Linear Algebra
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Lecture – 36
Matrix of a Linear transformation

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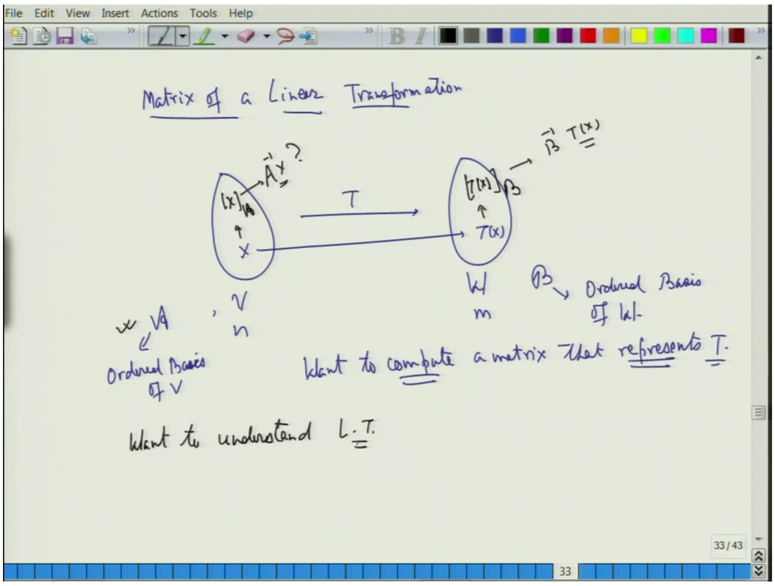

Alright, so let us start with the class today. What we are going to learn today is what is called matrix of a linear transform, alright. So, I have vector space V it is dimension is I am a starting with so, alright let it be n, fine.

I have another vector space W it is dimension is m and T is the linear transformation that I am looking at, but what is given to me, I have been given A here a basis ordered basis of V. I have been given another basis here which is the ordered basis of W, want to compute a matrix that represents T fine that is our aim here that we want to compute this, fine.

So, any x here is being sent here as T of x, fine. Now, x in this path looks like x of A which in some sense is A inverse of x alright, in some sense alright it is not exactly A inverse, but it is in some sense like this. Here I have T x with respect to B and therefore, I have B inverse of T x with me, is that ok? I want to relate these two. So, in the previous one when I went from A to B, two basis there I did not have T x, T x was x itself, now I want to understand these two, fine.

So, let us try to understand what we are trying to do, fine. So, I want to look at want to understand linear transformation, fine. So, let us do it slowly, so that you understand it.

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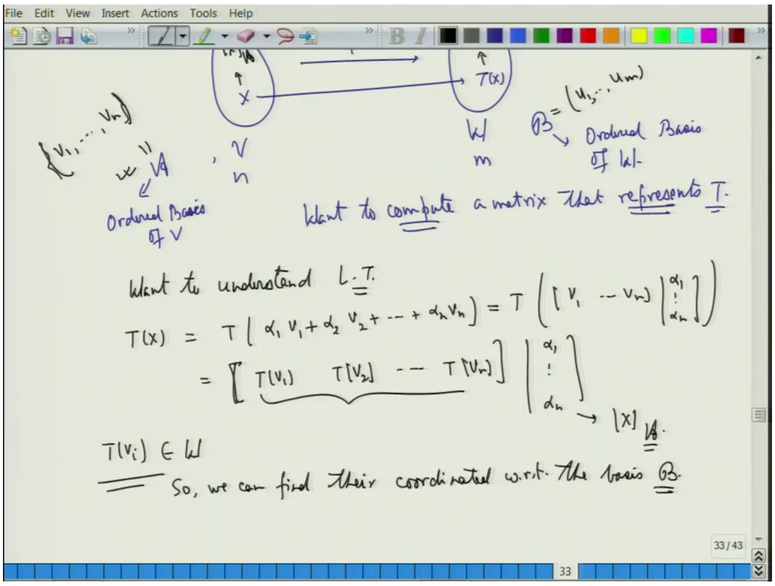

So, when I write T of x fine we write it as T of say some alpha 1. So, let the basis of this be v 1 to v n here and the basis here be this be; so, I think I should put this bracket. So, say u 1 till u m fine. So, alpha 1 v 1 plus alpha 2 v 2 plus alpha n v n this is, so that this is nothing, but T applied to recall yourself v 1 to v n and then it is alpha 1 to alpha n which was same as looking at T of v 1 T of v 2 T of v n and then, alpha 1 to alpha m, fine.

