Linear Algebra
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Lecture – 60
Spectral Theorem for Hermitian Matrices

So, we learned that if two matrices A and B are similar A is equal to SBS inverse. So, S is already invertible this is what we are writing invertible then A and B share the same eigenvalues, same characteristic polynomial share the same eigenvalues and so on the same eigenvalues geometric multiplicity is same as algebraic multiplicity and so on dot dot dot alright.

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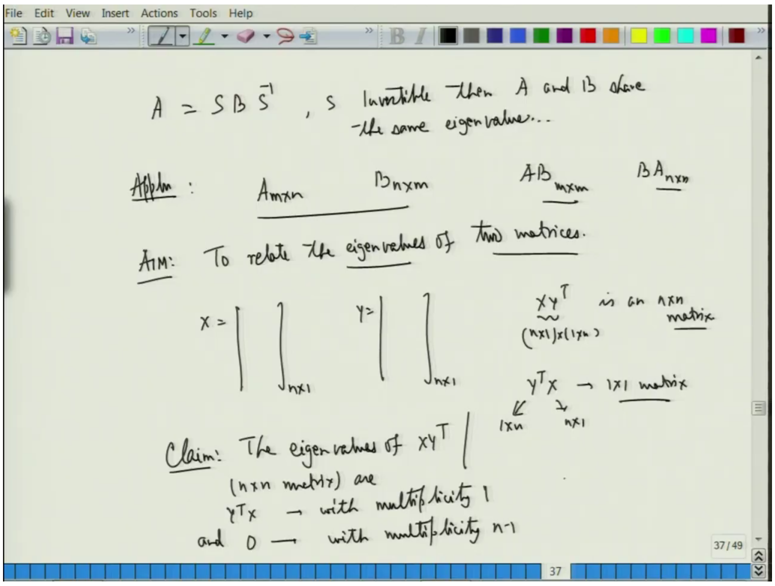

So, let us look at an application of this, alright. So, application. So, let A be an m cross n matrix; B be an n cross m matrix. So, if I look at A times B is defined which an m cross m matrix and B times A is also defined which is an n cross n matrix.

We want to relate the eigenvalues of the two matrices. So, aim to relate the eigenvalues of two matrices, fine. So, you can see that for example, if I look at this simple example that X is say a n cross 1 vector, Y is another n cross 1 vector fine, then what we see here is that XY transpose.

XY transpose if I look at this is a matrix of size X is n cross 1, Y transpose is 1 cross n. So, it is an n cross n matrix, 1 cross n. So, this is an n cross n matrix, fine. What about the transpose of this if I not the transpose look at the other way around if I want to look at say Y transpose X, Y transpose is 1 cross n alright and this is n cross 1. So, this is going to give me a 1 cross 1 matrix, fine.

So, we would like to claim so, claim the eigenvalues of XY transpose which is an n cross n matrix n cross n matrix, fine? Are Y transpose X with multiplicity 1 multiplicity 1 and eigenvalue 0 with multiplicity n minus 1, alright. So, it is an n cross n matrix so, I can do this.

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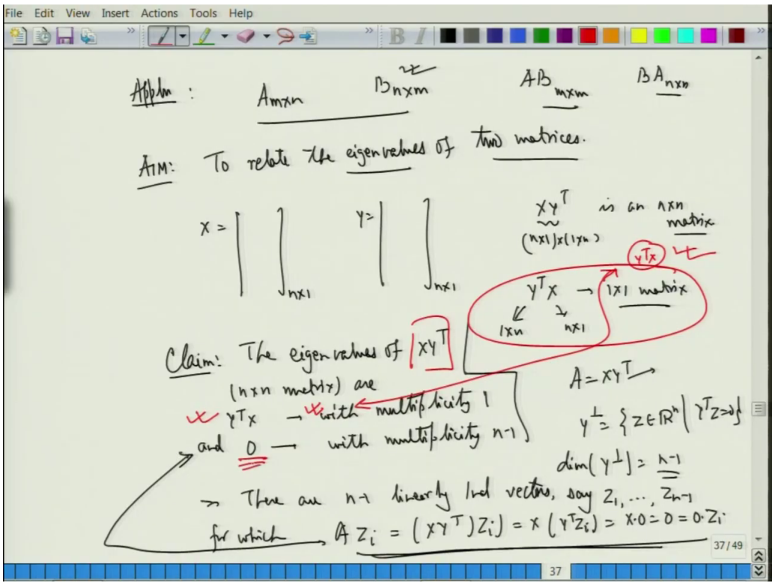

So, I would like you try that out yourself. At least you can see here that if I am looking at this matrix. So, I would define a as XY transpose corresponding to Y if I look at this set Y perp which is all Z belonging to R n such that Z is perpendicular to Y or Y transpose Z is 0, then its dimension of Y perp is n minus 1.

