Linear Algebra
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Lecture – 13
Row Reduced Echelon Form (RREF) Continued

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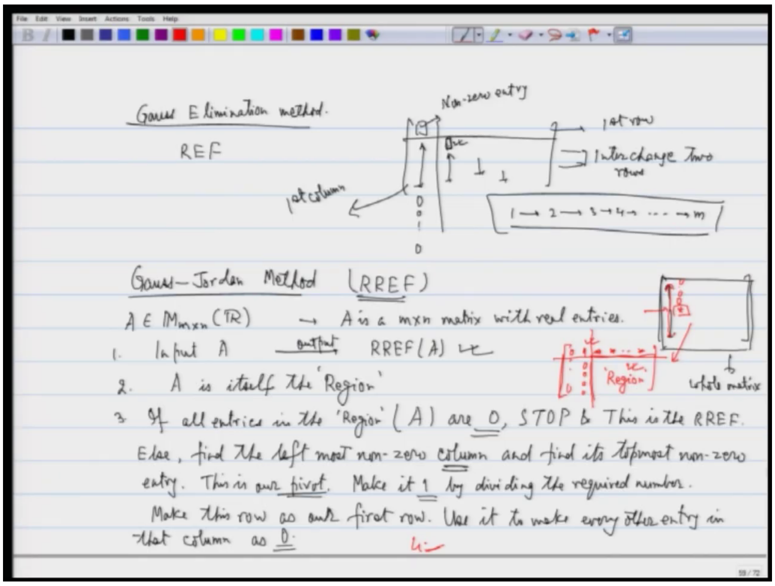

So recall that, in the last class we learnt is called the RREF. We have not yet been able to give an algorithm for that. So, in this class we will give an algorithm for that and what we will see is that it is a, it has lots and lots of application as per the understanding of system of linear equations is concerned. So, let us do what is called so, what we had learnt when solving system of equation was the Gauss Elimination Method.

We had learnt this, Gauss elimination method. And what was that Gauss elimination method? Where the matrix, you took the first one entry here which was nonzero entry. Used it to make every entry here to be 0. So, we made everything 0 here alright, that was the thing we did.

The next thing we looked at the second entry so we forgot about the 1st row and 1st column, then we came to the 2 2 entry of the new matrix. If this entry was nonzero, we took this as a pivot and made everything 0 fine.

If it turns out that this entry is 0, we can interchange two rows and then get a pivot and then proceed alright that was the idea of Gauss elimination method. But at each stage we just moved down, from the 1st row we went to the 2nd row, 2nd row to 3rd row, 3rd row to 4rth and so on, alright.

So, we just went down 1st to the 2nd, 2nd to 3rd, 3rd to 4th and so on, till the m th this is the way we proceeded alright. And, in the process we also looked at 2nd column, 3rd column, and so on fine. Now, would like to know what is called Gauss Jordan method. We want to understand what is called Gauss Jordan method alright.

So, Gauss elimination method gave us the REF; Row Equivalent Form alright. Now, this method Gauss Jordan method is going to give us the row reduced equivalent form alright. So, therefore, we need to be careful.

So let me write the actual without doing any mistake, the algorithms. So, algorithm is so, I have 1 matrix A which is given to be a matrix which is which has so may be a m cross n, with real entries or complex entries what it is.

So, A is m cross n matrix, with real entries fine. So, first thing is our input is A, input A and what will be the output? Output will be RREF of A. So whatever we get, final output will be the RREF of A. It will have all the properties of being RREF alright.

So, this is the input. The first step that we are going to look or the second step, whatever we want to say is the A itself A itself the region for us. Now, what is this region? Region means the place where I am going to work on. So, for me initially it is the whole matrix which is the region for me, the whole matrix alright.

Whole matrix is the region alright. Now, what do I do? Third, if all entries in the region or which is same as the A, all entries in A are 0, STOP and this is the RREF; this is the RREF alright. So, we have stopped if all the entries become 0 for us, alright fine.

Else, find the left most nonzero column. So looking at the column; the first column, second column, so and find its topmost nonzero entry alright. So what we do is that, if the matrix has all entry 0, we stop. We have the R R E F, if it is not, else you can find the leftmost column alright which has a nonzero entry.

In that column which has a nonzero entry, pick the topmost nonzero entry, that entry is our pivot. Take, find its topmost nonzero entry. This is our pivot; this is our pivot alright. Make this pivot 1, make it 1 by dividing by the number whatever it is, by dividing the required number alright.

So what we are doing? We have nonzero entry, this has become a pivot, we have also made it 1. Now, put it has or interchange this with the first row. So, interchange so make this the first. So make this row as our first row.

Use it to make every other entry in that column as 0 alright. So what we have done? I had look at this part, may be the first column consist of all 0's alright. So first column is all 0.

