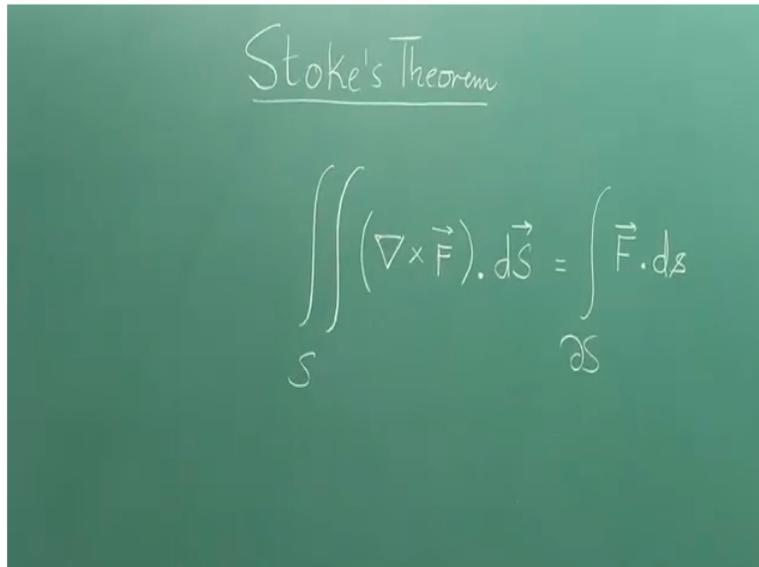


**Calculus of Several Real Variables**  
**Prof. Joydeep Dutta**  
**Department of Economic Sciences**  
**Indian Institute of Technology – Kanpur**

**Lecture – 34**  
**Stokes Theorem**

**(Refer Slide Time: 00:13)**



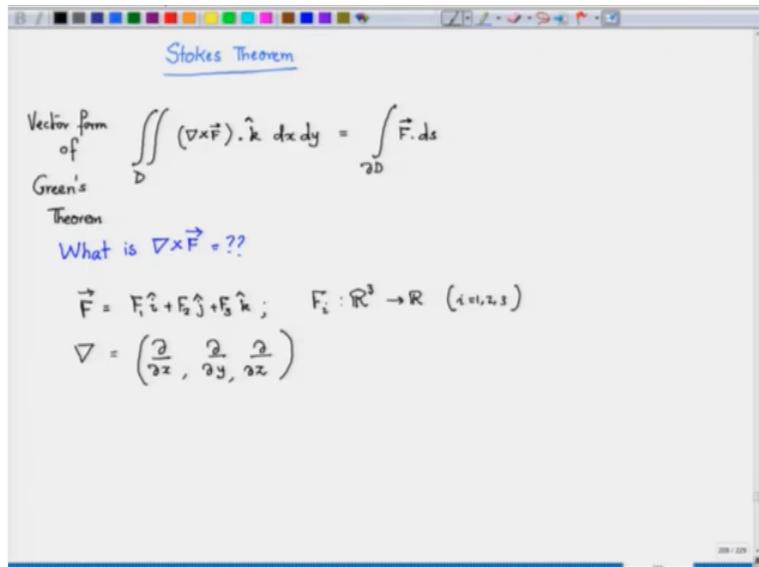
Stoke's Theorem

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

So today we are going to discuss one of the major integral theorems called as Stokes theorem and as I told you here I have written down the form of the Stokes theorem, it says curl of  $F$  dot  $dS$ ,  $dS$  is the surface area vector, double integral over that the surface integral basically the surface integral is now related to the line integral. So the surface of the boundary and on which if you take the line integral it is same as the surface integral of the curl.

Of course, this symbol curl I have not mentioned explicitly though I have used it yesterday. If you remember in the last lecture we have spoken about in the domain in the 2-dimensional domain  $D$ .

**(Refer Slide Time: 01:03)**



We have spoken about an alternative form of vector form of the Greens theorem. So vector form of Greens theorem, so here that it was shown that Greens theorem can be written also explicitly like this. I have told you what a cross product between 2 vectors is? And I have left it to an imagination to know what curl is? Because you would know what del is. And what F is? But today I am going to do it in a more formal way.