So, implies there are n minus 1 linearly independent vectors say Z 1, Z 2, Z n minus 1 Z 1 till Z n minus 1 for which a times Z I will be equal to X Y transpose times Z I which will be equal to X times Y transpose Z I which will be equal to X times 0 which is 0 which is same as 0 times Z I alright.

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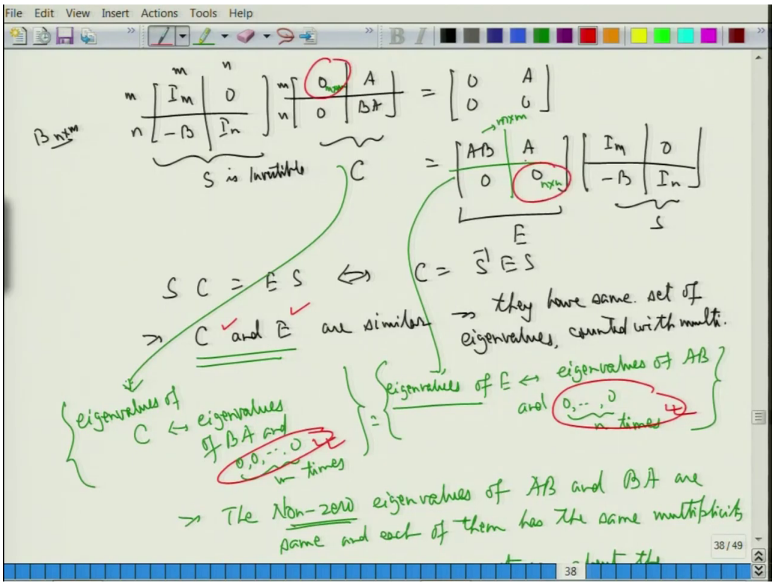

So, you can see that 0 is an eigenvalue n minus 1 times, this you can see, fine. So, the idea is to proceed with this and then build up the idea. So, let us look at AB now. So, I define my matrix.

So, I look at this product matrix product I m 0 minus B, alright. So, recall what is the size of B. So, I is m cross m. So, this is m this is m, B did I write? What is the size of B did I write? I think I wrote it as n cross m. So, I think it is n cross m I think B is n cross m,. So, I wrote it correctly.

So, B is this I want to multiply this matrix. So, this is my S that I am looking at in some sense which an invertible matrix because it is a lower triangular with diagonals 1. So, S is so this is S is invertible, fine. I am looking at this multiply this with this matrix 0, 0, 0, BA. So, look at the size of this matrix again B is here. So, B means B is the size of B is n cross m, we have n here n here. So, this matrix multiplication makes sense to you. So, this is same as just multiply it out.

So, this into this is 0. This into this if you look at you get A here this into this is 0 and minus BA plus BA is 0. So, this is this matrix I would like to write this also as something here, alright. So, this will go on the right. So, I want you write here I m here minus B here 0 here I n here. So, that this is again S for me that I am looking at. So, I want you to check here that this is nothing, but AB A 0 0, alright.

So, this is my matrix say C, this is my matrix E what we are saying is that SC is equal to EC which is same thing as saying that ES which is same thing as saying that C is equal to S inverse ES implies C and E are similar implies they have same set of eigenvalues counted with multiplicity alright whatever the multiplicity we need to count with respect to that, fine.

So, let us understand this now. So, what we are trying to say. So, we are saying that E and C are similar. So, let us look at the matrix E, fine, the size of AB is I think m cross m; this is n cross n. So, there are if I look at the eigenvalues of this eigenvalues of E they are eigenvalues of. So, they correspond to eigenvalues of AB and 0 coming how many times? n times, alright. This is what we see if I am looking at C. Now, if I go from C here, fine.

