Calculus of Several Real Variables
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology – Kanpur

Lecture - 35
Example of Stoke's Theorem

So, we come to the session on problem solving for Stoke's theorem rather doing some examples on Stoke's theorems, how to apply Stoke's theorem. Please understand this, one has to keep in mind while applying Stoke's theorem, we have just written down this expression made a proof when z is in, when the surface is a graph of some function and so and so forth. It looks pretty nice, it does not using, everything seems to very rigorous and nice.

But the certain things we need to mention, first of all the orientation, which we have been stating and which we have been very careful about.

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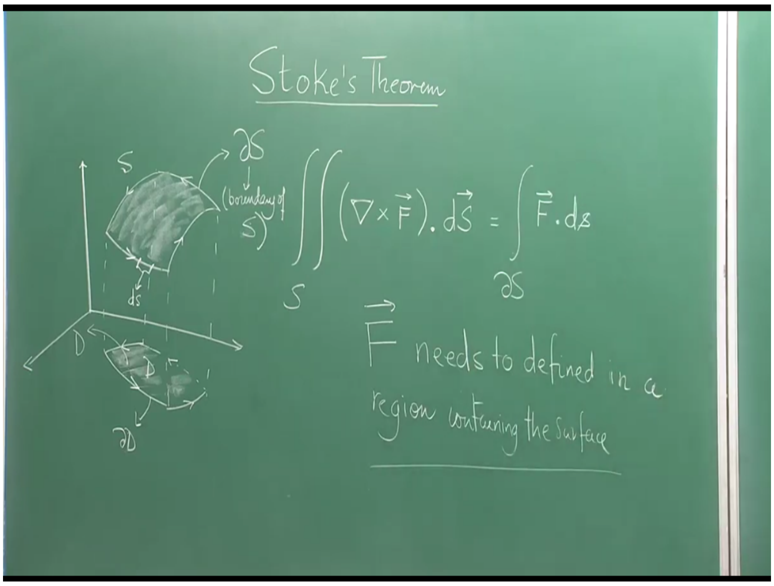

But you have to mention that this vector field F needs to be defined in a region containing the surface. This is a very important fact and if you miss this fact, things may go haywire. If you have a function F and you have a surface S and in the surface at every point of the surfaces the function is not defined as a function rather then this Stoke's theorem may not hold.

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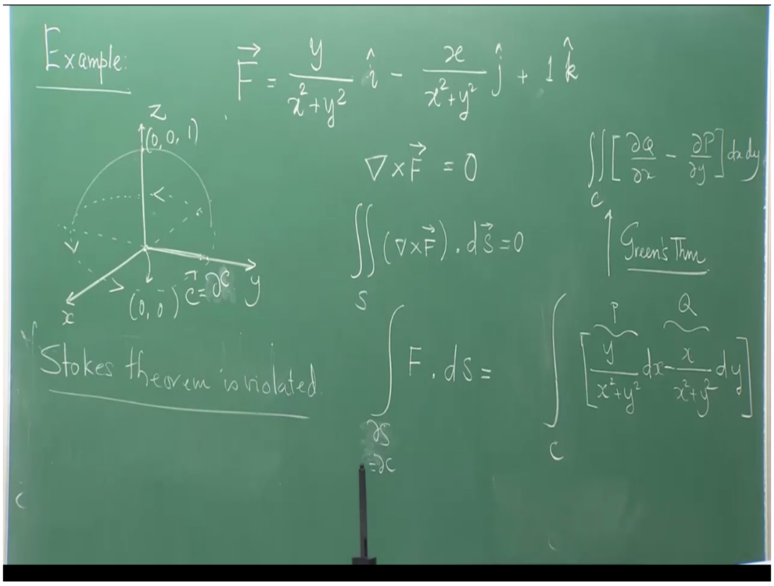

And that is exactly this example that I am going to give here now is the example which is given in the book of very good example which is very illustrative that if I put myself in a situation where there are points on the domain of D where the function F is not defined right or rather there are points where you cannot define the function of F at a given x, y, z, so then for example on this will show a function on the z-axis.