So you do not worry about it. It turns out that, suppose that the second column, I have a nonzero entry here; all these entries are 0. So what do we do? First, this is the pivot that I have, I multiply it by the number, put it back. So, what I get now is a matrix which has 0 here.

Now, there is a 1 here, and I do not know what they are. Some entries are there, I used this one to make every other entry 0 now, is that ok.

Once I have done that, I forget about this part and this part alright. At the next stage, that is at stage 4 alright, a step 4 what we are going to have is that, this becomes our new region alright. So, here we need to work on it. Is that ok? So, we go back to step 2; with this as our region and then do the algorithm again and again fine. That is what the algorithm is about.

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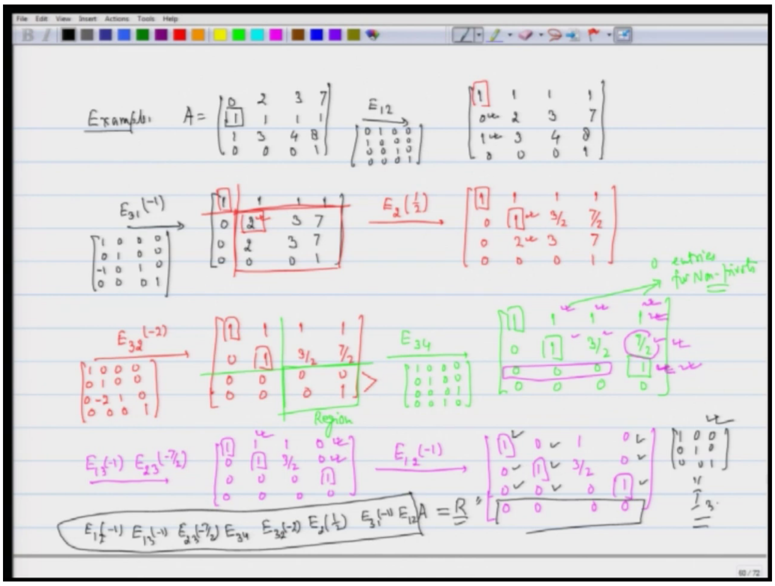

So, let us take an example to understand it better. So, example, so our matrix A is example, A is 0 2 3 7, 1 1 1 1, 1 3 4 8, 0 0 0 1. I have with me this fine.

So, at present what we see here is that, this has nonzero entry. So, the region is whole of A and the first nonzero entry here for us is this. So this is the pivot that we are going to look at.

So, what we are supposed to do at the next stage is, so do not write A here, because now the matrix is going to change. So, it is going to look like 1 1 1 1, 0 2 3 7, 1 3 4 8, 0 0 0 1 fine.

So what we have done exactly is we have interchanged the first and second rows fine. At the next stage, I already see that this entry is 0, we may to need to make this entry 0. So we just look at, what do we need to do? We need to replace the 3rd equation by doing something with the 1st equation.

And what do we do? Find this minus this; so minus 1. Is that ok? Which is same thing as looking at, if a from matrix point of view if I look at, here the matrix was 0 1 0 0 1 0 0 0 0 0 1 0 0 0 0 1. This was the matrix, 4 cross 4 matrix on the left.

Here I need to do; first row there is no change, so as it is. Second row there is no change, third row is minus 1 times the first row, third row is this, and this is the fourth row that we are looking at, fine. This gives us the matrix 1 1 1 1 0 2 3 7 0 3 minus 1 is 2 4 minus 1 is 3 8 minus 1 is 7 0 0 0 1, this is the matrix I get.

So, now at this stage we have taken care of this part fine. So once we have taken care of this, our new region is now only this part and we need to implement our algorithm only to this part fine. So here, what we see is that there is a 2 here, so we need to divide by 2 alright.

So, now we do the transformation which is the second row is being multiplied by 1 upon 2. So what I get is; 1 1 1 1 0 1 3 by 2 7 by 2 0 2 3 7 0 0 0 1 fine. At the next stage, oh I forgot to write the pivots here. So this is a pivot for me, this was the pivot, these are the 2 pivots that I have fine.

Now, I need to use this one to make this entry 0, the next entry is already 0. So, I need to look at the 3rd row is been replaced by again the 3rd row is been replaced by using the 2nd row, which is 1 here. So, I have to put minus 2 here fine.

So, it is just 1 as a pivot 1 1 1 0 1 3 by 2 7 by 2. I am multiplying by 2 and subtracting, so I get here 0 0 0 0 and the last one remains as it is, which is this fine. So here the matrix is going to be, there is no change in the first row; there is no change in the second row; the third row is being replaced by using the second row minus 2 1 0 fine this is what I have fine.

Now, what I see is that these two; the last row is supposed to be the 0 row, so this is 0 row. It has to be the last row and therefore, at this stage this is my region that I am looking at fine. And here, 1 is here, so we need to interchange, so let us do that.