So, eigenvalues of C of C are alright. So, 0 here is of A size m cross m. So, their eigenvalues of BA and 0 0 0 repeated m times, fine? So, we are saying that these two sets are same and therefore, what should happen? The nonzero eigenvalues are same that.

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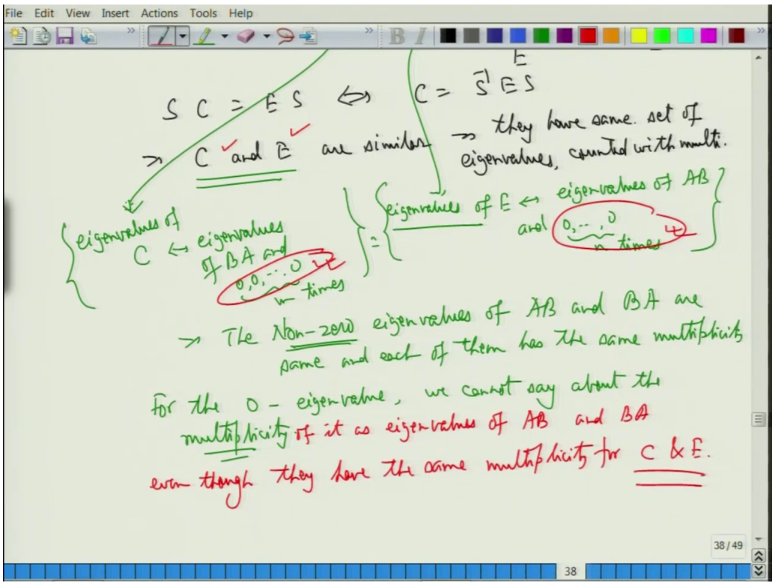

So, this implies the nonzero eigenvalues of AB and BA are same and each of them has the same multiplicity fine, have the same multiplicity and for 0 so, this is about the nonzero eigenvalues. For the 0 eigenvalue we cannot say about the multiplicity, why you cannot say about the multiplicity? Can we give an argument? So, the important idea is that you cannot talk about multiplicity because there is m of them here, n of them here.

So, the idea is that in C and E they are same, but when I go to AB and BA there is this which extra which is needs to be taken care of alright. So, for 0 eigenvalue you cannot say about the multiplicity of it as eigenvalues of AB and BA even though they are equal even though they have the same multiplicity for C and E, alright.

So, in C and E they have the same multiplicity, but when I come to BA and AB I have a problem alright I cannot guaranty it and this is what is reflected in the previous one that here this is have 1 cross 1 matrix. So, it is eigenvalue is only Y transpose X alright, but here if I look at this matrix which is XY transpose I wrote somewhere I think XY transpose, alright.

Then this is an eigenvalue. So, this is a nonzero eigenvalue or this could be the same Y transpose X could also be 0 for us I do not know what it is because that is not given to us, but what we see is that Y transpose X is an eigenvalue here and Y transpose X is also an eigenvalue here, alright. But, there is a packing of extra 0s here because of a B and B a and the extra 0s that we are putting in the proof I think this argument, alright. We are putting this extra 0s to get our argument, is that ok?

So, that extra 0 is been patched up to get our result and hence 0 may be there or may not be there as an eigenvalue of AB or BA, is that ok? So, that is all. Now, we will look at the next idea.

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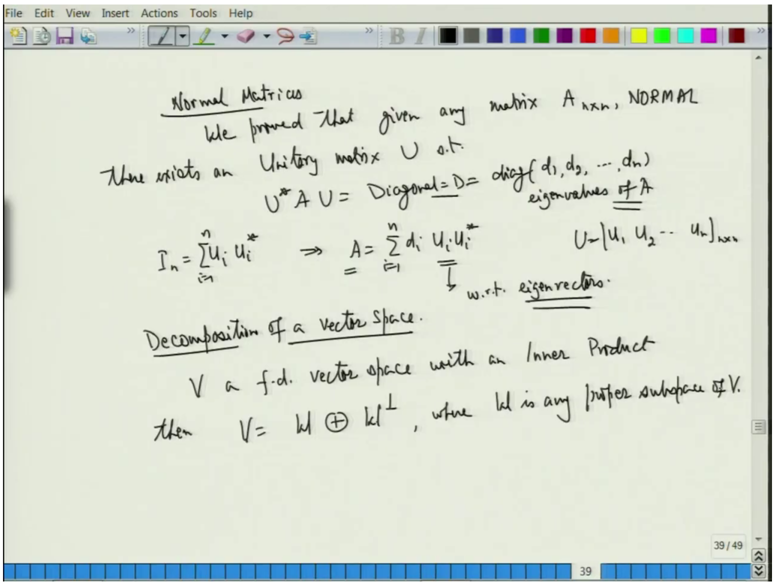