If I have a surface intersecting the z-axis and I have to construct a function which cannot be defined on the z-axis, then this thing may not hold true. Let us start with that example rather than doing the easier ones, let us start with an example like that. So, let me take a vector field given like this. Now, this function F, this vector function F is only defined, it only makes sense x square + y square is anyway greater than equal to 0.

But it will only make sense if x square + y square is strictly bigger than 0. Here, it is giving me a function from R3 to R3 with coordinates y this and 1 when a z is kept at the level 1. Now, if I look at the nature of the function, then this function is defined everywhere on the x-y plane except the point 0, 0. If x = 0 and y = 0, then the function is not defined. I would like students involved here were looking at these lectures.

For this particular case, calculate the curl of F. If you calculate the curl of F, you will realize by doing a little bit of computation that curl of F in this case is 0 okay. Now, suppose I talk about a circular zone S with unit circle, unit circle I take as the boundary of a surface. Let the surface be a hemisphere. Now, this hemisphere is intersecting the z-axis but for 0, 0 this function does not define any z but for 0, 0 this function the field has no value.

The field is not defined at this point 0, 0. Suppose, it cuts at 1 here, so F is not 0, 0, 1 at this point. This value if this 0, 0, 1 for example here does not correspond to F value because at this point 0, 0 this F is undefined. So, the function is not well-defined at a point on the surface. So, I have just taken it to be hemisphere but C is the boundary of the surface, so C is my del S.

So, by going by suppose going by the hemisphere does not matter because the curl is 0. One might show that is great, Stoke's theorem this will give me 0 but be careful. This little problem that you have that a function is not well-defined at that point 0, 0, 1. For example, if you take this point 0, 0, 1 which is a point on the z-axis. So, any point on the z-axis is 0, 0, 1 here. This is 1, always 1 so I am just looking at say hemisphere of radius 1.

So, if I look at the hemisphere of radius 1 and C is that domain where which is where we move in anti-clockwise fashion to remain positively oriented, then you observe that at that point 0, 0, 1 the function, here the field is not defined and that is the issue. In that case, the Stoke's theorem just might not occur but if you look at integral C F dot ds, so what did you give me?

So, obviously you have to do x = r cos theta, y = r sin theta and all kind of things like that and z = 0. So, if I write in terms of dx dy, so x is going from, so I have taken the unit circle 0 to 1, y is going from 0 to. So, if I take dx dy, x is going from - 1 to + 1, y is going from - 1 to + 1. See if I look at it, so I can write it in terms of if I compute this one, so what will give me, y of this is dx i vector + dz dy is j vector + 0 k vector.

You know should be able to compute this to, so you can apply again the Green's theorem which will be on that whole circular domain, you can apply the Green's theorem and you can compute out this. Double integral on this whole double integral on this whole region is actually computed out like this. So, sorry I made a mistake, so this is the way I can write it down F dot ds.

So, this is P, this is Q. I want you to actually go down and do this computation, apply the Green's theorem. Green's theorem means you have to do actually now instead of taking here I should see the unit circle, the unit circle C, this is actually your del S, my del C basically, del

S, this would be actually del C basically, I should write del C the unit circle is del C. Del C is the del S because this is equal to del C.

So, here you can write in terms of del Q del x in this case minus my del Q del x is minus of this okay. So, here in this case, you can apply the Green's theorem and right here the circular disc zone into so Q is minus of this, so del Q of del x - del P del y dx dy, dx is going from - 1 to + 1, y is going from - 1 to + 1. So, you apply the Green's theorem and you compute this. You can parameterize it that so or you can do it like this.

This particular case what you can do is you put x = cos t, y = sin t and z = 0 parameterize it like this. So, x = cos t, y = sin t and z = 0. So, then in this case cos t, sin t, 1, 1, sin square theta cos square t + sin square t is 1. So, it is sin t minus cos t plus, so the F dot ds is now ds is with just j, so it will become, my ds is dx + i dy, so now integral C F dot ds how do I write this one?

