Calculus of Several Real Variables
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Lecture – 37
Facts about Vector Fields

As we have started with an introduction, in a course we end with a kind of epilogue that was a prologue and it is an epilogue, so here I have actually summed up 2 major courses; one is a calculus and real variables and one is a calculus of many variables of course, whatever I have spoken about 2 variables and 3 variables can be generalized to Rn.

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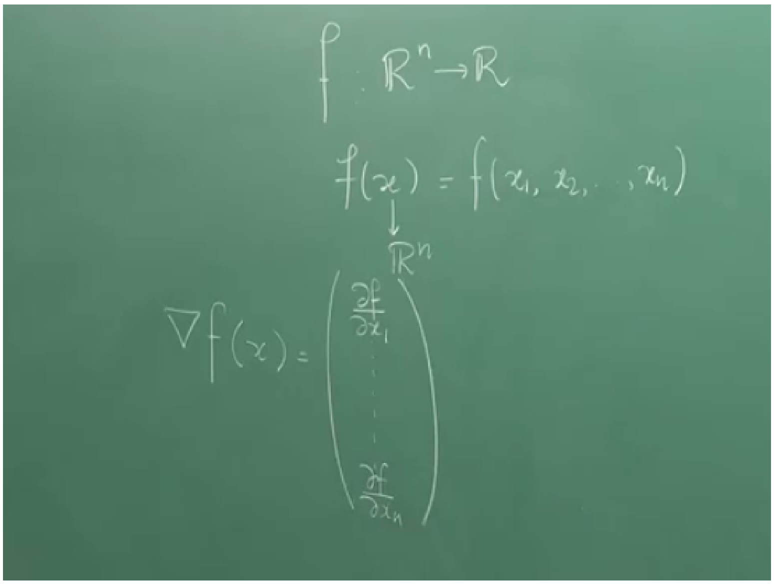

For example, if you take a function f from Rn to R, so it has n variables, so f of x where x is in Rn can be more descriptively written as which we do not do in general, so if you want to talk about a gradient of, f at a given point x, this is nothing but the vector of partial differentials, partial derivatives of first order; del f del x1, del f del x2 dot, dot, dot del f del xn.

So, whatever we have done here up to 3 dimensions can be now algebraically extended, geometry no longer gives you a view, algebra but algebraically, those ideas can be extended and so whatever we have learned for 2 dimensions, 3 dimensions most of the implicit function theorem Lagrange multiplier rule can be extended to many dimension; higher dimension, so these extension is possible that is what I want to say.

Summing up this whole 2 courses, it is a major course actually, I have hardly taught calculus in my own career maybe once and but it was fun to teach calculus to the extent that I have got received so much mails from various people and though I had promised a kind of notes in the first one, it has been difficult to write it down but what I now think is that I will put these notes in the form of a book which can be later on put on the net or put on the actually published as a book.

Because a lot of things, a lot of fantastic things can be told in one variable calculus at least the one variable calculus idea lectures can be formed into a book, there many, many things because I also would learn a lot as I go on because I believe in this dictum of Richard Feynman, the great American physicist Richard Feynman, many of you must have read surely looking for Mr. Feynman.

And hardly, there is a science student who does not know the name of Richard Feynman, is that there is nothing to unlearn, so this process so learning is not a learning process for me, it is not I am and all who is speaking here, I am also liable to make very, very fundamental errors, maybe there are some maybe I also fundamental misunderstandings in fact, I had always worried when I was a student.

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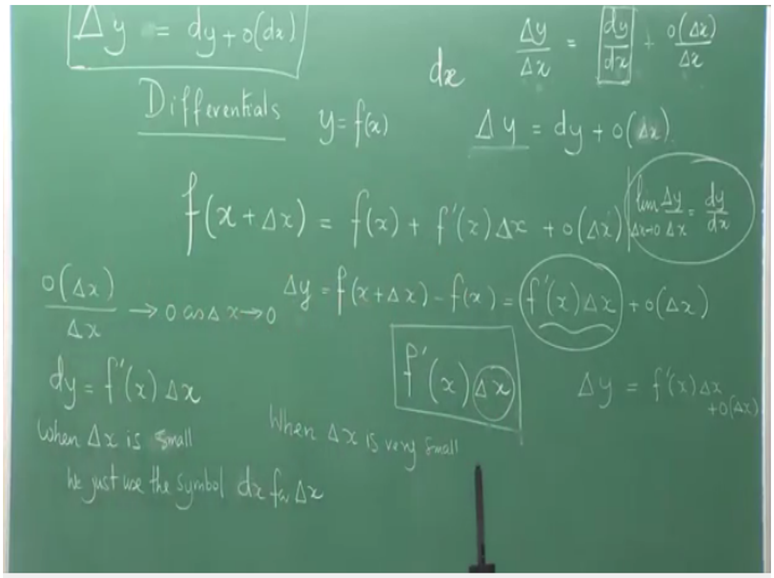

The people talked about differential, they handle this dx, these things as if they are playing with numbers, I always worried about these things, why do we talk like this and in fact Bertrand Russell, the great mathematician and philosopher who also had that feeling that calculus might not be so rigorous, though it is actually, it can be put in very, very rigorous framework.