So, let us recall that in the previous class we looked at normal matrices, alright. We proved there. So, we proved that given any matrix A n cross n normal there exists an unitary matrix U such that U star AU is diagonal which is D and this is nothing, but diagonal of say d 1, d 2, d n eigenvalues of A, fine. So, we wrote it in two way the first thing we wrote was identity we wrote identity as.

So, we took U as u 1, u 2, u n n cross n matrix and we wrote I n as u i u i star i is equal to 1 to n and when I multiply this to A this gave me A is equal to summation i is equal to 1 to n d i times u i u i star and from there we concluded that the action of A on u i is nothing, but multiplying by d i, fine. Now, so, here I am looking at with respect to everything is being done with respect to the eigenvector. So, here it is with respect to eigenvectors, alright.

So, let us recall what is called the decomposition of a vector space. So, decomposition of a vector space I did not do it separately, but what we said was that if V is a finite dimensional of vector space V a finite dimensional of vector space with an inner product then we can write V as W direct sum W perp or where W is any subspace or any proper subspace subspace of V.

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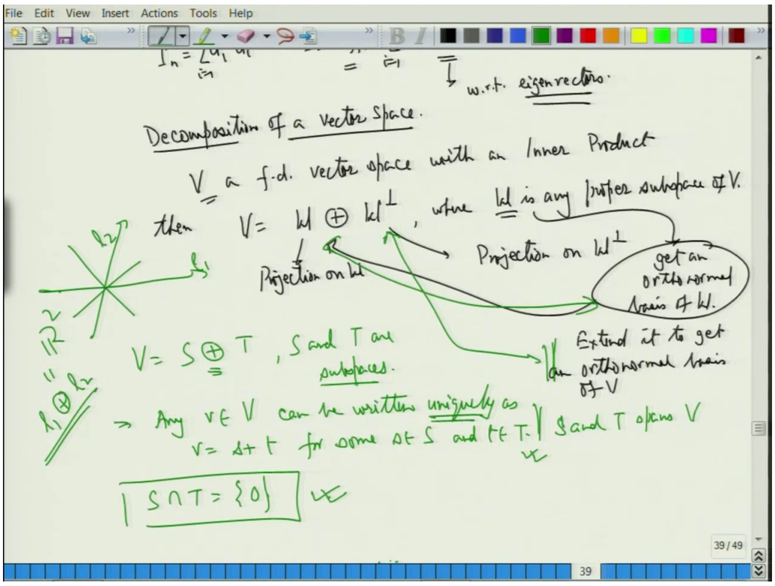

So, I did not write it this way directly but we had the notion of projection where the projection on W on W and projection on W perp. So, where the matrices of the projection that we had and the idea was that since V is finite dimensional W is finite dimensional. So, I can get a basis of it get an orthonormal basis normal basis of W extend it to get a basis of the whole space V extend it extend it to get an orthonormal basis of V, fine.

So, this part the linear span of this part is this alright the linear span of this part is this and this extension just remove the first part whatever is left out the extended part gives you the basis of this part is that ok. So, we have two things and what we are trying to say here is that when I write something as a if I write V as S direct sum T S and T are subspaces.

What basically I am trying to say is that this implies that any v belonging to V can be written uniquely can be written uniquely as v is equal to s plus t for some s belonging to S and t belonging to T that is one thing the other thing. So, S and T span the whole thing, alright.

So, this is what it mean by that S and T the spans V we also have that there is a direct sum here it means that S intersection T is just the 0 vector nothing else, is that ok? They are subspaces, fine? I am not saying they are orthogonal. I am nowhere using the word S and T are orthogonal. I am just saying that I can write them uniquely and the intersection is just the 0 vector, alright.

And, that intersection the 0 vector is the one which gives you that it is a direct sum or the expression here is unique is because of S intersection T being 0. So, example you have R 2 you can take R 2, this is your R 2; take any two line this and this, fine l 1 and l 2 then R 2 is linear span of l 1 l 2.