If you want to put F here as vector. Now this is the vector form of Greens theorem. So now my question is what is the curl of F? So let us first learn what is curl of F? That is the question I have also written down here. What is that is the question. Now let us understand this vector F has 3 components  $F_1 \hat{i}$  vector,  $F_2 \hat{j}$  vector,  $F_3 \hat{k}$  vector and each of those  $F_i$ s,  $i$  is running from 1 to 3 is a function from  $\mathbb{R}^3$  to  $\mathbb{R}$ .

That is a function of  $x, y, z$  and this operator nabla operator or the del operator is given as  $\nabla$ . It is an operator it is not a kind of real vector, please understand that  $\nabla \cdot \nabla x$  or  $\nabla \cdot \nabla y$  these are not vectors. Vector means these are all numerical quantities but  $\nabla \cdot \nabla x$  or  $\nabla \cdot \nabla y$  are not numerical quantities, they are operations. Now how do I define curl of F. Now to define curl of F I think this del as if it is a numerical quantity as a  $\nabla \cdot \nabla x$ ,  $\nabla \cdot \nabla y$ ,  $\nabla \cdot \nabla z$  is numerical quantity. And del is truly a kind of vector in  $\mathbb{R}^3$ .

**(Refer Slide Time: 04:03)**

What is curl of  $\vec{F} = \nabla \times \vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} - \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}$$

And then curl of F is just written as if we are writing a cross product with these 2 vectors. But the definition of a curl and when I write cross product of 2 vectors you know how to write a cross product i, j, k. So you find the determinant of these i, j, k. Now when you write even a cross product you know that these are actually vectors right. And so basically this is also a vector is a determinant what it gives out a vector.

Usually determinant gives out a number but here the determinant gives out a vector. You would be wondering why it is so? What we define it we it is really not a determinant but we actually work on it as if it is a determinant as if i, j, k were also numbers. So its operations the way you expand it you expand it exactly in the way we do a normal determinant del del x, del del y, del del z and F1, F2 and F3.

So this is the meaning of the curl and this can be now expanded as something into i vector. What is that something i vector into? So this part and this column in this row goes this minus this. So del as you go multiplying these 2. Ultimately it is an operation so we define it as del F3 del y - del F2 del z i vector minus now it just like the determinant. This into this, this into this del F3 into del x - del F1 into del z del del z of F1 j vector plus now remaining is this.

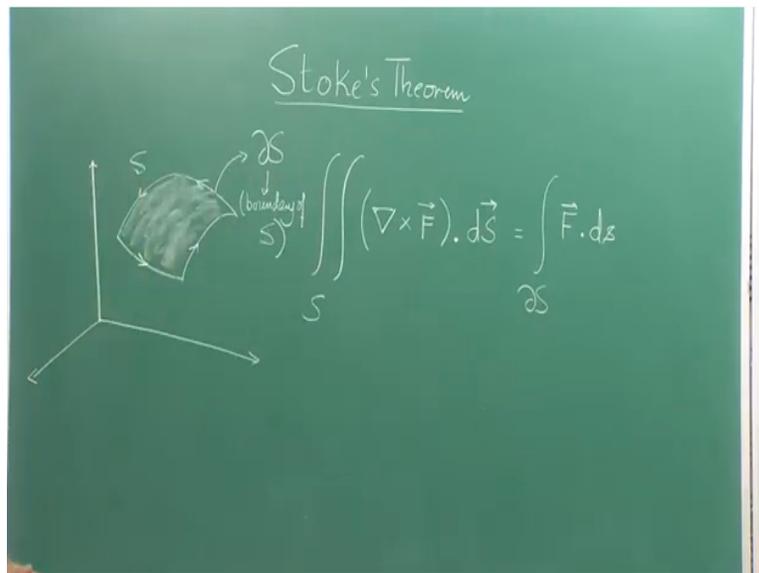
Because we take out this column, the last column and the first row. So it is del del x of F2 vector F2 - del del y of F1 into k vector. So if we expand it this is exactly the expression of the

determinant. This is what we mean by the determinant by the curl sorry not determined by the curl. So this is the meaning of a curl. The curl is a vector which is given like this, is a kind of cyclical thing 3, 2, 3, 1, 2, 1, y, z, x, z, x, y right.

