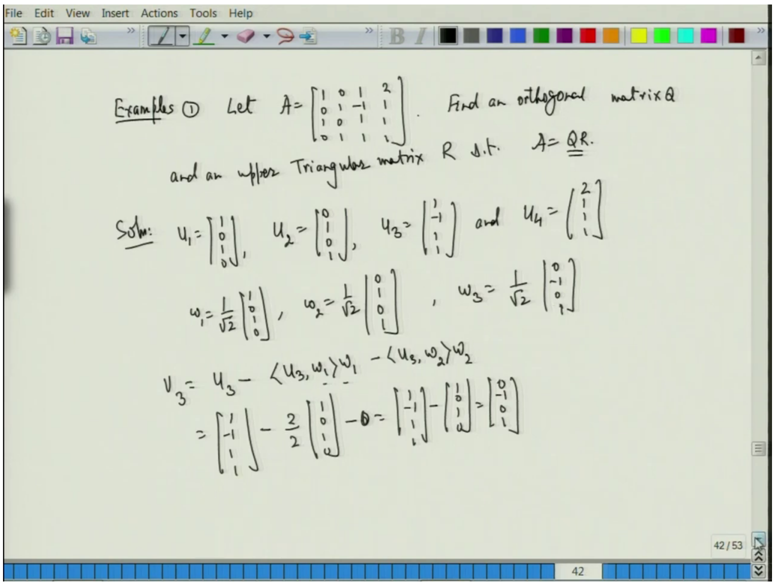
Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 49
Examples and Applications on QR-decomposition

So, we had done quite a few things in the previous few classes. What would like to do is that I would like to first do some exercises related with QR decomposition and if time permits then we will look at some ideas that I have missed otherwise I will do that in the next class alright. So, let me look at some of the examples for QR decomposition.

(Refer Slide Time: 00:40)



So, examples; so, the 1st example is let A is equal to 1 0 1 0 0 1 0 1 1 minus 1 1 1 2 1 1 1 alright, fine. Question is, find an orthonormal matrix Q. So, I write it as orthogonal because that is the way this written there in the orthogonal matrix Q and an upper triangular matrix R such that A is equal to QR alright.

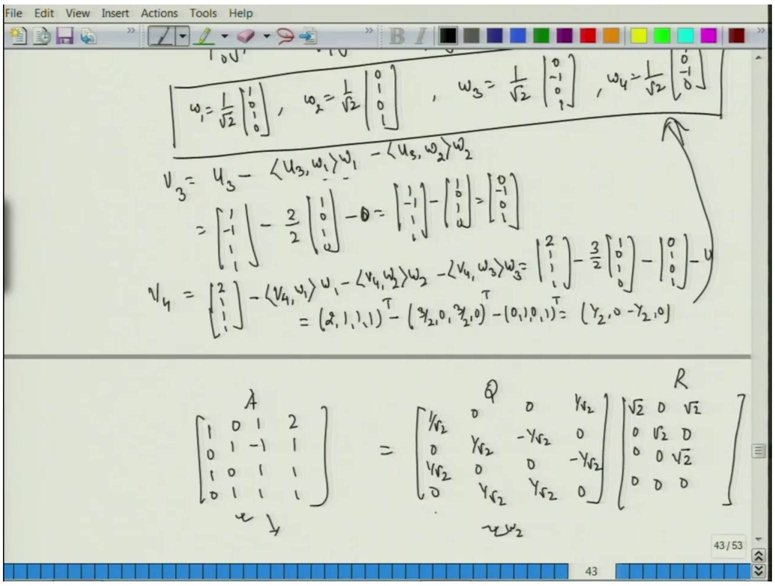
So, let us look at the solution. So, now, the ordering is fixed there is a first column, second column, third column and fourth column. So, we will have to go and do your calculation based on that itself. So, let me write u 1 as 1 0 1 0, u 2 as 0 1 0 1, u 3 as 1 minus 1 1 1, and u 4 as 2 1 1 1 alright. This is the way we are supposed to do. So, from here I get w 1 as 1 upon root 2 times 1 0 1 0. What about w 2? So, look at u 2 and this the inner product between u 1 and u 2 is already 0.

So, w 2 is equal to 1 upon root 2 times the same vector 1 upon root 2 times 0 1 0 1. If I look at u 3 alright, then u 3 is already perpendicular to u 2 look at this minus 1 1 and 1 1 that is 0. So, I have, but any how you have to compute what is v 2. So, let me compute v 2; v 2 is sorry not v 2 v 3.