Here, also we have used dx, dy, dz, so these differentials have to be clearly spelled out what are these, we have also used a term, infinitesimal, in mathematicians use this infinitesimal is a very scary thing it tells, it is smaller than the smallest number you can imagine, my goodness, smallest positive number that I can imagine, so if I do not think it is the only thing that it can be 0.

In fact, Euler; the great Euler has a calculus book, I think you would learn calculus first much better if you actually go and look at Euler's book, what Euler says is that guys, if when I; infinitesimally somebody talks about the infinitesimal the only thing that he said in this possible is true of course that the only possible infinitesimal is 0 but now if you look at the logical theory of infinitesimal the grades of infinitesimal and all those things.

But later on I found in a very beautiful book by written by a very famous mathematician Richard Courant and John; Fritz John, a great name in partial differential equations and also a famous name in optimization. So, here this; these 2 mathematicians written in a calculus book people; great, some great mathematicians have written books on calculus because calculus certain concepts needs actually very deep clarification.

So, they try to explain what is differential, so if you go by our standard notations, so if y is a function of x and if I make a Taylor's expansion of the first order; f of x + delta x is f of x + f dash x into delta x + some small o quantity of del x. So, what do I know that the only thing I know is that o of del x by del x goes to 0, as del x goes to 0, now I will bring this fx to this side and I can write f of x + del x - fx and this is what I will call as delta y.

Because y is equal to fx, the change in the value of y, the value of the function is f dash x + f dash x into delta x, f dash x and delta x plus the error, when delta x is very, very small, so what does this mean? This means that o of delta x runs to 0 or goes towards 0 faster than delta x can go to 0 because if delta x was going to 0 faster this value would actually keep on increasing, it will blow up towards infinity.

The fact that it goes to 0, tells us that this goes to 0 much faster than this can go to 0, so when del x is becomes very, very small, this is becoming also very, very small that is the key idea here. So, suppose I now look at only this quantity, so observe this quantity f dash x into del x. Now, when if you look at this quantity, when del x is very small, if del x equal to 1, this quantity is nothing but f dash x.

But when del x is very small, the change y is largely due to this factor because the error becomes very small, so the change in y is largely due to this factor. As a result, this thing, this factor is called the differential in y, it is defined as a differential in y and is symbolized as dy, so dy is defined as f dash x into delta x. So, when delta x is very small and then we also symbolize delta x, right.

We also write delta x, we write delta; we also give when delta x is small, we just use the symbol del x; dx for delta x and then that gives me and this one actually gives me dy is equal to f dash x dx, so actually it is f dash x into delta x. When delta x is 1, the differential is nothing but f dash y. So, when dx is; when delta x is very small, differential of y is the; differential of Y accounts for the main part in the change of the value of y.

So, delta y is actually the differential of y plus the error now, you can write here as o of dx also, this is; so this is the meaning of differential of y. So, if you not divided by delta of x, now if we have delta y by delta x, so it is dy by delta x, for delta x is dx and dx is very small so, o of dx in fact here delta x, we can just keep it as delta x; o of delta x, by I am writing; here I am writing dy dx, here I have delta x.

Now, take the limit so, if you take the limit as del x goes to 0, so dy represents this very small change, this change, this number is called; is denoted as dy, so this is nothing but dy dx + 0 because delta x by this; this part goes to 0, so it gives me back the definition of the derivative. So, this is the whole thing, so the key idea is that delta y, when x is very small, the main chunk of the change in y is affected by this part.

So, change in f or change in the function value is a differential of the function plus the error that is the idea that the change is nothing but the differential plus the error that is what it means, so this when I am talking about dy, I am essentially talking about a small number here and dx is a small number but when I am talking about ddx of y, here I am talking about this limit, this ratio now because this is; this will go to 0, this will go to 0.

This ratio you can say what happens in del x also goes to 0 but this is this is your f dash x because you have; this is your f dash x, you have actually divided of the del x, right, this is what happens. So, another way to; so delta y is equal to f dash x + delta x + o of delta x, you take divide by delta x and take the limit, it will give you f dash x. So, your f dash x, so here also you have the same thing.

So, delta y by delta x is if this plus this, so this does not no longer is depending on dx, you might say, Oh! Delta x goes to 0, dx also goes to 0 of course, so what here is the crux point that delta x is; this dx is just a symbolism for delta x, when delta x is very small and the error is 0 and that is why we write dy is f dash x into delta x. So, dy is not this dy, this is not dy, dy is this part.