So, how do I interchange? So Interchange this interchanging the 3rdand 4th row, which is nothing, but 1 0 0 0. No change 3rd row there is a change and there is a change in the 4th row. So this is a matrix that I am looking at. And the corresponding entry here is 1 1 1 1 0 1 3 by 2 7 by 2 0 0 0 1 and then 0 0 0 0 and this is the pivot that I have. I am interchanging these two.

Now, the idea is that I need to make so this was in some sense our Gauss elimination method in some sense, but making ones here as such fine. What we need to do is that, we need to make every entry here to be 0 here, 0 here also, sorry not here, 0 here and 0 here; 0 entry for non pivots. This is what we need to do alright.

So what do I do? At this stage, let us use another ink. So, I am going to write it at one go now. So, what I am going to do is, I am going to use this one to make this entries 0, and this entry 0 fine. So, I can multiply by, so I want to make this entry 0, the 2nd row; 2nd row is being replaced by 2nd row 3 fine.

And look at this, 7 by 2 here, so I have to look at minus 7 by 2 here, because there was 1 here fine. So this will give me, this part will be 0 fine and I can get this also 0 by looking at E 1 3 of minus 1 fine. I can do it together, basically because every entry here is 0 fine. So when I subtract, when I multiply this by 7 by 2 and subtract, nothing happens to the other two entries fine.

Similarly, when I multiply this by 1 and subtract, this will become 0, but these two entries will remain as it is fine. So therefore, I can do things at one go here itself. so I can write here 0 0 0 0, 0 0 0 1, 0 1 3 by 2 0 1 1 1 0 alright.

And not only that, you can also see that, if I want to play with this now, I can play and there will not be any change in these 0's now, alright fine. So at this stage, now I can do this, I want to make this entry 0. So the first row is going to be changed using the second one with a minus sign and I will get it as 1 0 1 0, 0 1 3 by 2 0 0 0 0 1 0 0 0 0 and these are the pivots that I have.

And you can see that these pivots they are 3 pivots, so I get 1 0 0 1 0 0 0 1 0 and then 0 0 1 here, so I get I 3 because there were 3 pivots alright. So please go through it very very important that the 0 rows are at the bottom. The first nonzero entry are the pivots and they are 1 for us fine. And every other entry in that row, in that column is a 0 and because of that we are getting this. So this is very very important.

That because of the condition that every other entry in the pivotal column is 0 implies that we get the requisite identity fine. And then we can write it as, so if I write this matrix as R, the final matrix as R then R is nothing but, just apply A was there, on A I applied E 1 2; then after E 1 2; I applied E 3 1 of minus 1.

Then I applied E 2 of half; then I applied E 3 2 of minus 2; then it was E 3 4; then it was E 3 4; then these 2 you can write it whatever way you want. E 2 3 of minus 7 by 2; E 1 3 of minus 1 and the last one is E 1 3 of minus 1 fine.

So, I have applied so many elementary transformations alright one A to get R. Is that ok? Now, if I look at these elementary matrices, they are product of elementary matrices as such, they are invertible and we know that product of invertible matrices are invertible.

So what we are saying is, let me rephrase it in the next thing. What we are saying is that, we have got this theorem, which says that theorem, right

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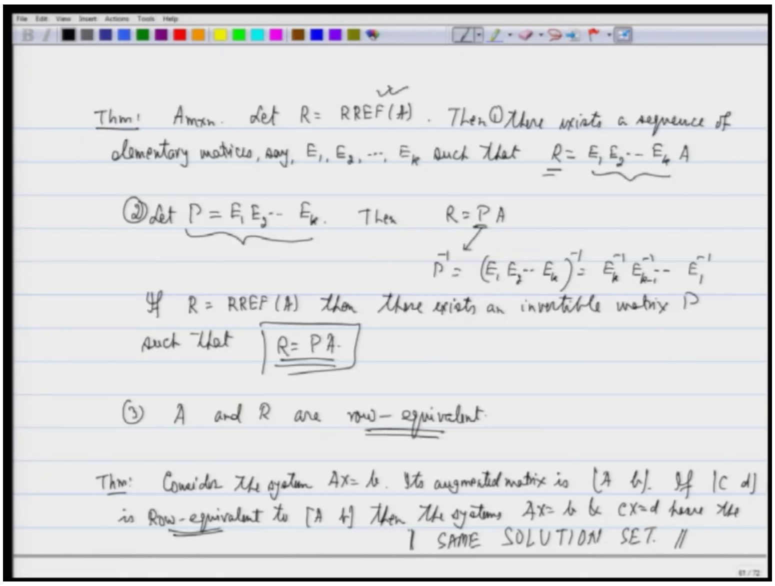

That A is m cross n. Let R be equal to RREF of A. Then there exists a sequence of elementary matrices, such say E 1 E 2 E k, such that R is equal to E 1 E 2 E k times A or we can say here is that, either this or this is one.