So that is the kind of thing that we have here. Now once I know what is curl of F, I would now like to write down, so what is the difference between the Stokes theorem that I have written here and the Stokes theorem that I have written down here with the Greens theorem in the vector form. You observe that in the Greens theorem in the vector form, we do not use the surface but we take a domain on R2 and work on R2.

But here we are on R3. So in R3, we are talking about a surface integral we need related to a line integral where the line integral the curl curve along which we move now is in the is outlining the surface. It is we are moving along the surface. So for example let us understand what do I mean.

**(Refer Slide Time: 08:21)**



Suppose I have a surface like this okay. This is my S. Now I need to understand what I mean by an orientation of S. So this is my del S, this is the boundary. This lines are the boundary. So this is my del S boundary of S. So when I am taking the line integral how do I traverse that boundary? That is also an important question. How do I traverse that boundary? This question is very important and needs to be answered.

Before we can get on to a hang of Stokes theorem. So if you look at this very carefully you would realize it is the same game that we are going to play. That when you walk along this boundary the surface so if there is a man who is walking along this boundary then the surface should be always on his left-hand side. So if I am walking along the boundary my surface okay I am on the hilltop but I am walking along the boundary but my surface should always remain on the left-hand side.

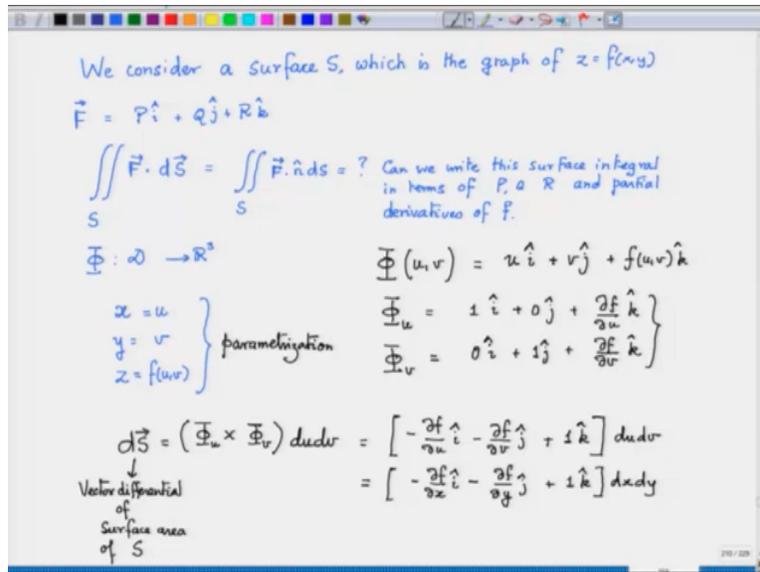
That is the very crucial thing. That is the kind of orientation we are talking about when we are evaluating this integral and that has to be kept in mind. You cannot just throw that kind of thing away okay. The book does an interesting job in the sense that it gives you a proof of Stokes theorem for a surface which has been given by a graph. But you can also parametrize the surface and you can talk about parameters where for whereas 2 different  $x$ s do not go to the same  $y$  kind of thing.

And surfaces which do not overlap for those kind of surfaces you can also prove the thing. So we shall follow the book. Because I am sure that many people have already got the book maybe from Liebchen or somewhere from the net and maybe following the book. So but you need a teacher because you also need things to be explained. Because everything may not be clear to you immediately.

So before we go and do anything certain things have to be made clear. Before I move to actually proving Stokes theorem. So the program of learning Stokes theorem is the follows. In the first part we do the proof. You might get bored and think that this guy is a mad guy, he simply wants to prove the stuff. Actually Stokes never prove the Stokes theorem it was already an idea was known to Lord Kelvin.

And if you Stokes just got it in a letter from him and later only give it an exam that is what the book tells I agree that I have I am here to really figure out the history. But I think I need to figure out this here what is really the stuff. But before I move along certain things have to be made clear.