So, this is the vector which is x with respect to A, fine. Now, each of these T of v i weight belong to W. So, we can find; we can find their coordinates with respect to the basis B, fine. So, again what we have done, we have to multiply it by B inverse in some sense and then take out B inverse outside and do the work, fine.

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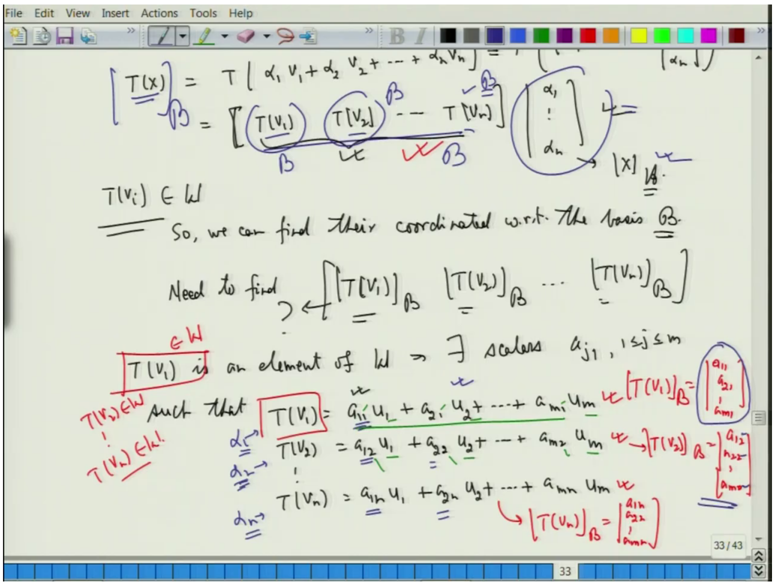

So, let us try to understand them nicely what we are looking at, fine. So, need to find T of v 1 with respect to B T of v 2 with respect to B and T of v n with respect to B, fine. This, what we need to find out and then something have will come out here and then will proceed, is that ok. This is what we had done for the ordered basis from A to B itself for the same vector space.

So, let me try to see it now in a different way, fine. So, what we have is T of v 1 is an element of W implies there exist a scalars a 1 1 or a i j or a j 1, 1 less than equal to j less than equal to m such that T of v 1 is equal to a 1 1 w 1 did I write it as basis as w 1 or u 1 u, alright. So, it is a 1 1 u 1 plus a 2 1 u 2 plus a m 1 u m.

Similarly, I can write T of v 2 as a 1 2 u 1 a 2 2 u 2 plus a m 2 u 1 1 u m 1 2 2 2 m 2 and so on till T of v n is a 1 n u 1 plus a 2 n u 2 plus a m n u m alright, understand what we have to trying to do? What we are trying to do is T of v 1 this is an element of W, similarly, T of v 2 will be an element of W each of these T of v n their element of W. So, I am writing them as linear combination of basis vectors, fine.

Once I have done that, then I can think of T of v 1 in the basis B as a 1 1 a 2 1 a m 1 this as T v 2 with respect to B will be equal to a 1 2 a 2 2 a m 2 and this will be T of v n with respect to B will be equal to a 1 n a 2 n and a m n fine, is that ok. And from there I can build up this matrix that I have started with, fine. So, let us do that one by one slowly. So, that you understand them nicely. You have to be very careful you have to understand it otherwise there will be problems, fine.

