in terms of real valued functions. So, again if your original function is real valued you will get the exact same result.

So, real valued Borel measurable functions anyway are extended real valued Borel measurable functions that we have left as an exercise earlier. So, you can, you could always apply this result for real valued Borel measurable functions. So, there are two parts to this theorem.

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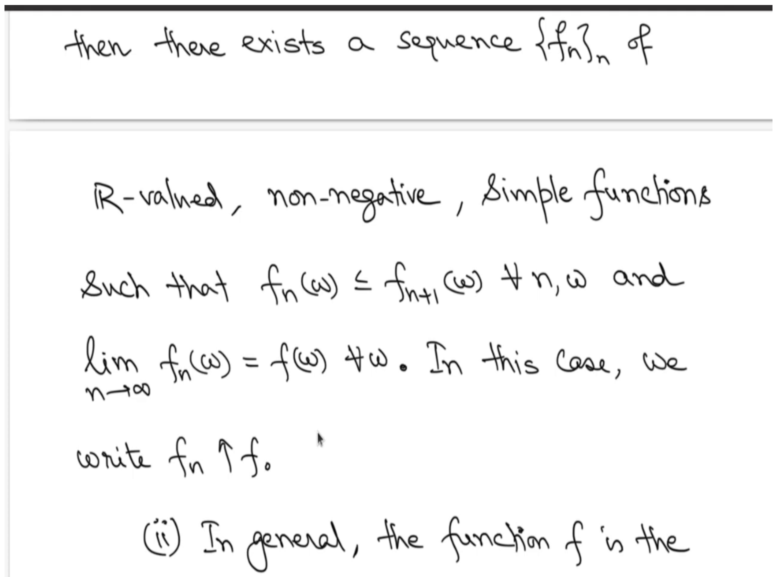

So, what do you do first so suppose your f takes only non-negative values. So, your f is a non negative Borel measurable function. Then you can find a sequence f_n of real valued non negative simple functions. Such that the simple functions increased pointwise and their limit is $f(\omega)$. So, there are a lot of things to digest here. So, first of all you are starting off with a non negative measurable function f, I am saying you are getting a sequence of simple functions.

These simple functions first of all are real valued you do not have to look at extended real valued functions and they take only non negative values. That is also an important information about this sequence of functions. If you look at the graph of the function f that is a non negative function it is the above the value 0. And what the simple functions are doing are it is approximating it from below.

So, f_n is increasing, $f_n(\omega)$ is increasing for each n as long as you increase the n it will give you slightly bigger value. So, and then at the end it will try to approximate the actual value of the function $f(\omega)$. So, these functions are increasing in n pointwise limit being the given function that you have started off with. So, you have this sequence of non decreasing functions, simple functions real valued and non negative. So, this situation we usually write it as f_n increasing to f. So, as long as you have a sequence of increasing functions you can write it in this way.

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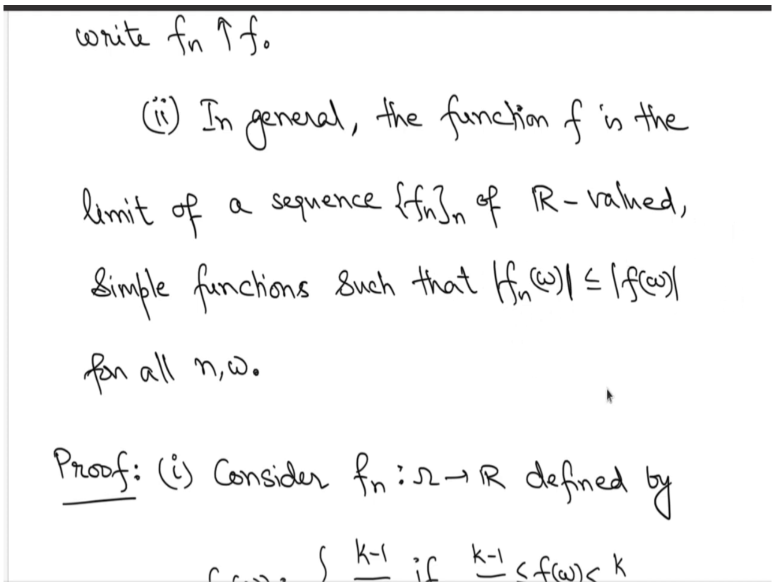