**(Refer Slide Time: 12:43)**



So we consider a surface  $S$  which is the graph of  $z = f(x, y)$ . Now if I have a vector field  $F$  which is given as  $P$  of  $i$  vector  $P(x, y, z)$  of course  $P$  is a function of  $x, y, z$ ,  $Q$  of  $j$  vector and  $R$  of  $k$  vector then can I write this surface integral the same as writing integral  $\int_S F \cdot n$  into  $dS$ ,  $n$  vector into  $dS$  and this unit normal along the surface area vector and the question is can we write this surface integral in terms of  $P, Q, R$  and partial derivatives of  $F$ .

This will actually go in long way in handling our situation. First thing to realize that  $dS$  is very easily computed by using a parameterization. So we have to now talk about a parameterization  $\phi$  for the graph which you already know. So whatever be the domain  $D$  it does not matter and what that is the parameterization. So but if you have a graph of a function  $z = f(x, y)$  and the parameterization is given as  $x = u, y = v$  and  $z = f(u, v)$  is equal to  $f(x, y)$ .

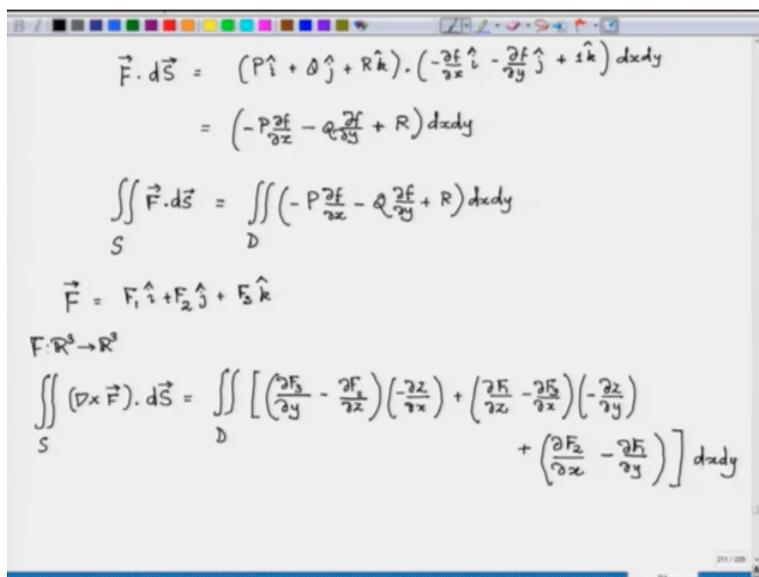
So this is the parameterization which means here if  $z$  is a function of  $x, y$  or all of  $\mathbb{R}^2$ . It does not have  $D$  could be some domain or all of  $\mathbb{R}^2$ . Then  $\phi$  this vector  $\phi(u, v)$  is actually  $u$  vector  $v$  vector and  $f(u, v)$   $k$  vector. But you can replace always  $u, v$  and all this stuffs by  $x, y$ . So again just for your practice I am doing  $\phi$  of  $u$  which is  $1$  into  $i$  vector  $+ 0$  into  $j$  vector  $+ \frac{\partial f}{\partial u}$  into  $k$  vector.

And similarly  $\phi$  of  $v$  is  $0$  into  $i$  vector  $+ 1$  into  $j$  vector plus  $\frac{\partial f}{\partial v}$  into  $k$  vector okay. So we have already been studying this fact that  $d$  of  $S$  vector is  $\phi$  of  $u$  cross  $\phi$  of  $v$  into  $du dv$ . This is

the area differential right. Actually I should write it is an area vector differential of the area of surface area of S. Now in this particular case what is it? What is my phi u cross phi v? If I do it. So that turns out to be - del f del u i vector - del f del v j vector + 1 into k vector into del u del v.

But if you look into this very carefully I can replace u with x. So I can write - del f del x i vector - del f del y j vector at time I would ask you do actually compute this cross product which you can do + 1 into k vector into del u del v can also be written as del x del y as u is x. So dS vector is this.