So, v 3 is u 3 minus u 3 w 1 w 1 minus u 3 w 2 w 2 which is same as 1 minus 1 1 1 minus inner product of u 3 with w 1 u 3 with w 1 is 1 into 1 plus 1 into 1 that is 2 and this square root of 2 comes twice. So, you divide by 2 here, fine and multiply by that. So, it is 1 0 1 0 minus u 3 with w 2.

u 3 with w 2 if you look. So, 0 times 1 1 times minus 1 and 1 times 1 will give me 0. So, it is a 0 for us. It does not give me anything. So, it is nothing, but 1 minus 1 1 1 minus 1 0 1 0 which is equal to 1 minus 1 is 0 minus 1 minus is minus 1 1 minus 1 is 0 and 1 here fine. So, be careful I always do mistakes as I said. So, I define w 3 is equal to 1 upon root of 0 minus 1 0 1.

(Refer Slide Time: 04:24)



So, now, I have to compute what is v 4. So, let us look at what is v 4. So, v 4 is 2 1 1 1 minus v 4 with w 1 w 1 v 4 with w 2 w 2 and v 4 with w 3 w 3. So, this is equal to this vector 2 1 1 1 minus v 4 with w 1. So, it is 2 plus 1 I think 2 plus 1 is 3, 3 upon 2 times 1 0 1 0 minus v 4 with w 2; v 4 with w 2 will be I hope I have done correctly 2 plus 1 sorry 1 plus 1 is 2.

So, it is just 0 1 0 1 minus v 4 with w 3; v 4 with w 3 is minus 1 here and 2 minus 1 is. So, 0 0 minus 1 0 once it is 0 here, fine. So, this is equal to let me write in terms of the row vector 2 1 1 1 transpose minus 3 upon 2 alright. So, 3 upon 2 0 3 upon 2 0 minus 0 1 0 1 which is equal to 2 minus 3 by 2 is half alright. 2 minus 3 by 2 is half, 1 minus 1 is 0, 1 minus 3 by 2 is minus 1 so, minus half, 1 minus 3 by 2 is minus half and that is 0, 1 minus 1 is 0. So, this is what I get, fine.

So, this will imply that w 4 is 1 upon root 2 times 1, 0 or since I am writing in columns so, let me write column itself with a space 1 0 minus 1 0 alright. So, this is what I get w 1 w 3 and this fine w 4 is that ok? So, therefore, what I get is Q 4 or the matrix Q is nothing, but look at. So, I think I should write somewhere here.

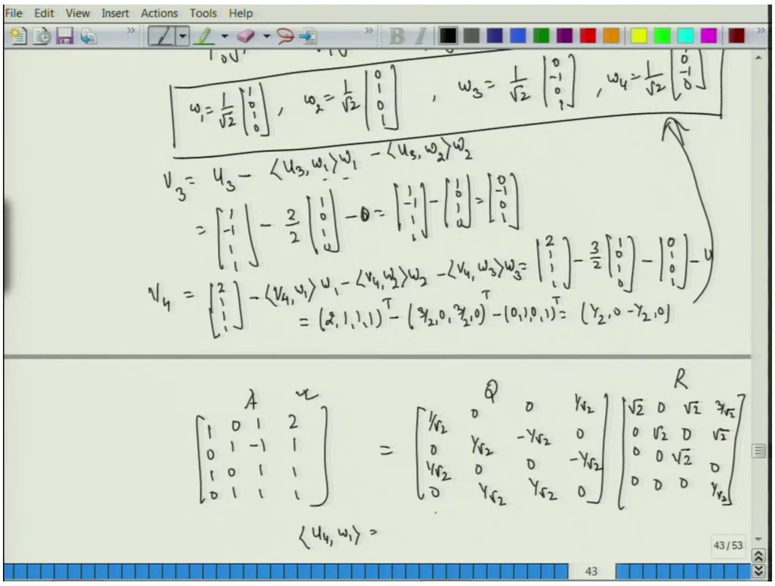
So, Q is 1 upon root 2 0 1 upon root 2 0 that is the first vector; second vector is 0 1 upon root 2 0 1 upon root 2. The third vector w 3 is 0 minus 1 upon root 2, 0 1 upon root 2 and the fourth vector is 1 upon root 2 0 minus 1 upon root 2 0. This is what Q is. I need an R here and I have an A here.

A is given to be equal to 1 0 1 0 0 1 0 1 1 minus 1 1 1 and 2 1 1 1 alright. So, to get the value of R as I said these are orthonormal. So, you just have to look at inner product of Q the first column with the first column here and so on, alright. So, if you look at the dot product here it is root 2 for u alright, fine. So, look at 1 upon root 2 plus 1 upon root 2. So, you get root 2.