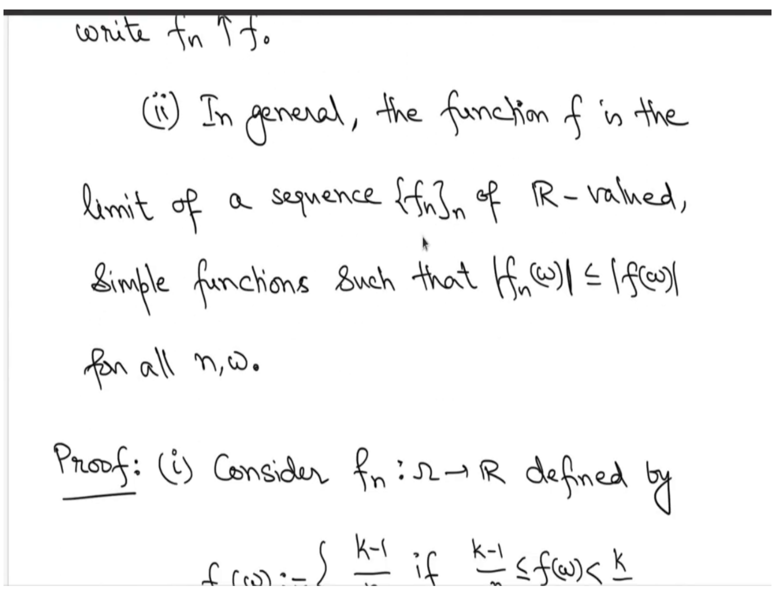

And the second half of this is for general measurable functions which could take positive and negative values. So, here we are saying that if you are having this function. Then again you will get a sequence of simple functions. These simple functions will no more be non negative they will be the real valued so again you are restricting your attention to only real valued simple functions such that you have a different condition on the sequence now.

So, this $f_n(\omega)$ will be in modulus will be dominated by a $|f|$. So, for each n and ω these functions are bounded above by $|f|$. So, that is basically the idea. So, for the non negative measurable functions f you can get approximate in sequence that approximates the function from below.

In the general function's case, you get a sequence again it will turn out to be a pointwise limit. But then you get a different bound for the approximating sequence. So how do you prove this? So, consider the case of the function, the given function f when it is non negative and measurable.

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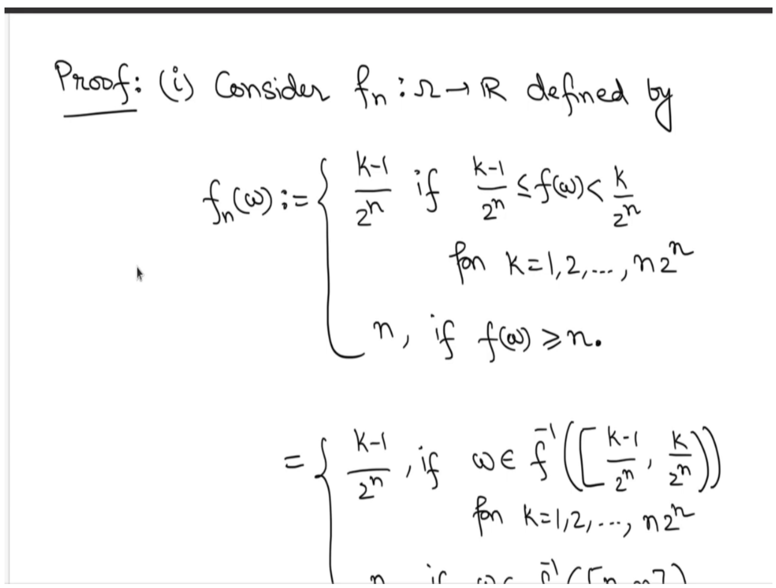

So, here we are supposed to figure out a sequence of functions f_n which are simple functions. So, therefore they are defined on the given domain Ω taking non negative values. So, for the moment just let us just write them as real valued but you would like to specify non negative values for these functions. And we also want to show that pointwise for every fixed set sample point the function values f_n converges to the given function value $f(\omega)$.

So, that is what we are going to do. So, this is what we are going to construct explicitly. So, what do you do? So, you first fix a integer n and look at a sample point ω. So, now if the actual function value $f(\omega)$ is exceeding that integer n if it is greater or equals to n then you specify the functional value for the approximating sequence as n. What remains? So, in that case $f(\omega)$ takes values between $[0, n)$.