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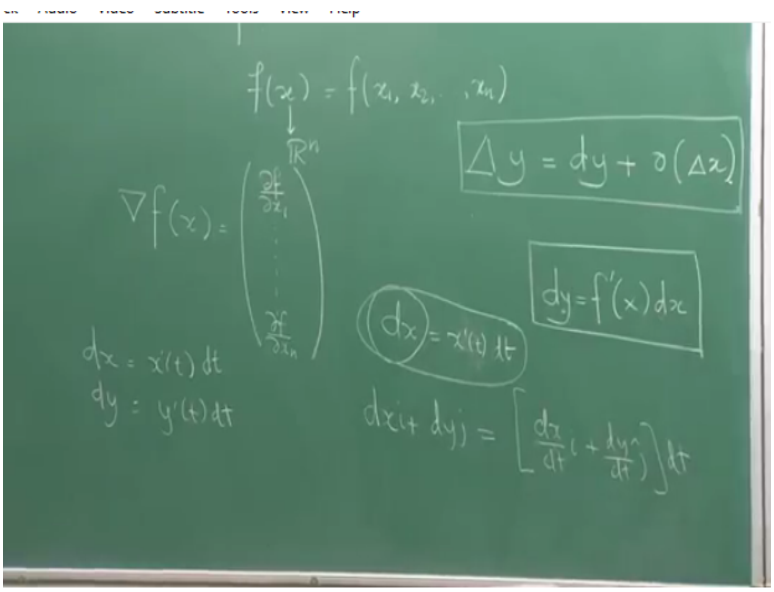

So, again I want to write that you have to remember here that dy, so this is nothing but this dy + o of delta x, this is something one has to remember, the change in the function value is the differential plus the error, so if I sum up the course, this is a very important thing that you have to keep in mind and then only we can you can be much more rigorous while you are working, right.

So, suppose people write so, dx is dx dt, dt, so the differential of x is the derivative of x with respect to t into dt, just this definition, so when suddenly you have dx here and people would say, you have dx, you have dxi vector plus dyj vector and these are functions of t, the x and y's, you will suddenly see them written as dx dt i vector plus dy dt j vector dt (()) (16:00), what happened you just multiplied that no, no it is not like that.

When you write dx, you are talking about the differential of the function x which is a function of t, so dx actually means x dash t dx dt into dt and dy means y dash t into dt and that is exactly what is written, the x dash t into dt y dash t into dt, take the dt out and here x dash t is nothing but dx dt, y dash t is nothing but dy dt, so that has to be; you have always applying this definition, please understand.

This is the definition of the differential, the differential is f dash x into the change in x that is the definition of the differential of the function value, so that is exactly what you are applying, it is not that multiplying dt here and dt here no, you are applying this definition whether it is dy, dx, we are applying this definition because here x is a function of t, here y is a function of t, it is the differential that we are talking about.

So, we are talking about the differential right, so when I am writing dx, I am talking about the differential, I am essentially looking at the large chunk of change in y, I am not looking at the actual change in y, I am looking at the change in y effected by the derivative. How much the derivative, to what extent the derivative plays a role in the change that change of value that y has or y that has taken place, what component; what is the contribution of the derivative to the change of the value of the y?

So, the value of the y has changed because x has changed, so when you look at that how much the y value has changed, the differential tells you how much what is the role of the derivative, how much is the role of the derivative in that change, how much change has been effected by the derivative and that is the key idea of calculus that we always look at the change in a function value through its derivative.

So that is and that is why derivative is a fantastic way to measure change and that is a very handy tool and that is exactly what we keep on doing, when we are looking for a change, when we write dx, we are essentially eager to know how much is the x value, y value or z value is changing as a function of t, how much in terms of the derivative, how much to what extent how much is the derivative part of the change.

So, you know that it is change in y is the differential plus dx, order of this; the error so, I am not bother about the error, I am essentially telling that I want to know how much here the derivative is there, it is here. So, how much change the derivative is effecting in the change of y, it is not giving me the total change but it is giving me a large part of the change because when del x is very small and the error is very small.

This is the key idea of calculus and this is what you have to keep in mind and so friends, it is time for me to say goodbye, it is time for me to thank all the people who has supported me this giving this huge course over 2 years of calculus 1 and calculus 2; 2018, 2019. I thank all the people in the office who had been kind to me, I have been late in many times arriving at, arriving for the lectures people have been very kind.

They have stood up with all my; they have tolerated all my idiosyncrasies, I have completed 5 more courses and that ends my relationship with the MOOC program, so as I go out of the MOOC program, I would like to thank everybody and if my behaviour has any way sadden anybody, I really want to say, apologize if in mind. If any student have been at the receiving end of my any of my behaviour maybe I have not answered their mail, maybe I have not satisfied them with their answer, they can still write back to me I would like to answer them back.

So, because there are so many other work, I might not be answering them right away but maybe because my email ID is public, you can definitely answer me back whenever you want you write me back whenever you want, so thank you goodbye have an excellent career ahead of, I am sure all the people who are looking at this course are very young brilliant sharp minds.

And I am sure you do very well in your chosen path whether it is engineering, mathematics, economics, physics, chemistry even biology, biology needs a lot of math nowadays, please understand do not; biology is also kind of mathematical science. So, thank you once again, so I would just tell you my last words to you would be love mathematics and mathematics would love you and help in your work and that is the only thing I can say and I am happy that I could share my love for mathematics with you, thank you