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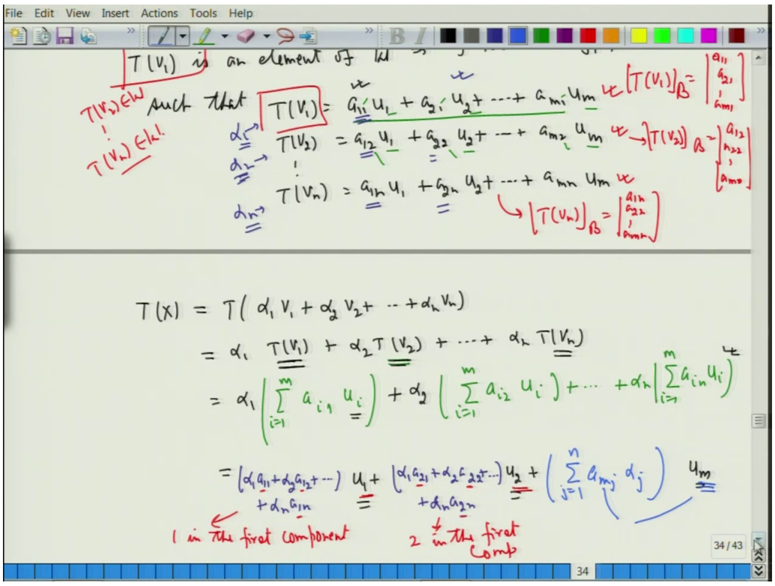

So, when I am writing T of I think I wrote it as T of v T of x I wrote T of x. So, T of x was equal to T of alpha 1 v 1 plus alpha 2 v 2 plus alpha n v n by linearity of T I have here alpha 1 T of v 1 plus alpha 2 T of v 2 plus alpha n T of v n, alright. This is what we had T of v 1 T of v 2 T of v n.

So, again look at here T of v n this is what we wrote, fine. So, that there is no confusion for you otherwise, there will be problems. So, this is same as alpha 1, now what is T of v 1? Look at what is T of v 1, T of v 1 was let us write this as linear combination. So, what I have is summation something a here u i here i is going from 1 to m u 1 u 2 u m. So, i is going from 1 to m that is, fine.

Look at this 1 is here, 1 is here, 1 is here and the first part it changing. So, I have writing a i 1 here, is that ok. So, this is my T of v 1 similarly, I have plus alpha 2 fine, but then if I look at

T of v 2, T of v 2 is u 1 u 2 u m. So, again u i comes into play i is equal to 1 to m and these 2, 2 the second component is as it is, so I have this.

So, plus so on plus alpha n I going from 1 to m a i n u i fine, this is what we do. I do not want to get into, so let us understand slightly more now, now what I have done is that I have written T v 1 T v 2 T v n in terms of the basis of the right hand side of the codomain, fine.

Now, I want to write what are the coefficient of u 1. So, what is the coefficient of u 1 I want to find that, I want to find the coefficient of u 2 and I want to coefficient of u m their vectors u 1 u 2 u m with me I want to find their components, alright. So, look at here this is getting multiplied with alpha 1.

So, this is getting multiplied by alpha 1, this is by alpha 2, this is by alpha n. So, I will get alpha 1 times a 1 1 here alpha 1 times a 1 1 will come here a 1 1 plus alpha 2 will get with a 1 2 alpha 2 with a 1 2 plus so on plus alpha n will get multiplied with a 1 n alpha n with a 1 n, fine.

Similarly, when I look at u 2 here alpha 1 is getting multiplied to alpha 1 is getting multiplied to a 2 1, alpha 2 is multiplied to a 2 2 plus so on, plus alpha n is getting multiplied to a 2 n alright. So, understand it nicely here very very important I am looking at 1. So, look at 1 where 1 comes, 1 comes here, 1 is here, 1 is here. So, 1 is appearing in the first component 1 in the first component, fine. This is u 2. So, 2 should appear in the first component, fine; so, 2 in the first component.

Similarly, here I should have. So, I can write here in summation j equal to alpha 1 alpha 2 till alpha n 1 to n, fine I can write it. So, 2 is fixed. So, a n j alpha j, can you see this? So, look at again nicely here I am looking at sorry, I am looking at a m. So, a m has to come so this should be m j a m j. So, this m is there. So, there has to be this m the first component, if you see here 2 2 2 here. So, this is m. So, m m m has to come and this alpha 1 is with 1 2 with 2 n with n, fine. So, m with m so, this is what I am going to get, fine.