What do you do you split this interval $[0, n)$ into $\frac{1}{2^n}$ length sub intervals. So, how many such intervals you are going to get? You are going to get $n2^n$ many such sub intervals. These sub intervals will be from $[\frac{k-1}{2^n}, \frac{k}{2^n})$. So, $\frac{k-1}{2^n}$, $\frac{k}{2^n}$ excluded. So, this is left closed right open intervals and if you look at for all possible k like this.

So, 1 2 3 up to n times 2^n then you are going to cover that interval $[0, n)$. So, now what you do? So, you will look at the actual function value $f(\omega)$. It is now in $[0, n)$. So, then in particular it will be one of these sub intervals for some fix at k. What do you do? You specify the function value for the approximating sequence for that sample point as the left hand point value.

So, we will see why we need this but we specify this as the left hand point value $\frac{k-1}{2^n}$. So, that is your function f_n for each function value, each sample point you have defined the function value $f_n(\omega)$. So, now you are going to check whether you have defined it for all possible ω observe that the original function f is non negative valued. So, you have the function values taking between $[0, \infty]$.

So therefore, what will happen is that you are basically going to split the interval $[0, \infty]$ included that range of function values for the original function f into these intervals. So, one of the intervals is $[n, \infty]$. So, $[n, \infty]$ so that is one interval. And the interval $[0, n)$ you have splitted into these parts. So, that therefore you have considered all possible values of the function f.

And therefore, you have considered all possible sample points on the domain side Ω. So, therefore you have defined it for all possible sample points ω in your domain. So, f_n is well defined now. But I am going to claim that this is a simple function.

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$$f_n(\omega) := \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \le f(\omega) < \frac{k}{2^n} \\ & \text{for } k = 1, 2, \ldots, n2^n \\ n, & \text{if } f(\omega) \geq n. \end{cases}$$

$$= \begin{cases} \frac{k-1}{2^n}, & \text{if } \omega \in f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right) \\ & \text{for } k = 1, 2, \ldots, n2^n \\ n, & \text{if } \omega \in f^{-1}([n, \infty]) \end{cases}$$

$$= \sum^{n2^n} \frac{k-1}{} 1 \; (\omega)$$

What is the reason for this? So, the reason for this is very simple. So, when you are looking at the original function value from this left closed right open intervals. You are just rewriting this condition in another format. So, you are simply saying that the function value $f(\omega)$ lies in this interval $[\frac{k-1}{2^n}, \frac{k}{2^n})$. But you can write it in this different format that ω belongs to the preimage of this left closed right open interval. So, on this you have specified the approximating sequence f_n value as $\frac{k-1}{2^n}$

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$$\begin{cases} & \text{for } k = 1, 2, \ldots, n2^n \\ n, & \text{if } \omega \in f^{-1}([n, \infty]) \end{cases}$$

$$= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right)}(\omega)$$

$$+ n 1_{f^{-1}([n, \infty])}(\omega).$$

The verification of the relevant

properties is straight-forward.

Proof: (i) Consider $f_n : \Omega \to \mathbb{R}$ defined by

$$f_n(\omega) := \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \le f(\omega) < \frac{k}{2^n} \\ & \text{for } k = 1, 2, \dots, n2^n \\ n, & \text{if } f(\omega) \geq n. \end{cases}$$

$$= \begin{cases} \frac{k-1}{2^n}, & \text{if } \omega \in f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right) \\ & \text{for } k = 1, 2, \dots, n2^n \end{cases}$$

On the rest of the portion, when you are a function value is from n to ∞. So, we are looking at ω as it belongs to f inverse the preimage of this side, preimage of this interval. So, therefore on that interval on that preimage you are specifying the value n. So, therefore you are now going to write that conditions in terms of indicators. So, that is as simple as that. So, as long as ω belongs here this indicator will supply you the value 1.

Otherwise, it will supply you the value 0. So, all these intervals to $\frac{k}{2^n}$. So, for different values of k and n to ∞ that interval these are all pairwise disjoint intervals. So, their preimages are also pairwise disjoint and therefore this collection that you see here it is basically splitting up the domain Ω into all these preimages of the original function f.

So, that is what we are doing. And all these individual preimages we are specifying the function values of f_n. So, now we have written it in terms of indicators which are all involving pairwise disjoint sets and it basically covers the whole domain. So, therefore you have written the original function $f_n(\omega)$ in terms of these combinations of indicators. And therefore, this is a simple function.