So, this entry is root 2 and since I am looking at Gram–Schmidt process where the linear span remains the same till the i-th stage. So, I get this as 0 here there is no contribution from the second third and fourth column fine. Now, if I want to look at the second entry here, I will have to look at inner product of this with the second 1 and this with the second one. So, this with the second one is already 0; this only this one which is playing the role.

So, is nothing, but 0 root 2 0 0. Similarly, let us look at now the third one. So, I have to look at the inner product of this with this one. Inner product of this with this one is 1 upon root 2 plus 1 upon root 2 which is root 2. Inner product of the second vector that is w 2 with u 3 is already 0. This what we had seen. So, this part will be 0, fine. Similarly I would like you to see that this is 0 here, 0 here. Fourth one – let us look at the fourth one.

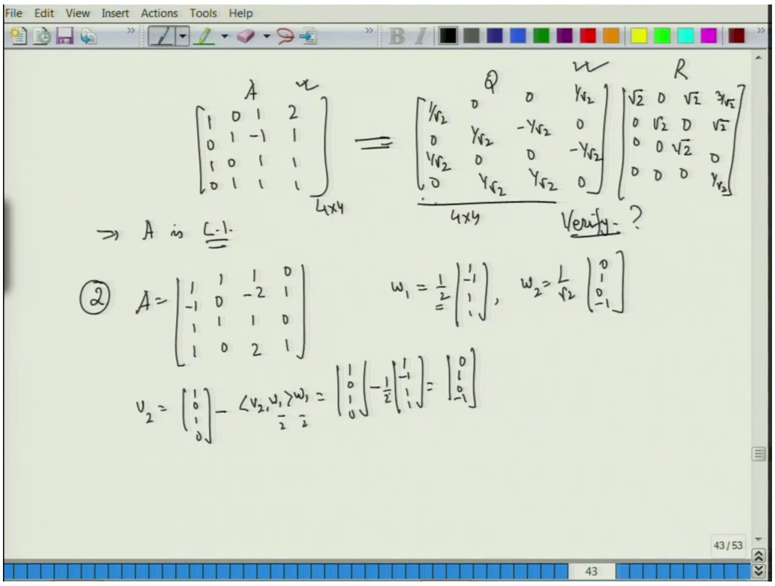
(Refer Slide Time: 09:11)



So, fourth one has lot of entries. So, be careful. So, I have to compute fourth one this is a fourth one. So, u 4 inner product with w 1; so, with w 1 the inner product of this is 2 0 1; so, 2 plus 1, 3 upon root 2 alright. So, it is going to be I am looking at this. So, this into this should be 3 upon root 2, 2 plus 1 did I write it correctly. I think I have written it wrongly there.

So, it should be 3 upon root 2 here fine. This with this one will give me 2 1 0 root 2 should give me root 2, fine. So, basically you have to compute things this with this will give me 0 minus root 2 0 minus here. So, that goes off this is 2 plus 1. So, it is 1 upon root 3 here, fine. So, 1 upon root 2 just look at it 2 0 1. So, 2 upon root 2 and 1 upon root 2 is 1 upon root 2 0 alright.

(Refer Slide Time: 10:15)



So, verify I may have done mistake as verify it because my notes show something else. So, I am sure that I have done a mistake. So, verify it alright very very important verification alright. You have to do that. So, what you have seen here is that here A I have got a Q which is 4 cross 4. This Q is also 4 cross 4 and therefore, this implies that A is linearly independent. This implies A is linearly independent, fine.