This is nothing but what is F here, F here is cos t i vector - sin t sorry x is so it is y sorry sin t i vector - cos t j vector + 1 into k vector and what is dx here? Actually, dx dt basically dx dt, dt we will take out. So, dx dt is it might just cut, I will just maybe I will try to do it in a separate place okay. So, either you can apply the Green's theorem or you can do it here and what is here?

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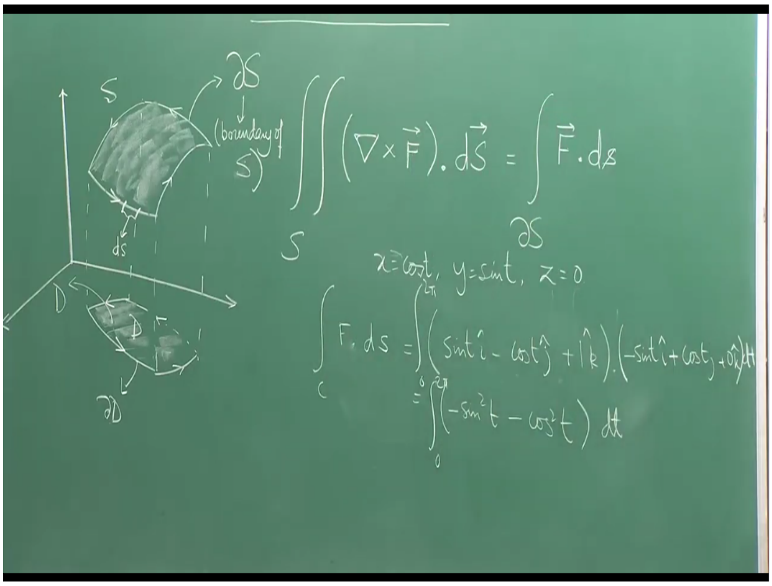

It is so x = cos t parameterized, so I told you that we can also parameterize and put z = 0. So, here integral C, t is varying from 0 to 2 pi. So, this can be written as x square + y square is 1, r is 1 here, so y in this particular case is sin t i vector - cos t j vector + 1 k vector and dx dt is ds is dx i vector + dy j vector + 0 k vector. So, dx dt is what? Dx dt is - sin t i vector + dy dt cos t j vector + 0 k vector this into dt that will give you the dx.

So, basically this will become - sin square t - cos square t, this is now 0 to 2 pi because cos square t and dt. Now, if you integrate this out, I am not going to integrate this out for you, you can integrate this out yourself and integrate this out, the answer would be minus 2 pi which is not equal to 0 and that is the violation of the Stoke's theorem. So, what we conclude that in this region where F is not defined at the point 0, 0, 1 on the hemisphere, Stoke's theorem is violated.

This example is very important that it is not that Stoke's theorem just holds, you tell something whatever you dot does not matter about the F, write some F and the Stoke's would know, it holds only the F is absolutely properly defined on the surface on which your actually should be defined on a region which contains a surface that makes it much more safer to work with okay.

Now, some easier examples which we will do it here on this kind of stuff, so take this example, now let me change the color.