Now, it is a easy fact to check that for n and $n + 1$ the simple functions that you are going to get f_n and f_{n+1} these values can be compared. So, how do you do this? So, to do this what you need to do is to look at the function values for n and $n + 1$ look at the corresponding function value

$f(\omega)$ and just compare the corresponding prescribed value and in the corresponding prescribed value the nice thing is that you are approximating the actual function value $f(\omega)$ by this left endpoint of the interval.

So therefore, you are making a error of at most $\frac{1}{2^n}$. So, that is why it is also that approximating sequence as n increases your error decreases. So, that is the error that you are looking at here the difference between these two values. So, you can check these increasing properties. And by definition of course this f_n the functions, simple functions are taking non negative values. So, that is my definition.

So, but it is important that these functions f_n taking, are not taking values in the extended real numbers these are not taking the value $+\infty$. Even if any of the original functions f is taking the value $+\infty$ you can approximate by really valued non negative simple functions here. So, that is the proof of the first part.

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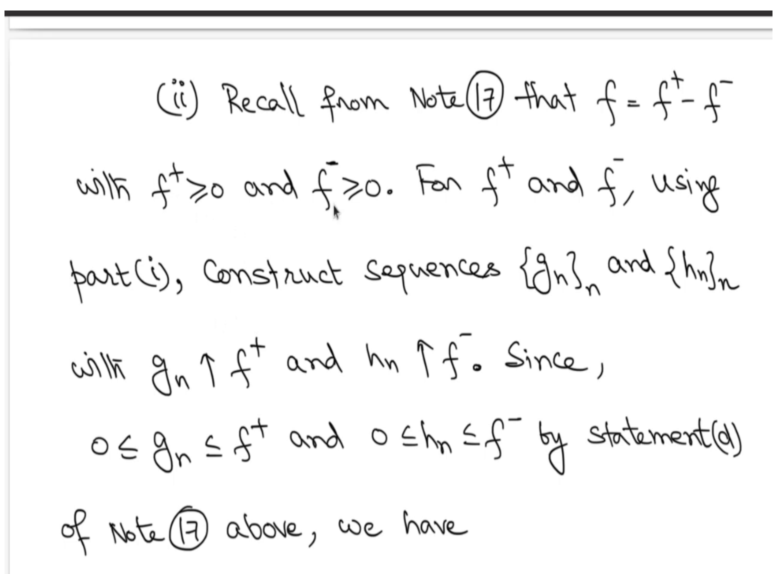

So now, what was the second part? Second part was for general measurable functions which could be taking values $\pm\infty$. And they could be taking positive or negative values also. So, here what we have observed earlier is that we could write the function as $f^+ - f^-$. So, this we have discussed a few minutes back before discussing this theorem. So, now these two functions f^+ and f^- are non negative measurable functions.

So, for these two individual functions f^+ and f^- by the first part you can now construct this sequence of functions gn and hn each of which will approximate it from below. And these gn's and hn's should be simple functions. So, this is by the first part. So, f^+ and f^- are non negative measurable functions and therefore you can do this.

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$0 \le g_n \le f^+$ and $0 \le h_n \le f^-$ by statement(d) of Note 17 above, we have

$$\{\omega \mid g_n(\omega) > 0\} \cap \{\omega \mid h_n(\omega) > 0\} = \phi \; \forall n$$

and hence $f(\omega) = \lim_{n \to \infty} [g_n(\omega) - h_n(\omega)], \forall \omega.$

Here, "$\infty - \infty$" situation does not appear on the right hand side.

Take $f_n := g_n - h_n \; \forall n$. Check that f_n is

But then you observe that gn's fall between 0 and f^+ and hn's fall between 0 and f^- that simply by the conditions. And then you can talk about this difference now, so gn minus hn. So, if you look at gn minus hn, gn approximates f^+, hn approximates f^-. So, therefore their limit value will be actually$f^+ - f^-$ that will be $f(\omega)$ you can try to check the details.

So, the idea here is this for each fixed n when you are looking at $g_n - h_n$ it will not happen that both of the functions gn and hn are positive. So, for each fixed n if you look at the points ω says a $g_n(\omega)$ is strictly positive, $h_n(\omega)$ is strictly positive that will be an empty set.