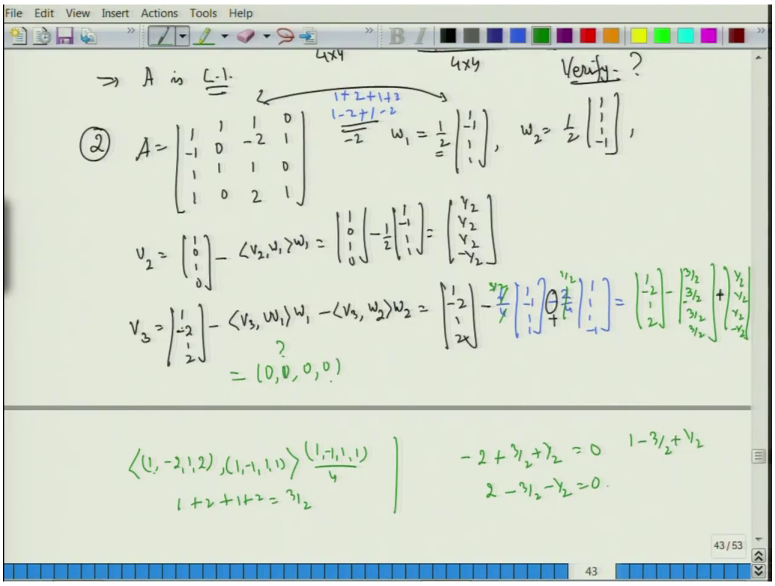
So, it gives me linear independence as well as a matrix which is orthonormal, Q is orthonormal and I have got the matrix Q fine. Let us look at another example slight change. So, example 2; I take A as 1 minus 1 1 1 1 0 1 0 1 minus 2 1 2 0 1 0 1. So, again I have got four of them – first, second, third, fourth alright. We will see what we need we want Q and R again, alright.

So, for me w 1 will be equal to 1 plus 1 2 plus 3 plus 1 4. So, 1 upon 2 times 1 minus 1 1 1. If I look at these two alright these two they are not perpendicular. So, I will have to compute v 2. So, v 2 is 1 0 1 0 minus dot product of these two which is 1 plus 1 2 basically with this w 1. So, v 2 w 1 w 1 which is equal to 1 0 1 0 minus this with this is 1 plus 1 is 2 upon 2 is 1.

So, it is 1 minus 1 1 1 which is 1 minus 1 is 0 1 1 minus 1 is 0 and this is minus 1. So, w 2 is 1 upon root 2 times 0 1 0 minus 1. I hope I am doing correctly because as I said I am good at it no I have got something else here in my notes v 2 is 1 0 1 0, yes. So, I get v 2 as 1 1 1 minus 1. So, where is the mistake? So, this with this I have to do v 2 with w 1.

So, 1 here 1 here; there will be 1 upon 4 here. See w 2 1 is a 2 here. So, there will be a 2 coming here 2 coming here. So, it will be 1 upon 4. So, it should be half here alright. So, as I said I am good at mistakes. So, I may have done similar mistake earlier also. So, be careful.

(Refer Slide Time: 13:01)



So, it is 1 minus half is half 0 minus minus plus is half 1 minus half is half and 1 minus 0 minus half is minus half alright, fine. Now, it is correct now. So, w 2 here is again. So, it is 1 1 1 minus 1. So, it will be 1 upon 2 times 1 1 1 minus 1 alright. So, this time my notes are correct I think. So, I have got this here, fine. So, I have got w 2.

Let us get w 3. So, I have to go to v 3 v 3 is 1 minus 2 1 2 minus v 3 with w 1 w 1 and v 3 with w 2 w 2 which is equal to 1 minus 2 1 2 minus this with this will give me. So, let us do this part 1. So, I am good at mistakes. So, let me do it I think 1 into 1 is 1, minus 2 and minus 1 is plus 2, 1 plus 1 is 1 and 2 here; so, 2 4 6, 6 upon 4 times 1 minus 1 1 1, fine.

With w 2 if I see it is 1 here minus 2 here. So, 1 minus 2 1 minus 2 plus 1 and 2 minus 2 here; so, 1 plus 1 is 2 2 minus 2 is 0; so, minus 2 upon 4 of 1 1 1 minus 1. So, this turns out to be I have written it as 0 1 0 1. So, let us see. So, this will cancel out. I get 3 upon 2 here, this will give me half.

So, I have got 1 minus 2 1 2 minus of 3 upon 2 minus minus plus 3 upon 2. So, with a minus sign here, then it is 3 upon 2 and 3 upon 2 minus half half half minus half and this is equal to 1 minus 3 by 2 and minus half that is minus 1 I think I get here minus 2 plus. So, it is plus 3 by 2. So, it will again be half.