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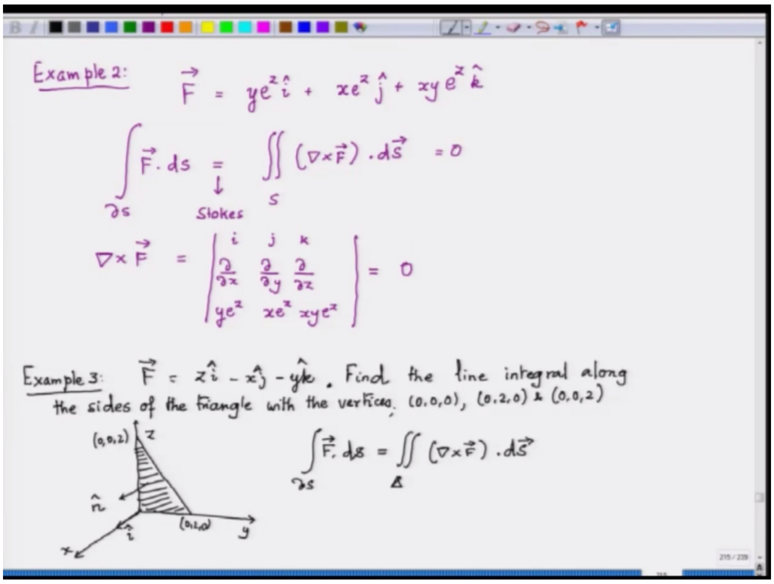

Example 2 actually in this case, let F be now take it from the book, x e to the power z j vector + xy e to the power z k vector. This is what we have. Now, I ask you to compute the line integral around the boundary of any given surface S, oriented surface, surface means in this form. How will you compute the line integral? To compute the line integral, applying Stoke's theorem, apply Stoke's dot dS vector.

Now, so first you have to compute this which is i j k del del x del del y del del z or y e to the power z x e to the power z xy e to the power z. So, here the answer turns out to be 0. You can check out this. The curl of F is 0. So, it does not matter what surface you take, if it is properly oriented surface, then this surface integral is 0. Now, this line integral on the boundary of the surface is 0, it does not matter.

So, the second example, example 3, so a field is given, say this field would be defined in any surface, it does not harm. There is no division by 0 or this field, there in example 2 is defined. So, that is why Stoke's theorem can we applied closing your eyes. Now, F is given by zi vector, xj vector – yk vector. If I take this vector and then I am telling that okay find the line integral, so find the line integral, this is also well defined, this F, it depend on what is the chore of your coordinates.

Find the line integral along the sides of the triangle with vertices, triangle with the vertices 0, 0, 0, 0, 2, 0 and 0, 0, 2. So, let me just draw it out. It was not much easier to draw it on the board, things look much better. So, what happens here is that okay 0, 0, 2 so this is my triangle. The normal vector from any point x, y, z, this normal vector from any point is actually parallel to the i vector.

I can take i vector as the normal vector. So, to compute the line integral, I will again use the Stoke's theorem. So, del S F dot ds is over this triangle which I am writing like this dot dS vector. Now, let us first compute the curl in this particular case.

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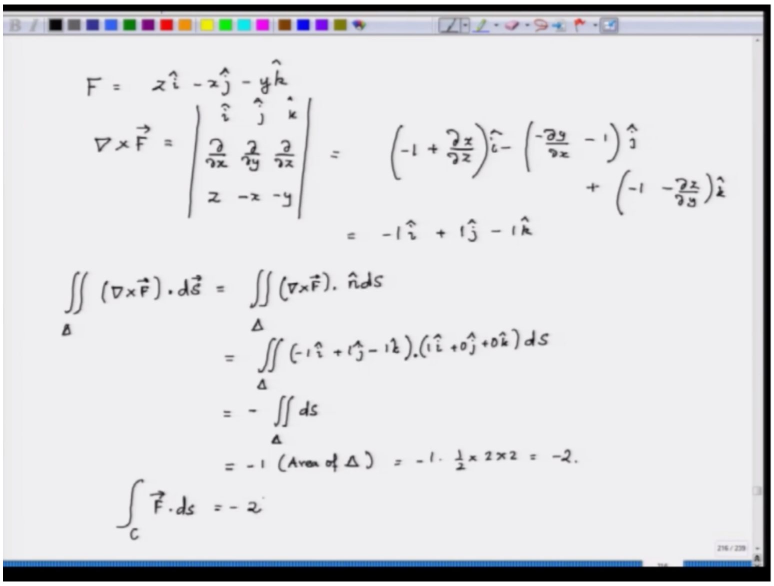

Let us just remind that F is and then in that case my curl becomes i vector sorry i vector j vector k vector del del x del del y del del z. So, F1 F2 F3 that is z - x – y. So, it is if I do the whole job, then I go by this job, then it is del y into - del - 1 + del z del x sorry del x del z, minus minus becomes plus, i vector but x, y, z they depend on del x, del z is actually 0 because x, y, z are independent variables minus del minus del y del x minus 1 j vector plus minus 1 minus del z del y k vector.