So, here you will not have an $\infty - \infty$ situation. So, therefore this difference is making sense. So, the limit will make sense because pointwise limits for gn and hn and make sense. So, you will get the function $f(\omega)$ because g_n approximates f^+, h_n approximates f^-.

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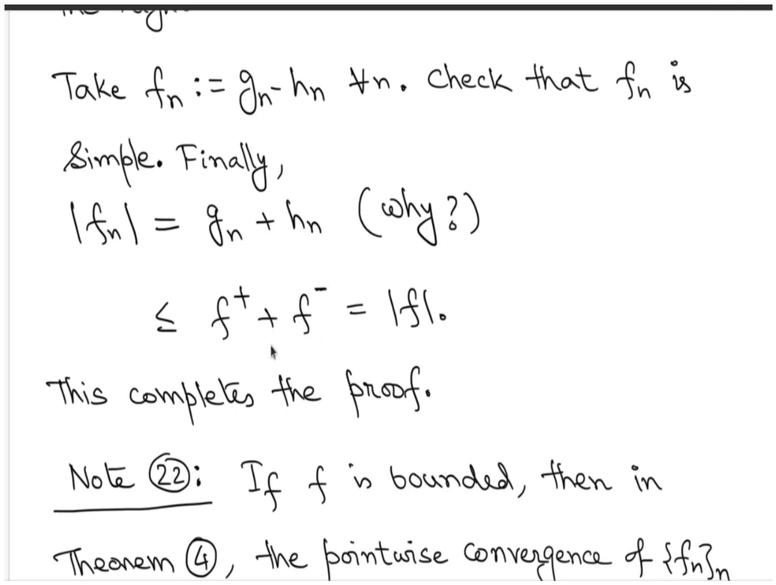

So therefore, you get this difference gn minus hn. But then this $g_n - h_n$ let us call it f_n, but then as we have mentioned that this is now nothing but our linear combinations of indicators, gn was a simple function, hn was a simple function. So, each of which are linear combinations of indicators. So, if you are looking at gn minus hn it is also a linear combination of indicators. So, therefore f_n will turn out to be simple by appropriate adjustment that you have mentioned earlier.

And then finally you check this condition that $|f_n|$ which is $g_n + h_n$. So, you can check this why this is happening. But then gn will be dominated from above by f^+, hn is dominated from above by f^-, add them up you get mod f. So, this was also mentioned earlier in a note 17. So, please check this why this is true, why this equality holds. So, with that we have completed the identification that given are non negative measurable function.

You can get this appropriating sequence of simple functions non negative which appropriates the function from below. If you are having a real valued or extended real valued function taking positive or negative values in which case you will get a appropriating sequence again real valued. But it will no more be non negative.

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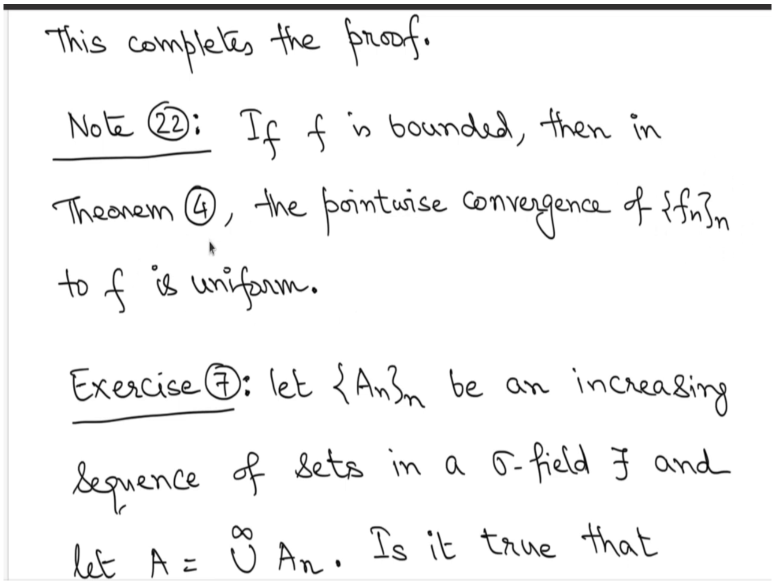

So, another important observation is that if you go through the prove you can actually try to show that if a original function is given to be bounded. So, if it is a real valued and it is bounded then you can actually show that this pointwise convergence of these simple functions is actually uniform. So, this f_n's that you constructed that will actually give you a uniform appropriation to the function.