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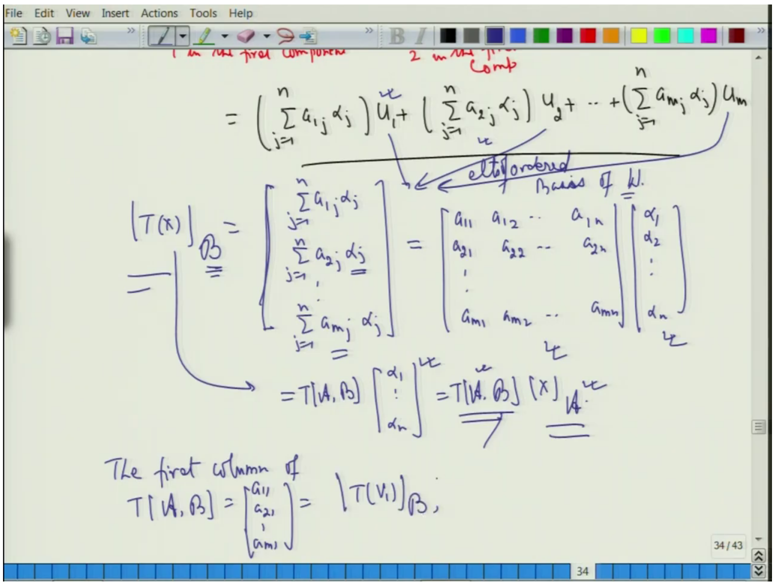

So, therefore, this is equal to the first part is fine. I do not know how to write for you. So, let me write use this notation itself. So, summation a 1 j so, a 1 j alpha j, j is equal to 1 to n times u 1 plus j is equal to 1 to n a 2 j alpha j u 2 plus so on plus j is equal to 1 2 m a m j alpha j of u m, fine.

So, once I have got T of x here as this now what are these? These are elements of ordered basis of W, fine. Since, there are ordered basis so, T x with respect to the ordered basis B will look like alright, it will look like i j equal to 1 to n a 1 j alpha j the first element of u 1 is this. So, you have this, u 2 will give me this part summation j is equal to 1 to n a 2 j alpha j and the last one will be j is equal to 1 to n a m j alpha j alright, this is what I will get.

So, this is equal to look at the matrix product, it is a 1 1 a 1 2 a 1 1 alpha 1 a 1 2 alpha 2 fine, a 1 3 alpha 3 till a 1 n alpha n. So, this what it is, alpha 1 alpha 2 alpha n. Look at this part it says a 2 1 alpha 1 1 a 2 2 alpha 2 a 2 n alpha n and similarly, the last one is a m 1 alpha 1 a m 2 alpha 2 a m n alpha n, alright. So, I have got that T of x looks like this matrix times this, fine. So, this is equal to so any linear transformation T of x looks like a matrix A times alpha 1 to alpha n and this was nothing, but a times this vector was x of A, alright.

So, I do not want to confuse this A with that A. So, let me write different notation here the notation that we follow is T of A and B here, fine. So, it is T of A and B times x of A. So, recall what you used to get cancelled out, the top one is to cancel out. So, this will cancel out with this one; you will be left out with this. So, this what you are left out with, you want to compute with respect to B you look at the domain, domain has x A. So, A has to be written in terms of B and then you have to compute it fine.

So, let us again go back and understand it nicely, this is the crux of the things you have to be careful you have got T of A, B, what is T of A, B? So, what was a 1 1 a 2 1 a m 1? So, look at this part, this was the first row sorry, the first column. So, T v 1 of B is a first column. So, I would like you to see here that the first column of T of this is a 1 1 a 2 1 a m 1 which is nothing, but T of v 1 with respect to B, alright.

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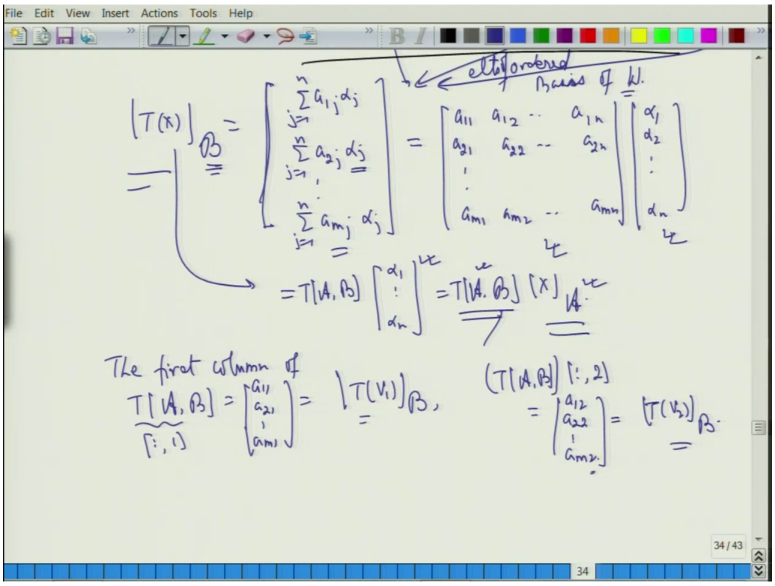