So, just write it l 1 direct sum l 2 alright because l 1 itself is a subspace l 2 is also a subspace. So, you can write any element of R 2 in terms of an element of l 1 and an element of l 2, alright. The way we do for X and Y similarly you can do it here also, fine.

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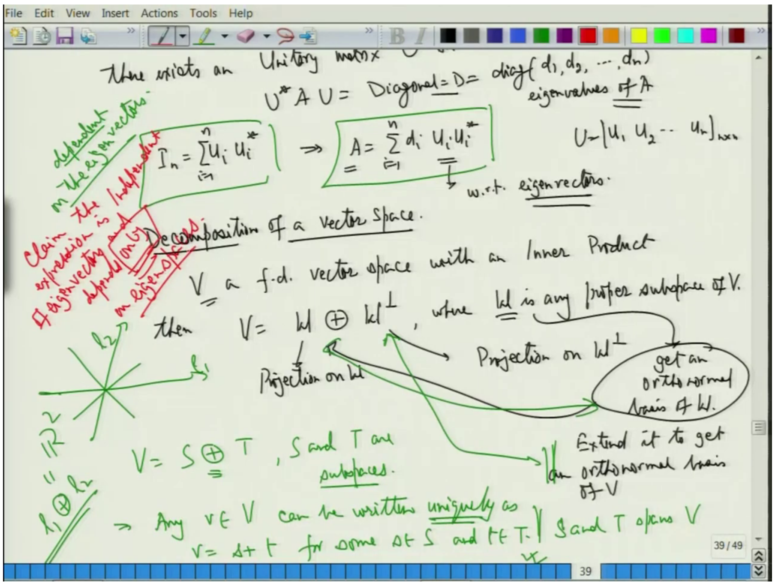

What we want to stress here is that this part that you wrote here this one or this one was dependent on the eigenvectors. So, this was dependent on the eigenvectors. So, alright what I would like to say is that they are not only dependent on eigenvectors in place of that we would like to say that they are independent of the eigenvectors. So, we want to claim.

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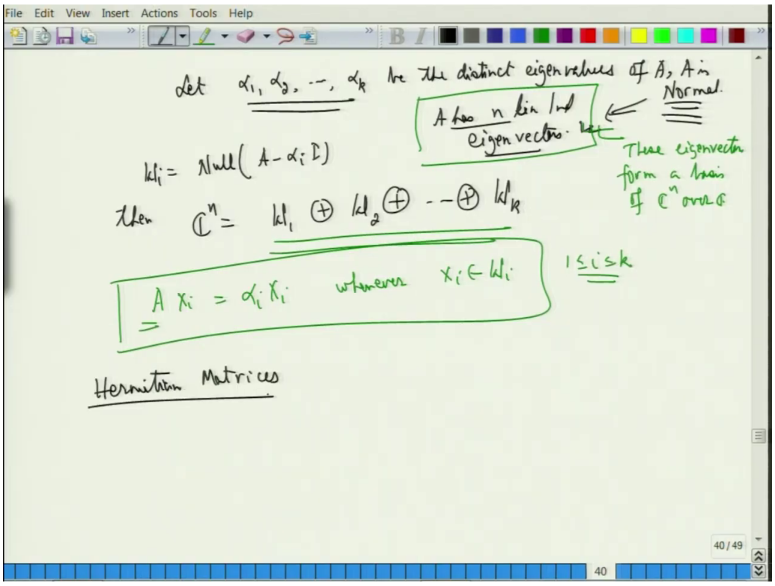

So, claim the expression is independent of eigenvectors and depends only on eigen spaces, fine. So, let us do that part. So, what we are saying you all just I will just write it down alright. So, define let alpha 1, alpha 2, alpha k be the distinct eigenvalues of A, values of A. So, these are the distinct eigenvalues of A and we are saying that A is normal that is also given to me, fine. So, A is normal implies A has n eigen vectors n linearly independent eigen vectors, fine.

So, therefore, if I look at I define W i is equal to null space of A minus alpha i of I then dimension of W i. So, the so, what can then I can write R n or no we are looking at C n then we can write C n as W 1 direct sum W 2 direct sum W k alright, this is what is important because they will give me certain eigen vectors. So, each of them it has n linearly independent eigen vectors.