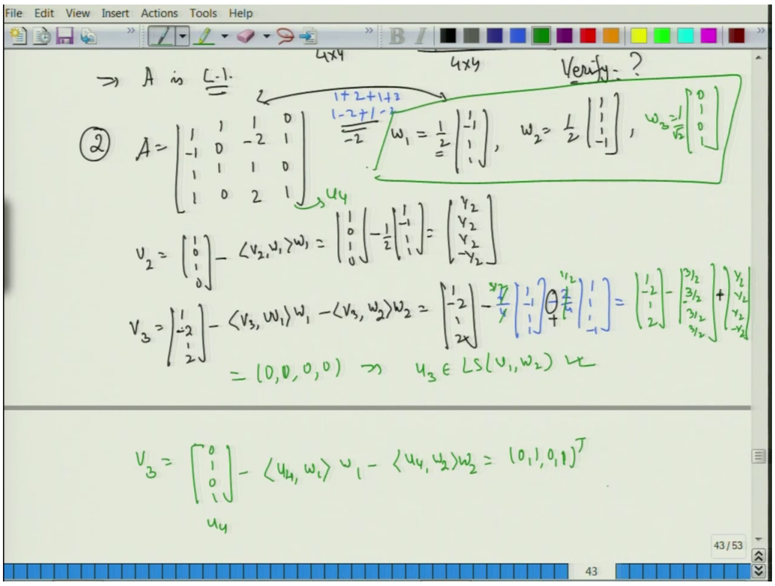
So, I think there is some mistake I think again 1 minus 3 by 2 minus half I am getting here and my note says something else. So, let me do it again I think as usual. So, I wrote it correctly I think 1 minus 2 1 2, fine. So, I have to look at inner product of 1 minus 2 1 2 inner product with 1 minus 1 1 1 times 1 minus 1 1 1 divided by 4.

So, this is 1 here plus 2 plus 1 plus 2 which is 3 by 2. So, minus 3 by 2 I wrote here 1 minus 1 1 minus 1 I wrote it correctly this part the other part was v 3 with this. So, it is 1 1 1. So, 1 minus 2 1 minus 2, so, minus half; so, minus minus plus. So, there is a mistake here. It should have been plus here because there is a minus minus here. So, there is a minus 2 here. So, it should have been plus here alright.

So, therefore, there is a plus here. So, 1 minus 3 by 2 and plus half is 0 fine. Then 1 minus minus plus 3 by 2, 3 by 2 and plus here, fine. So, look at this 1 plus 3 by 2 and plus half is that ok. So, it should be 2 here for me 3 by 2 plus 1. So, sorry it will be 1 here. Then what am I doing? Let me write it minus 2 plus 3 by 2 plus half which is 0. Why did I write here 1 here?

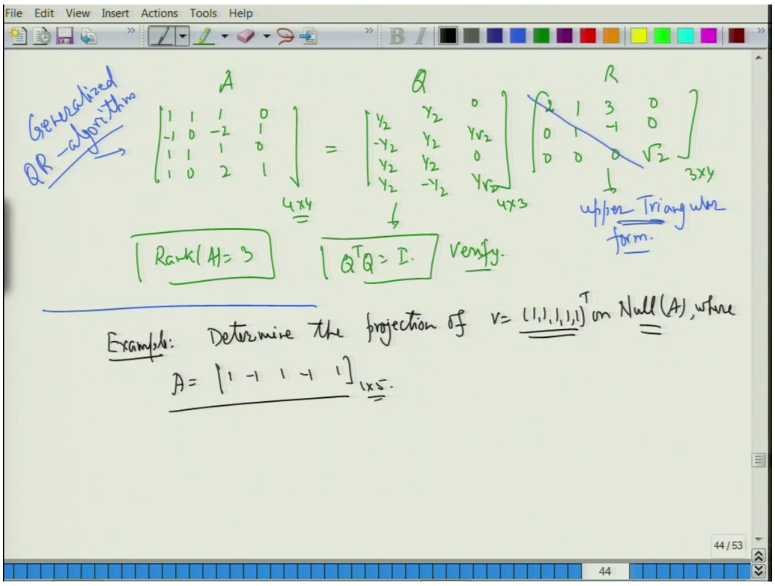
Again some mistake is there I think 1 1 minus 1 half I got it as minus 2. So, minus minus plus and half of half will come into play, that is minus 3 by 2 1 minus 1 1 1 here 0 1. So, let us see what it is fine. So, 1 plus 1 minus 3 by 2 whatever it is 1 minus 3 by 2 plus half 3 by 2. So, it is 0 here. So, check what it is this entry alright and 2 minus 3 by 2 minus half which is 0 here. So, sorry it is 0. So, everything is 0 here alright. So, everything is 0. So, it means the calculation was correct fine my notes are correct it means.

(Refer Slide Time: 17:56)



So, v 3 is 0 implies that u 3 belongs to linear span of w 1 w 2, this what we will see also afterwards. So, I take the next vector as my v 3 as 0 1 0 1 minus. So, this was my u four. So, I am looking at u 4 here; u 4 with w 1 w 1 minus u 4 with w 2 w 2 and this gives me my notes says that this is equal to 0, 1, 0, 1 transpose. So, I hope it is correct. So, I get that w 3 is 0, 1, 0, 1 and then there is this. So, 1 upon root 2 take care of this part. So, this is what my w 1 w is. So, try that out yourself I think.