The second column, the first column when I write it is basically this comma 1 that I am looking at, the second column alright is nothing, but a 1 2 a 2 2 a m 2 which is nothing, but equal to again same thing T of v 2 with respect to B, alright. So, see here 2 2 is there on the second part, alright. So, let us go back and see here this is what it is, 2 2 the second part is the same, is that ok.

So, you can see that what we have got makes sense. So, what you are doing is, so we wanted to compute T of x. So, we wrote T v 1 T v 2 T v n, this was our x of A that we already wrote here, we have to write this with respect to B, alright. So, if I wanted to have this B here I have to write this with respect to B here, is that ok. So, we are writing B with respect to B means I am writing this with respect to B this with respect to B, this with respect to B, alright. So, it is v 1 v 2 v n which are playing the role, fine. So, this is the rows which are coming from here, fine.

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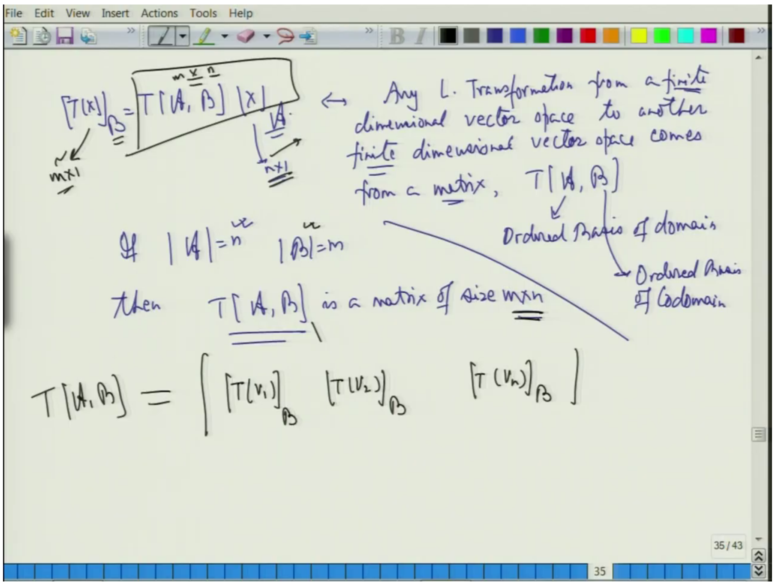

So, in the nutshell what I am trying to say is that there is a notion of what is called T of A, B. So, T of A, B means, so what we wrote was T of x with respect to B was equal to this times x with respect to B. So, we are saying that that any linear transformation; any linear transformation from a finite dimensional vector space to another finite dimensional vector space alright, one finite dimensional to another finite dimensional vector space comes from a matrix.

Now, what is this matrix? T of A, B A is ordered basis of domain, B is ordered basis of codomain, fine. And, so if number of elements in A is n, number of elements in B is m then T of A, B is a matrix of size m cross n, alright. How do you get the sizes, again understand there was a mistake here alright, it should have been A here because x is an element of the domain. So, there is a mistake there. So, T A, B here. So, x is an element of A it has got size n here. So, this is an n cross 1 matrix fine, this is an n cross 1 matrix.

So, if I want to multiply these two n cross 1. So, there has to be n columns here, I have, alright. This is a matrix of size m cross 1 because B has m elements here. So, this is a matrix of size m cross 1. So, this n comes from this n and this m will give to this. So, it is an m cross n matrix, is that ok. So, therefore, I got this as an m cross n matrix is that ok.

What are the elements of T A, B? So, the elements here are, look at elements of A, elements of A are v 1 v 2 v n alright write them with respect to B and close it. So, this is what you have T of A of B is, fine. So, I would like you to understand them nicely fine let us take some examples, alright.