So, each of these eigen vectors they have to correspond to some alpha I and therefore, there will be in some null space W i and hence you can write C n as this. So, n linearly independent eigen vectors means that these eigen vectors form a basis of C n over C alright.

So, this is what we are writing, fine. So, therefore, we can write A in terms of that and the matrix A what we are saying is that the matrix A here looks like. So, the action of A on any X i is alpha i of X i whenever X i belongs to W i, alright. So, this what we are saying 1 less than equal to i less than equal to k. So, what saying is that action of a on W i is by multiplying by alpha i, is that ok?

So, now, let us rewrite this in the language of Hermitian matrices, alright. So, Hermitian now what happens to Hermitian matrices? So, now, let us. So, we did it for normal matrices we want to write it for Hermitian matrices.

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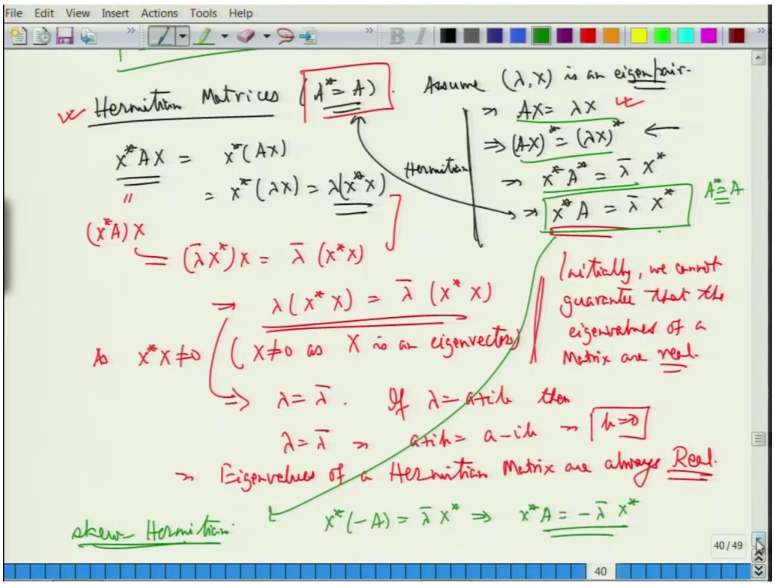

So, first thing I would like to say is that if A is Hermitian means A star is same as A, then what happens is that look at this expression X star AX, alright. So, I should also write here I think Hermitian matrix assume lambda comma X is an eigen pair, fine.

So, this implies AX is equal to lambda X, fine. This also implies just take the star on both the sides. So, AX star is equal to lambda X star and this implies X star A star is equal to lambda bar X star which is same thing as saying that X star A is equal to lambda bar X star, alright, fine.

So, look at carefully we are saying that lambda comma X is an eigen pair. So, that implies that AX is equal to lambda X we are taking the star on both the sides. Once I have done that then the star tells me that I can rewrite it as X star A star is equal to lambda bar X star, but A star is same as A. This is the way we are using Hermitian, alright till that stage we are not used Hermitian. So, we get X star A is equal to lambda bar X star.

So, now I want to compute this X star AX by two methods. So, one is look at X star AX like this which gives me X star AX is lambda X. So, it gives me lambda times X star AX I get this part, fine. I also want to use this now. See if I want to use this I can write this as X star A of X. So, my associativity is like this. I write this as X star A is lambda by X star of X which is same as lambda bar X star X.

So, we are saying if you look at these two we are saying that this implies lambda times X star X is equal to lambda bar X star X what does this imply. So, initially when we solve for matrices we are looking at eigenvalues at the initial stage we do not know whether the eigenvalues are real or complex, alright. So, initially we cannot guarantee that the eigenvalues of a matrix are real I cannot guarantee this part.

But, what it tells me that, alright what it tells me as X star X is not equal to 0, why it is not equal to 0? X is not equal to 0 as X is an eigen vector, fine. Since X is an eigen vector so, therefore, X star X is not 0 and therefore, I get from here that lambda is equal to lambda bar. So, this implies that so, if I take lambda as.

So, this implies. So, if lambda is equal to a plus ib implies this, then lambda is equal to lambda bar implies a plus ib is equal to a minus ib and this implies b has to be 0, alright. So, implies eigenvalues of a Hermitian matrix always real fine. So, I would like you to see that nicely nothing is special here we just had a Hermitian matrix.