(Refer Slide Time: 18:48)



So, what I get here is that I have the matrix A here, Q here, R here. So, Q for me is 1 upon 2 minus 1 upon 2 1 upon 2 1 upon 2; then it is 1 upon 2 1 upon 2 1 upon 2 minus 1 upon 2 0 1 upon root 2; 0 1 upon root 2 this what I have. A is already given to me that 1 minus 1 1 1 1 0 1 0 1 minus 2 1 2 0 1 0 1. So, I have to find this matrix Q. Again, look at the dot product of this and this and verify that this turns out to be 2 0 0 1 1 0 3 minus 1 0 and 0 0 root 2 alright.

So, please verify this yourself. So, what we are seeing here is that this is a 4 cross 4 matrix. This turns out to be a 4 cross 3 matrix and this is 3 cross 4, fine. So, this part tells me that the rank of A is 3 and I would like you to verify that Q transpose Q is identity just verify it verify alright. And, you can see that this matrix in general it is not upper triangular, but it has an upper triangular form. It has an upper triangular form, is that ok?

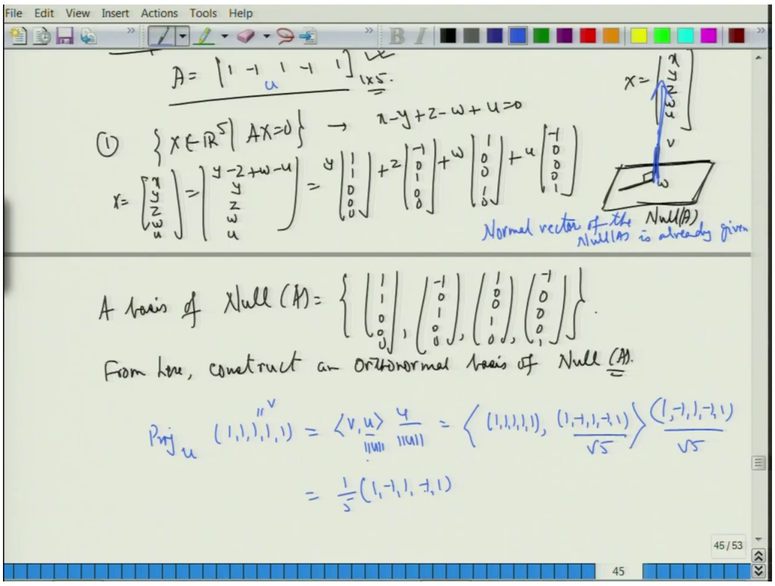
So, this is what you have. This what you have to be careful about that you should know how to do it and understand these ideas that even though I am looking at orthonormality and so on, at the back of my mind independence is always there. So, even if I have independent thing here in example 2 sorry, example 1 where it was linearly independent here I got Q which was of the same order as A there was no issue and similarly R was also of the same order or similar order you can say.

But, when it is dependent, then the vectors in Q become less the number of columns in Q become less that is equal to the rank and you do this, alright. So, this part that I had applied here is called the generalized QR algorithm QR algorithm and the previous one was the actual one, the QR algorithm. Is that ok? So let me write what exactly I did alright, fine or let me go to one more example and then I think I can end this and then look at general ideas alright.

The next thing that I would like to now understand is what was this question we did about projection; so, one example about projection example. So, determine the projection of v which is 1, 1, 1, 1 transpose on null space of A where A is the matrix 1 minus 1 1 minus 1 1 alright.

So, this matrix A is 1 cross 5 and I am supposed to look at the I think I should have written one more because it is have five components there should be five components here also fine. So, I have to look at the projection of this projection of v on A null space of A, is that ok? So, let us compute what is the null space of A and then proceed, is that ok? So, we will do it by a two method if I have time.

(Refer Slide Time: 22:39)



So, let us see. So, one is so, method 1 is look at what is null space. So, null space of A if I want to look at X belonging to R 5 such that AX is 0 this is the null space of A. So, this gives me x minus y plus z minus w plus u is 0. So, I am writing x as x y z w u I always do mistakes. So, I will do again a mistake. So, be careful.

So, what I get here is X is equal to x y z w u will be equal to X is minus X is y minus z plus w y minus z plus w minus u; y remains as it is, z remains as it is w remains as u remains. So, from here what I get is y times 1 1 0 0 0 plus z times minus 1 0 1 0 0 plus w times 1 0 0 1 0 and plus u times minus 1 0 0 0 1, fine.