All these terms del x del z del y these are all 0 because these are independent variables. So, I have minus i vector sorry - 1 into i vector + 1 into j vector - 1 into k vector. This is my curl and what is integral of S or in this case the triangle, so you have to see all the normals are parallel to i vector, so i vector can be taken as the normal. Here, n vector is i vector, so it can be written as this thing dot dS and be written as dot n dS.

Delta now it is minus i vector plus 1 j vector minus 1 k vector into n vector is 1 into i vector plus 0 into j vector plus 0 into k vector that dS. Now, if you take the dot product here I simply have minus 1, so minus of dS, so it is nothing but minus 1 into area of the triangle. The area of the triangle is 2, so it is 2 is the base length, 2 is the altitude. So, minus 1 into 1/2 into area of S. So, 1/2 base into altitude, so this is 2, the base is 2.

If you go by the picture, the base is 2 because this is 0, 2, 0 and the base into altitude, the altitude is also 0, 0, 2 is 2, so 2 centimeter or whatever any, 1/2 base into altitude. So, it is minus 2, so minus 2 is the answer. So, this is the, so the line integral that you wanted to compute C F dot ds is minus 2. Even though my orientation along this triangle is in this direction, so I have applied the Stoke's theorem keeping the counter clockwise orientation.

So, this is a kind of example which is pretty interesting. So, let me take another example before I end this series of computing examples so that Stoke's theorem can be applied. So, now let us take an exercise from the book.

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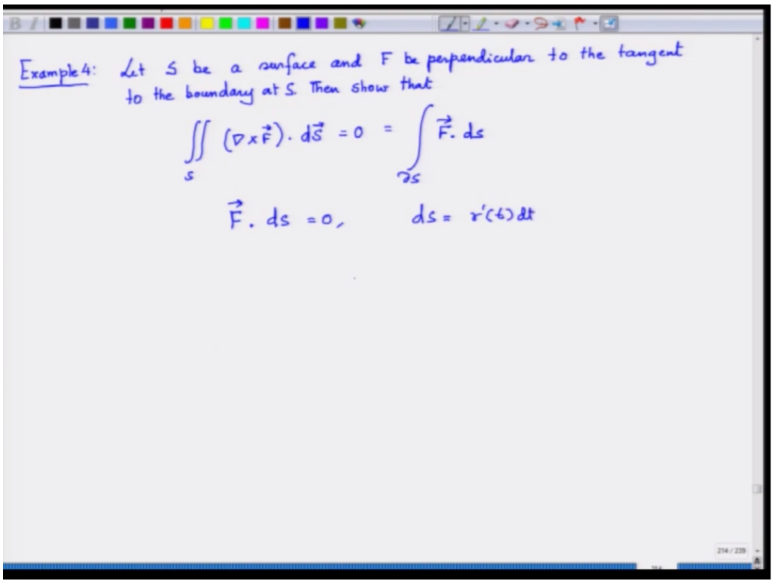

Example 4, let S be a surface and F be perpendicular and F the field be perpendicular to the tangent to the boundary of S. F be perpendicular to the tangent to the boundary at S. Then, show that this is a simple application of Stoke's theorem. So, what is the meaning of tangent to the boundary?