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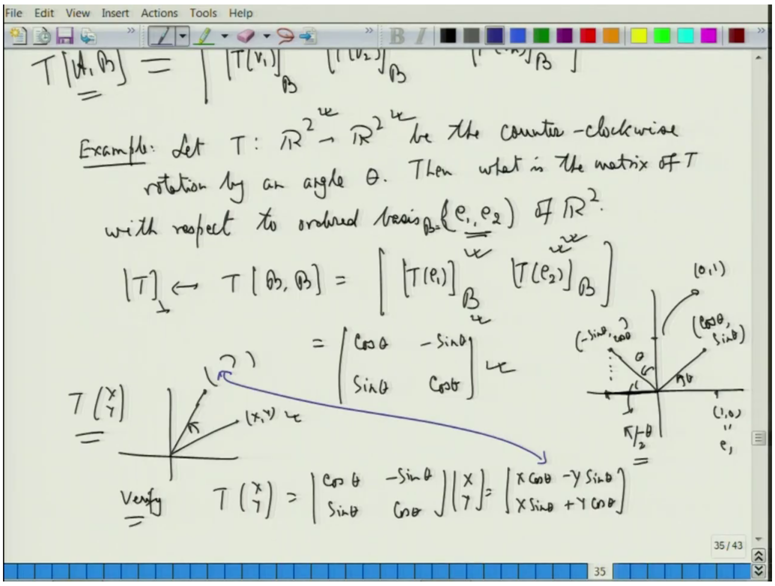

So, I will not go like this I will do it in different ways. So, let T be the let T from R 2 from R 2 to R 2 be the counter clockwise; be the counter clockwise rotation by an angle theta. Then, what is the matrix of T with respect to ordered basis e 1 e 2 of R 2, alright. So, what exactly we need to do? We need to compute so, if I want this matrix T. So, ordered basis I will not write anything, here I wrote T of A, B.

So, this will indicate this T of; so, if I write this as B T of B, B because I am going from R 2 to R2 itself so, B, B itself. So, R 2 supposed to look at T of e 1 I am supposed to look at T of e 1 with respect to B T of e 2 with respect to B this is the matrix that I am going to look at, this is what we saw T of v 1 T of v 2 T of v n. So, T of e 1 T of e 2 and with respect to B so, what is this?

So, let us go back here I have the vertex here as 1 0 this is my e 1, fine. Now, if I rotate it by angle theta counter clockwise this is my theta, then this point will be cos theta sin theta, fine. So, therefore, this vector in the basis B, what is the basis B? A standard basis itself it is nothing, but cos theta and sin theta, fine. The second vector is e 2 so this is my e 2 which is my 0 comma 1, I would like to rotate it by angle theta here. So, what is this vector? Fine.

So, if this vector is theta so this is nothing, but pi upon 2 minus theta and therefore, this vector is. So, if I look at this the x component here is this part fine, which is negative here and it is nothing, but cos of this and cos of this means it is minus of sin of this. So, it is minus sin theta because cos is negative in this component, fine. Similarly, the vertical component will be sin of this; sin of this is same as cos theta, fine.

So, this vector is nothing, but complete it yourself, again see x component I am looking at x component is negative because this is the origin is negative this is the angle pi upon 2 minus theta. So, it is cos of this; cos of this is nothing, but sin theta with the negative sin. So, I get minus sin theta and similarly I will get cos theta here, alright. So, overall if I want to see it is pi minus theta, but I am just looking at this part so, this is pi by 2 minus theta.

So, I get this as a matrix is that ok. So, you can easily see that if I want to if I had wanted to compute T of X Y for a general setup it would have been a big job for me, because I will have to look at any point X Y here rotate it by angle theta and then find out the coordinates would have been difficult. But, in place of looking at any X Y it says that I just have to look at a standard basis and when I am looking at a standard basis e 1 and e 2 is a standard basis for me or any basis as such that I could do it.

So, just verify that verify T of X Y since I am looking at standard basis it is nothing, but cos theta minus sin theta sin theta cos theta times X Y which is nothing, but X cos theta minus Y sin theta and X sin theta and plus Y cos theta. So, I would like you to verify that these points are same, the point here and the point here they are the same, is that ok. You can do it differently also, but that is the way we do it, fine.

One more example or I will do the next example in the next class, alright.

Thank you.