So, I have got a basis of the null space. So, a basis of null space of A is 1 1 0 0 0 minus 1 0 1 0 0 1 0 0 1 0 and minus 1 0 0 0 1 fine this is the basis of null space, fine that is one way of going about it. We will see the use of this; from there we will get an orthonormal basis and then proceed. So, from here construct an orthonormal basis of null space of A, fine.

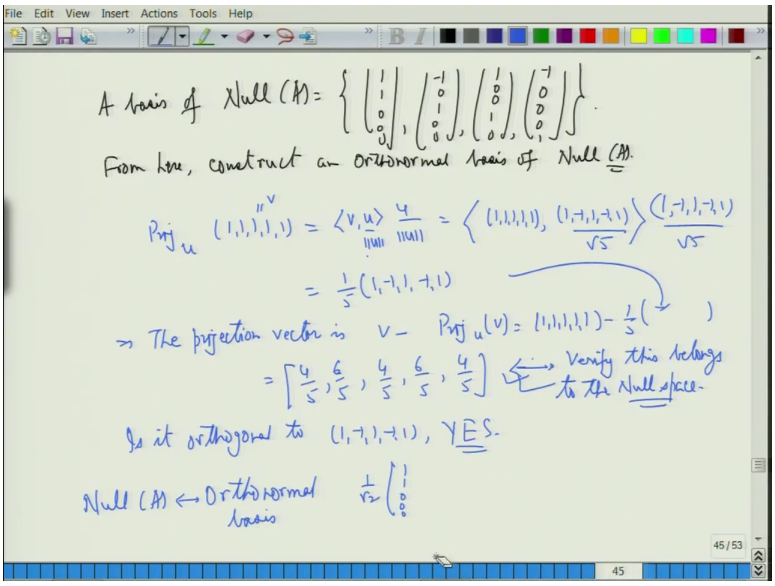
The other way is, so, as I said if you remember that I have this space with me. I have the vector v here; I am dropping a perpendicular here. So, this is 90 degree that I have. So, to get this vector w there are two ways – one is just look at the projection directly, the other way is compute this part alright. So, if I want to compute this part that is much easy for me because for this plane the null space of A alright there is only one vector which is perpendicular to it which is normal to it and there is already given to me alright.

So, the normal vector of this plane so, normal vector of the null space of A is already given alright. So, this vector this perpendicular vector the direction cosines of this is already given to me I can use this to get a projection onto this alright on this, fine on this vector and then subtract it that is another way of doing it. So, let us do that part first.

So, look at projection of the vector 1 1 1 1 1 on this vector whatever that is A for me is, fine. So, not let me not write a here. So, whatever it is. So some vector which is let me write it as u fine. So, I want to look at this. So, this was if you remember this was nothing, but. So, this is my v. So, it is v comma u divided by length of u into u upon length of u, this is what I was supposed to do fine.

So, here if I look at this into this is give me 1 minus 1 1 minus 1 1. So, it is 1 upon alright this is 1 upon something u was this is my u vector. So, the length of this vector is 5. So, it is 1 upon. So, let me write I think otherwise there is always a confusion. So, I am writing here as 1 1 1 1 1 with 1 minus 1 1 minus 1 1 divided by root 5 into 1 minus 1 1 minus 1 1 divided by root 5 which is equal to as I said 1 upon 5 times 1 minus 1 1 minus 1 1 alright; this is what it is.

(Refer Slide Time: 27:06)