The second thing we can say is that, I can take this matrix define P equal to, let P is equal to E 1 E 2 E k alright. Then R is equal P times A and what is P? If I look at this matrix P, it is a product of elementary matrices. All of them are invertible, so the product is invertible hence P is also invertible.

So, nothing, but P inverse is nothing but E 1 E 2 E k whole inverse, which is E k inverse E k minus 1 inverse, E 1 inverse. So P is also, so what we are saying is that if R is equal to RREF of A, then there exists an invertible matrix P, such that R is equal P times A alright.

So, I will be using this idea again and again that given a matrix A, if I want to look at the RREF then that RREF is nothing but P times A for some invertible matrix P. I will not be writing it as product of elementary matrices. I will just say that I have an invertible matrix P so that P times A is the RREF. Is that ok? That is very important for us, and this also tells me that as a third (Refer Time: 21:00) that if you look at this RREF is nothing, but obtained through elementary matrices.

So the matrices A and R are row equivalent. So what was the definition of row equivalent? Go back to the definition, it says that A and b are row equivalent if I can obtain one from the other by multiplying by left multiplying alright or multiplying on the left by elementary matrices.

So this is what we have got. We have obtained R through multiplying elementary matrices on the left of A and therefore, the matrix A and R are row equivalent. And the idea is that whenever you have row equivalence, the solution set does not change.

I am not proving it, but I am just going to write this as a theorem that consider the system A x is equal to b, if so consider system this. Its augmented matrix is A b fine. So, if C d is row equivalent to A b then the system A X is equal to b and C X is equal to d have the same solution set alright.

This is very important for us, that whenever I go from one system to another system and the two systems are row equivalent, now this important, row equivalent that I am applying matrices on the left. I am multiplying on the left by elementary matrices which are invertible matrices, the solution set does not change. Is that ok? Fine.

So, the as the final I thing would like to state here and I will use it again in the next class is what is called that this theorem which I will not be proving it, which is actually very important that.

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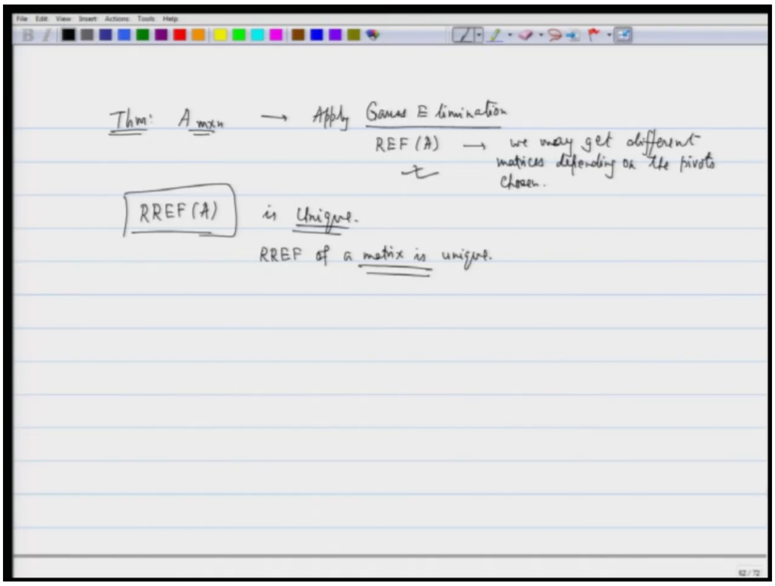

So, what happens, suppose you have matrix A which is in m which is a m cross n matrix. If you just apply, apply Gauss elimination or you just write the corresponding REF. So compute R RREF of A, then it may happen that we may get different matrices depending on the pivots chosen.

So, what we are trying to say here is that, if I look at my this matrix A, it may happen that I have a pivot in the first row, but if I have the pivot in the first row there is a huge number or there is a number which needs to be whosemultiplication may result in lot of fractions alright.

But there could be a row in which they are all ones as in the previous example, if they are all ones, if I have to subtract things or add things it comes a way much easier. So to do our process we may like to interchange fine different rows at whatever order we want and that will result in different types of matrices that I have.

So we may get different row equivalent forms for a given matrix A fine. It turns out that, that is not true as per as the RREF is concerned. So, whatever way you compute your RREF of A, whatever way you compute alright this matrix is unique alright. So, in whatever way you compute, whatever you do this matrix will always be unique.

And this very very important. We will be using this idea again and again that the RREF of a matrix; RREF of a matrix is unique alright. So keep track of these. This is very important, we will be using it again and again to understand lot of ideas alright. So, please note that we will be using this RREF idea again and again. So, keep track of that.

Thank you.