**(Refer Slide Time: 19:38)**



$$\begin{aligned}\vec{F} \cdot d\vec{S} &= (P\hat{i} + Q\hat{j} + R\hat{k}) \cdot \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right) dx dy \\ &= \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R\right) dx dy \\ \iint_S \vec{F} \cdot d\vec{S} &= \iint_D \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R\right) dx dy \\ \vec{F} &= F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \\ F: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\ \iint_S (\nabla \cdot \vec{F}) \cdot d\vec{S} &= \iint_D \left[ \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \right] dx dy\end{aligned}$$

So now what is my F dot dS this is nothing but P i vector, Q j vector + Rk vector dot product of - del f del x i vector - del f del y j vector + 1 into k vector dx dy. So this gives me - P of del f del x i del f del x - Q of we are doing the inner product now between these 2, del F this vector and this vector and these are scalar multiplied with it differential of the area on the x y plane. Del f del y + R dx dy.

So integral F dot dS a surface integral over S can be written as integral over some domain D which I have said whatever with the domain could be R2 also but let us take some domain - P of del f del x - Q of del f del y + R dx dy okay. So this is what we have. So if you want to progress at all trying to prove Stokes theorem this idea should be in our minds. Now here in our story my vector field is not given by any function F written as P, Q, R but in terms of the curl.

So suppose now I have  $F$  vector which is given as  $F_1 \mathbf{i}$  vector say  $F$  is a function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ ,  $F_2 \mathbf{j}$  vector and  $F_3 \mathbf{k}$  vector. So  $F$  just if I take  $F$  is a function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  see when I write in a function mode then I do not put an arrow. So when you say  $F$  is  $\mathbb{R}^3$  to  $\mathbb{R}^3$  it is assumed it is called out to be a vector function or when I write in the  $i, j, k$  mode in the that kind of mode then the physicists mode I would say then I will put an arrow.

And I am putting here you may or may not put arrow that is up to you it does not matter. But here you need to put an arrow to say that okay product of 2 vectors. So now our vector field  $F$  here is curl of  $F$  okay. So I am now going to write down what do I mean by this assuming that my surface area surface is given by the graph of a function. So I am trying to see whether I can establish the Stokes identity Stokes equality through for that particular case.

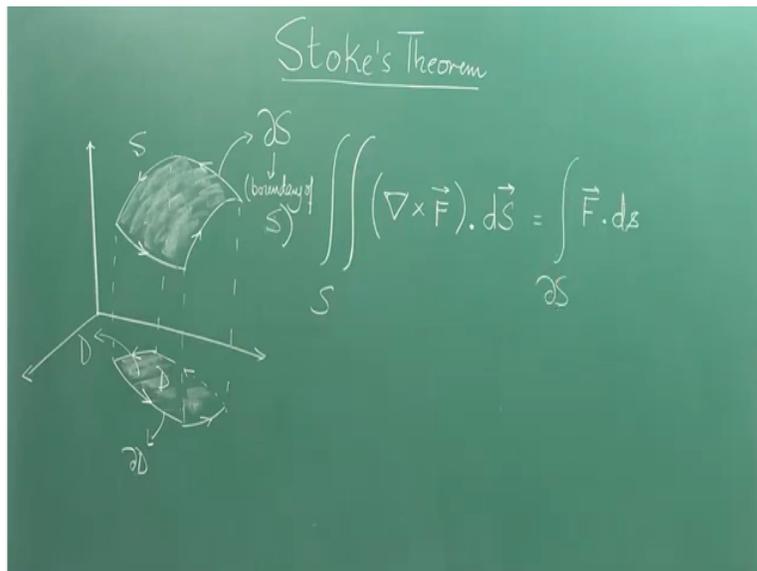
Harder case I leave it to you, you have to understand the teachers and I am becoming pretty old. So not very old 50 fifty years will be soon 51. So I am taking hard work of me, you are young guys you young guys and girls and you should be able to do the hard work. I am sorry for telling this but I am doing quite a bit of hard work. I am sure you would agree to that. So now if I look into this I know what I have done my  $P$  is this, my  $Q$  is minus of this, my  $R$  is this.

So this I will go by this story what I have just written down here integral of  $d$  my  $P$  is this one. So I will just mark with the chalk that this is my  $P$  when I take curl of  $F$  when I write it as a vector field. This is my  $P$  and this is my  $Q$ . This the  $Q$  is minus of this. So this is your  $Q$  this whole thing and this is my  $R$ . So what I have is if I want to write  $+Q$ ,  $Q$  would be  $\text{del } F_1 \text{ del } z - \text{del } F_3 \text{ del } x$  you can write it like this.