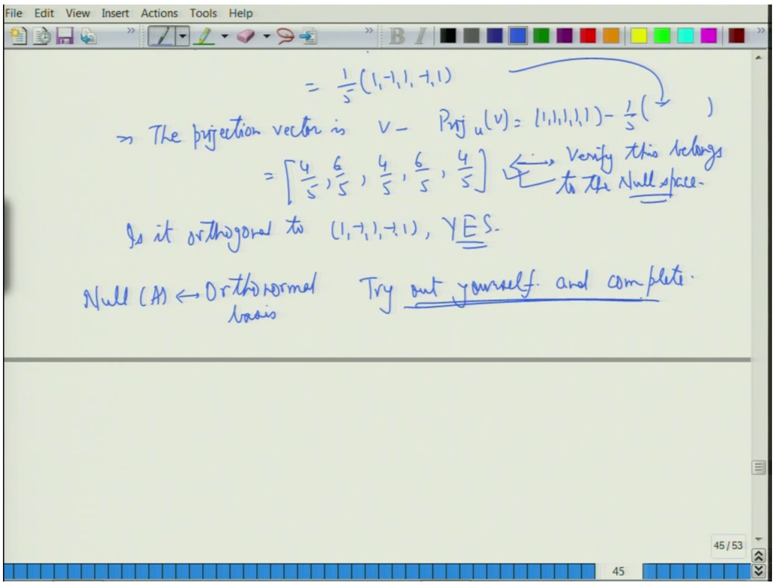


And, therefore, the vector that I am asking for the projection vector is the v minus this projection which is v was 1 1 1 1 1 minus 1 upon 5 times this vector whatever it is. And therefore, this turns out to be 1 minus 1 upon 5 is 4 upon 5 then 1 plus 5. So, 6 upon 5; then again 4 upon 5, 6 upon 5 and 4 upon 5, is that ok.

So, I would like you to see that this vector is indeed in the null space, so, alright. So, verify this belongs to the null space alright. So for verification what you are supposed to do? Just check whether this vector is perpendicular to this vector or not alright, fine. So, is this is it orthogonal to 1 minus 1 1 minus 1 1? Answer is yes. 12 minus 12 is 0, is that ok?

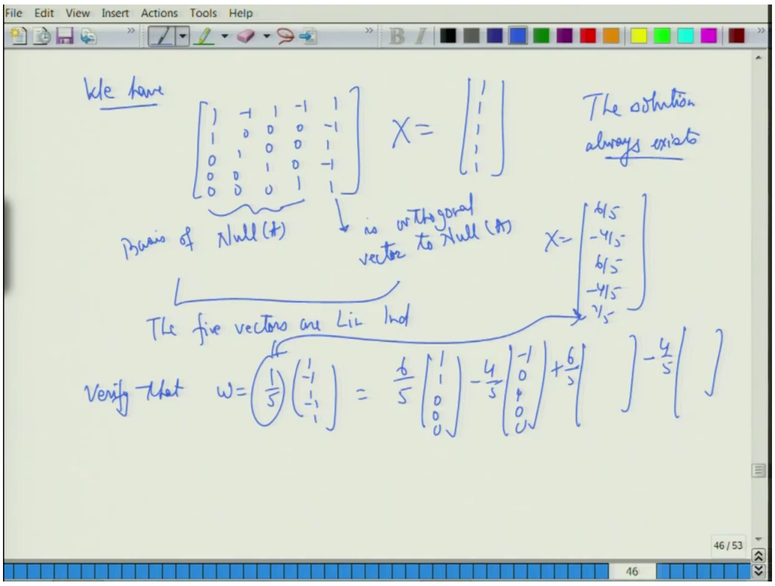
The other was look at the orthonormal basis of null space. So, I have computed that orthonormal basis null space of a orthonormal basis. So, orthonormal basis turns out to be why did I write here 1 upon root 2 times 1 1 0 0 0 or you can do. So, no I will just because time is not there.

(Refer Slide Time: 28:57)



So, try it out yourself try out yourself alright and complete. I will give you another idea which is also important. Therefore, I think I am asking you to do this.

(Refer Slide Time: 29:14)



The third idea is that we have null space of a the basis of null space of a as 1 1 0 0 0, another element was minus 1 1 and then 1 0 0 and that part minus 1, alright. So, I had minus 1 0 1 0 0 then I had 1 0 0 1 0 minus 1 0 0 0 1. So, this was about the null space of A and the vector that was given to me on which we will looking at things the null space was 1 minus 1 1 minus 1 1 alright.

So, if I look at this. So, this comes from the null space basis of null space of null space and this is orthogonal vector to null space of A or is it normal of it, alright. So, therefore, these five vectors, they are the five vectors are linearly independent alright. Since these are linearly independent, I can always solve the system this is equal to 1 1 1 1 1. So, the solution always exists the solution always exists alright.

So, I can solve it and it turns out that X is equal to. So, X turns out to be equal to 6 upon 5 minus 4 upon 5 6 upon 5 minus 4 upon 5 and 1 upon 5, this is what it turns out to be. So, I would like you to verify here. So, verify that w vector is already 1 upon 5 times 1 minus 1 1 minus 1 1 and this 1 upon 5 is coming as the last entry here.

This is also equal to 6 upon 5 times the first vector minus 4 upon 5 times the second vector, then plus 6 upon 5 times the next vector and minus 4 upon 5 the last vector is that ok. So, complete this argument yourself and verify. So, there are different ways of computing these things. I would like to try that out alright and learn it.