So, the condition was A star is A; from there using that they say there is an eigen pair we wrote A is equal to lambda X, took the star we got this, wrote it this and then we replaced A star by this is what we did. And, then just a small calculation tells me that lambda equal to lambda bar because X is not the 0, vector alright fine.

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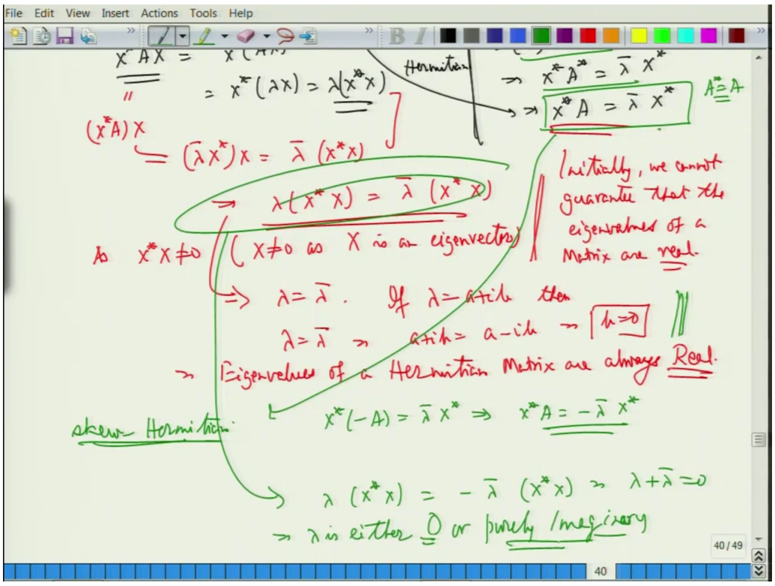

If I had started with in place of Hermitian if I had started with skew Hermitian fine. The same thing would have been true here other than something here. So, for a skew Hermitian this part will become X star of A it will become minus A here will be equal to lambda bar of X star and this will imply that I am looking at X star of A is equal to minus of lambda bar X star, that is all I will have, alright.

And, therefore, when I want to look at this part this part will tell me that lambda times X star X will be equal to minus lambda bar X star X this will imply that lambda plus lambda bar is 0 and therefore, I will get that this implies lambda is either 0 is either 0 or purely imaginary is that. So, this is what you have to be careful about. There is the same calculation nothing special.

We are doing the same thing, but things have changed because of this minus coming into play and the argument is the same that I used here B became 0 because a plus ib equal to a minus ib.

There it will become the other way around it will become a plus ib is equal to minus a plus ib. So, minus a and plus a will cancel out I will get that b has to be either 0 means a has to be 0 so, b can also be 0. If a and b are both 0 then I get 0; if b is not 0, I get purely imaginary, is that ok?

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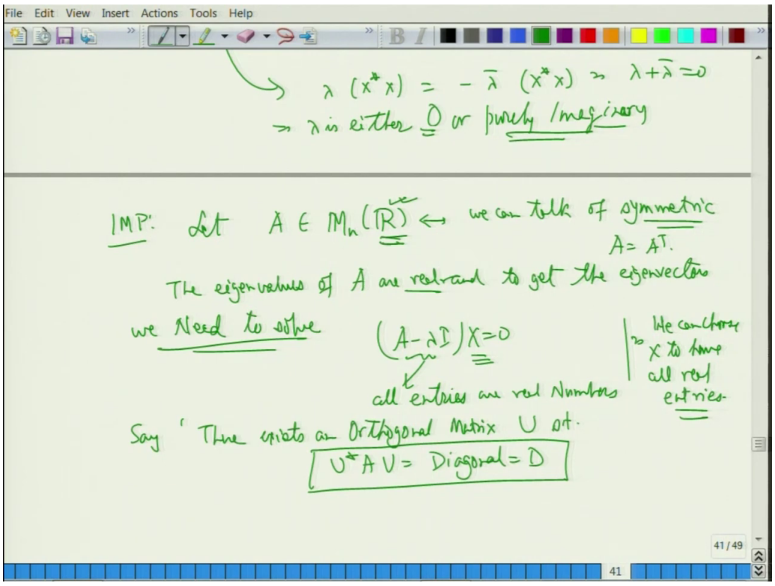

So, you have to be careful a small implication of this that I would like to write here is implication or important. Let A belong to say M n of R, fine. Since it is a real number so, we can talk of; we can talk of symmetry symmetric means A is equal to A transpose, fine. What we have seen what we have seen here is that the eigenvalues of A are real fine and to get the eigen vectors and to get the eigen vectors we need to solve what we need to solve? We need to solve A minus lambda I of X is equal to 0.