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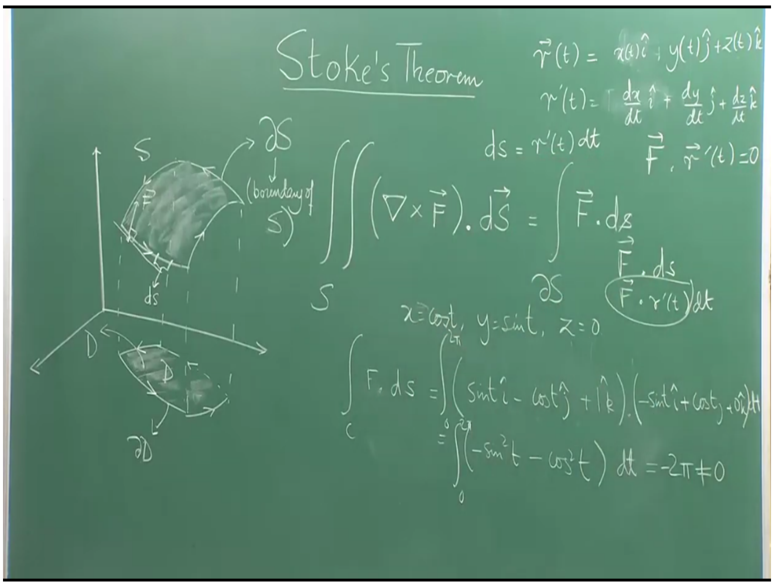

So, here suppose I take this point here and I draw a tangent line here at this point, so the field F here is perpendicular to the boundary. This is the field say, so if I parameterize the curve, so if this is my curve and suppose I parameterize the curve, then so what is my r vector here, r vector here is, so in this case if I am parameterizing let me just go ahead and just do it here. Here, my r vector is xi vector xt i vector yt j vector and zt k vector where t is the parameterization of this boundary.

Because here parameter has to be in terms of xt, yt, zt, this is my rt. So, r dash t is the tangent right. So, what is dS, what is dS here. It is r dash t dt right. So, what is r dash t here? R dash t here is dx dt i vector + dy dt j vector + dz dt k vector. So, what is the meaning that F is perpendicular which means here F dot r dash t, F is always parameterized by t, r dash t is 0, so that is exactly the meaning.

So, F dot ds is F dot r dash t dt, so this thing is 0 and so the line integral is actually 0 and here and that is exactly this whole the line integral here on the boundary because F dot ds here is 0 and as we have seen ds here is nothing but r dash t dt and r dash t is whatever it is, is a radius vector. So, that is the radius vector and the change of the radius vector, the velocity, the tangent line is given by the velocity.

So, r dash t it is described, is a parameterization of the curve xt, yt, zt is the radius vector, the parameterization and then the tangent vector is nothing but r dash t, r dash t is basically talking about the distance by moving particle on that surface and the velocity. Velocity is the tangent, tangent is given by the velocity and so velocity is nothing but r dash t. So, sometimes it is good to think in terms of mechanics.

So, well we have come towards the end of these lectures. Tomorrow is the last session, so we are going to talk about Gauss's divergence theorem which we have. Once we have spoken about Gauss's divergence theorem, we are going to define it, give an example, we will try our best to see if we can prove it but proving it would be quite a feat, which I do not want you to get into.

But then we will continue talking about many other things, about fields, about conservative vector fields and many other issues and how lot of these are related to the fundamental theorem of calculus that you have learnt in one variable plus we are going to talk more and more about we will try to do more and some more problems tomorrow about Gauss divergence theorem may be some application of Stoke's and Green's.

So, tomorrow after stating the Gauss divergence theorem and problem solving and some ideas about field's etcetera, we are going to wind up the course. So, tomorrow's lecture would be a full one-hour lecture. So, that would cover the full course that would be the end of the course. Please do not go by the exact divisions I have done but the course material is completely been taken into account.

So, I would say a goodbye to all of you. I am sure that you have enjoyed the course as I have enjoyed and learnt a lot while doing this course. Every teaching is learning something new, it also excites me to look into some newer aspects which I have not studied in my (()) (37:15). Thank you very much for being patient and if there are any questions, please put it on the portal, we would be very happy to get back. Thank you.