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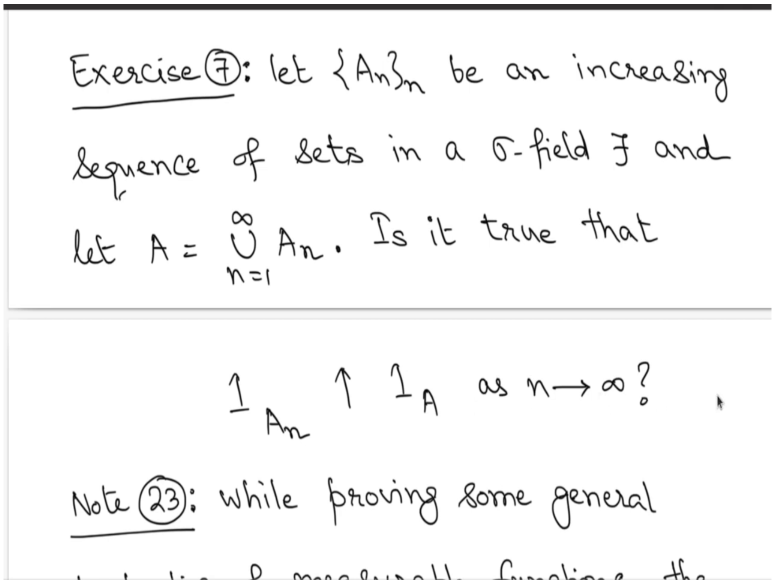

We now discussed certain exercise. So, if $\{A_n\}$ is an increasing sequence of sets, look at the union A. Now you can ask whether $1_{A_n} \uparrow 1_A$. So, these are all non negative functions. You are asking whether $1_{A_n} \uparrow 1_A$. So, please check this.

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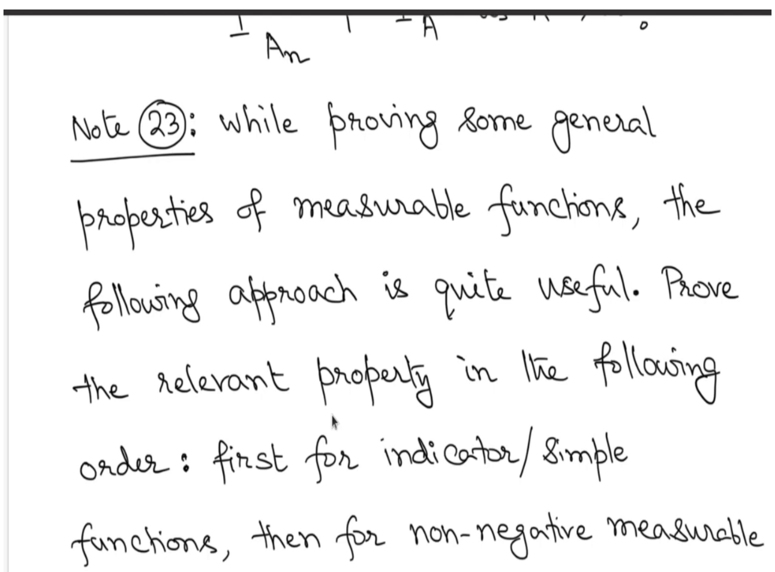

So now, one final comment as we stop. So, while proving these general properties of measurable functions you have this general principle of working with such functions. So, in case you are unable to verify the required property directly for all measurable functions. What you should do is to following approach.

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following approach is quite useful. Prove the relevant property in the following order: first for indicator/Simple functions, then for non-negative measurable functions and finally for measurable functions.

First you prove it for indicators or simple functions. Then you go by approximations and prove it for non negative measurable functions. Because non negative measurable functions are now pointwise limits of simple functions. So, you hope that your property will be carried over by these pointwise limits. And finally, you approximate it for measurable functions. Because measurable functions are now difference between two non negative measurable functions.

So, as we have observed if f is a real valued or extended real valued measurable function f you can write it as $f^{+} - f^{-}$. So, if you prove it for f^{+} and f^{-} you hope that whatever property you want also holds as you take the subtraction between these two. So, this is a general principle that will be used in discussions later on in this course. So, this is a general property that also helpful in proving other properties. So, we stop here and we shall discuss, we will continue this discussion in the next lecture.