Now, everything here all entries here are real numbers and this will imply that. So, this implies that we can choose X to have all real entries, fine. So, I would like you to see that for symmetric matrix A is symmetric means with real entries that real entries is very very important. So, A is symmetric with real entries then A is Hermitian. So, A has real eigenvalues. So, I can choose my X here alright the eigen vectors to be having real entries and hence in place of saying that A is unitarily diagonalizable I can say that there exists a unitary.

So, there exists so you can say there exists an orthogonal matrix U such that U star AU is diagonal which is D the eigenvalues, alright. So, in place of saying that I can get it using unitary we are saying something more we are saying that we can do it using orthogonal matrices, alright. So, orthogonal matrices means I have got all the entries to be real. So, over real fields itself I can do it I do not have to go to complex numbers, is that ok?

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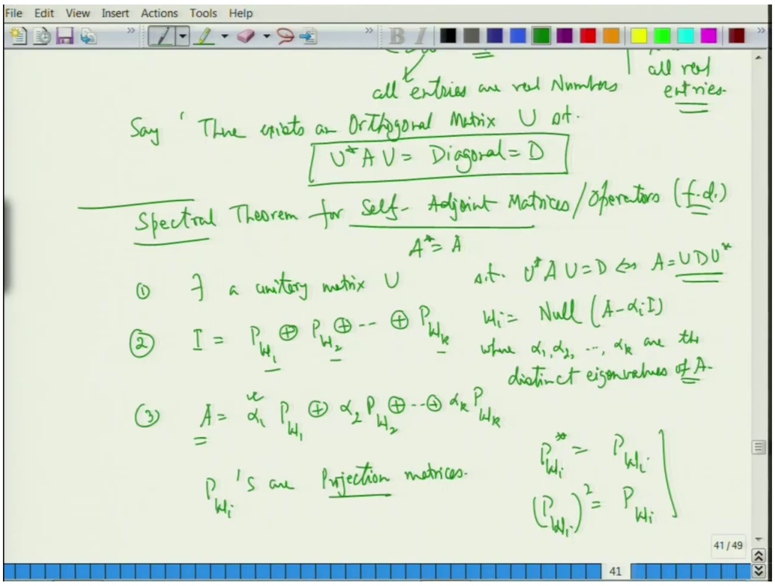

So, that finishes the idea of looking at things from the point of view of normality and so on. At a last step what I would like to just tell you that what is called the spectral theorem for self advanced operators a spectral theorem for self adjoint matrices or operators we are looking at finite dimensional that is very very important. So, self adjoint basically means A star is A. So, it is a Hermitian that is all we are saying.

So, what we are saying is that one there exists a unitary matrix U such that U star AU is diagonal which is same thing as saying that A is equal just look at this fine. So, it is UDU star, is that ok?

That is one thing; 2, I can decompose identity as projection into W 1 projection into W 2 direct sum projection into W k and what are W 1, W i, W k? So, W i is null space of A minus alpha i of k where alpha 1, alpha 2 alpha k are the distinct eigenvalues of A and we are doing it over R, alright. So, if a was symmetric it is over R, if it is Hermitian it has complex entries then it is over over C r.

3rd A is nothing, but so, we are writing like this A is equal to alpha 1 times P of W 1 projection onto W 1 with multiplied by alpha 1 alpha 2 p of W 2 plus 1 plus alpha k p of W k, fine. So, the A only depends on the eigen spaces, it does not depend on the eigen vector that is very important and we are able to write decompose them in this form that is more important that there is this i and what are these P i’s?

So, understand P W i's are projection matrices and projection matrices means so, here for us we are looking at orthogonal projectors only. So, we are saying that P W transpose or here it will be a star because you are looking complex is same as P of W i and P of W i square is equal to P of W i. So, so they are idempotent and symmetric, is that ok?

So, that is all for this lecture. Next lecture we will look at something more, alright.

Thank you.