So  $P$  can be written as like  $Q$ ,  $Q$  is minus of this. So again now I am writing down so instead of just  $F$  as  $P, Q, R$  I am writing down what is  $P$  in my case of curl when  $F$  is given as  $F_1, F_2, F_3$ . Now so I will write down the first  $P$  component  $\text{del } F_3 \text{ del } y - \text{del } F_2 \text{ del } z$ . Now  $\text{del } F \text{ del } x$  instead of  $\text{del } F \text{ del } x$  I will write  $\text{del } z \text{ del } x$  because we will soon have to apply the chain rule and because I just want to tell you that you see what would happen here is that  $F_1, F_2$  and  $F_3$  all are functions of  $x, y, z$  to  $\mathbb{R}$ .

So just to maintain that kind of symmetry and uniformity now I will instead of F I will use z. So I will have  $-\frac{\partial z}{\partial x}$  plus so here Q is minus of this. So  $\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}$ . So  $\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}$  into  $-\frac{\partial F}{\partial y}$  not  $\frac{\partial F}{\partial y} \frac{\partial z}{\partial y}$ . F is z actually, z itself is a function of x, y. So F is a function of x, y and z itself a function of x, y. That is what the chain rule it should come plus R. So R is what? R is this one.  $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$  close the brackets dx dy that is what we have reduced this whole thing to.

**(Refer Slide Time: 27:35)**



Now this S that we had drawn is here if you look at it is related to a domain D on the 2D plane could be some domain like this. Now how I am how in this D how is D oriented? How am I moving along D? D is again whenever we are writing anything conventionally we will always take it is positively oriented as we have done in the case of Greens theorem. Positively oriented means if I walk along the boundary of D along so this is my D and this is my del of D boundary.

So if I walk along the boundary of D as I have already shown my area should so if I am walking along it my area should always beyond my left hand. So this is something you have to keep on repeating to yourself when you are working with these kind of things those who read differential geometry in the future these things would become very useful. So now here we will come to del D and we will soon see that we will convert we will now try to do something. We have now written down what is the thing for curl of F?

It is written here. So it is done. I have computed this side of Stokes theorem. Now I start computing this side of Stokes theorem what I would do here is to apply the Greens theorem. In Greens theorem if I convert the line integral over this line that is circulating around the boundary of the surface line integral over the boundary of the surface by the Greens integral can be brought up into the line integral over the boundary of the domain.

And that is what I am now trying to do. So my next step is I will parameterize this domain  $\partial D$  sorry the boundary of the domain  $\partial D$  that is the first step that I am now I am trying to compute this. Please understand I am trying to compute this now and I am going to show that when I compute this, apply Greens theorem here I like this and this would become equal whatever I had here. Whatever I had so that will come to chain rule. So here parameterization would become important.

**(Refer Slide Time: 30:50)**

$a \leq t \leq b$  and let  $(x(t), y(t))$  be a parametrization of  $\partial D$   
 Then  $c(t) = (x(t), y(t), f(x(t), y(t)))$ ,  $a \leq t \leq b$   
 is a parametrization of  $\partial S$ .  

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_a^b \left[ F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \quad (\text{By chain rule})$$

$$= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

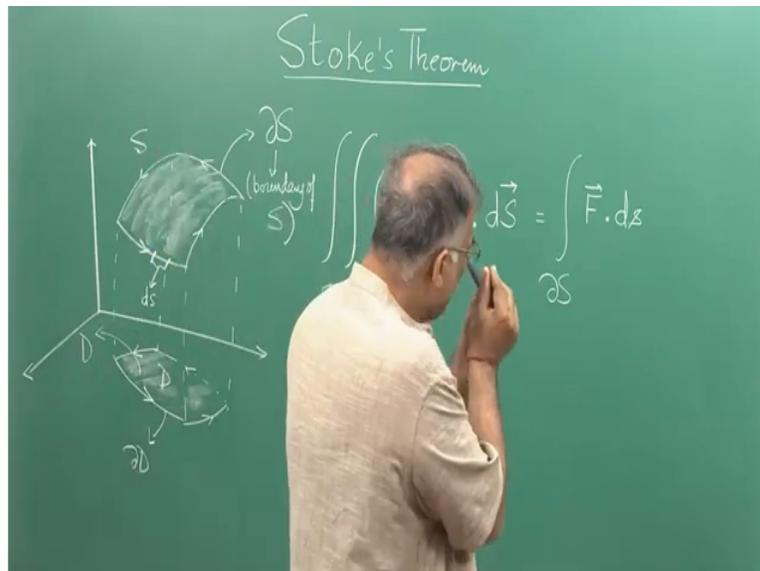
$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_a^b \left[ \left( F_1 + F_3 \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left( F_2 + F_3 \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt$$

$$= \int_{\partial D} \left[ \left( F_1 + F_3 \frac{\partial z}{\partial x} \right) dx + \left( F_2 + F_3 \frac{\partial z}{\partial y} \right) dy \right]$$

So let take  $t$  between  $b$  and  $a$  and let so here the points are  $x, y$  on this curve right this so and let  $x(t), y(t)$  be a parameterization of  $\partial D$ . Then  $c(t)$  were at  $c(t)$  is  $x(t), y(t)$  while using that parameterization that we have learned of  $\phi$  and  $f$  of  $x(t), y(t)$  then this becomes a parameterization in terms of  $t$  of  $\partial S$ . Then  $c(t)$  is a parameterization of  $\partial S$ . So  $\partial S$  is this one the boundary of the surface not the boundary of the domain is a parametrization of  $\partial S$ .

Let me look at this quantity itself here  $S$  is the arc length on  $\partial S$  and  $dS$  here is  $dS \, dx \, i$  vector  $dS \, dy \, j$  vector  $dS \, dz \, k$  vector. Here  $dS$  you have to understand I am now going to compute and in the bracket I am just writing that here I mean arc length in 3 dimension is basically  $dR \, dx \, i$  vector  $dy \, j$  vector  $+ dz \, k$  vector. This is the change in the radius vectors but basically a change here  $dS$  is  $dS$  simply represents this small change here.

**(Refer Slide Time: 33:48)**



So this is represented as  $dS$ . So here this basically difference between the 2 radius vectors and that when the difference between the 2 radius vectors is very small. This difference is very small  $dR$  is very small. Then  $dS$  and  $dR$  are almost nearly the same. So that is  $dS$  is taken to be  $dR$ . The straight line very well approximates that curve. So now I can now write  $dS$  and I can just allow a now instead of going through writing  $\partial S$  I can write through a parameterization in this form.

$F_1 \, dx \, dt + F_2$  so  $dS$  I write as now this can be written as  $dS \, dt \, dt$ . So at  $dx \, dt \, i$  vector  $dy \, dt \, j$  vector  $dz \, dt \, k$  vector and so I can write that thing here in this form once I do that dot product, dot product you know how to do it. So it is you can show is it really dot product is that here really number see a very small number so we are doing the simple dot products this is a vector actually. Now if I go by the chain rule  $\partial z \, dt = \partial f \, dt = \partial f \, \partial x$ ,  $x$  is just 1 variable with respect to  $t$ .

So it is  $dx \, dt + \partial f \, \partial y \, dy \, dt$  this is by chain rule. Now once you know this so you put it here put  $dz \, dt$  here and then integral  $dS$  now I will do the manipulations you check the manipulation

but I am just writing it down it is very simple it is you do not even you can just do it without just by observing this. So you will  $F_1$  with  $dz dt$  will  $F_3$  you will have this one. So it  $dx dt$  you will have to combine this part.

So if there will be  $F_3$  multiplied here. So let us see what happen? So  $dz dt$  is now  $\frac{\partial f}{\partial x} dx dt + \frac{\partial f}{\partial y} dy dt$ . So  $F_3$  gets multiplied into 2 parts and then they are combined up with the  $F_1$  and  $F_2$ . So  $F_1 + F_3 \frac{\partial f}{\partial x}$  so this can be written as  $\frac{\partial z}{\partial x} dx dt$  and  $f$  is  $z$  actually  $\frac{\partial z}{\partial y} dy dt$ . So  $\frac{\partial f}{\partial x}$ , so  $F_3 \frac{\partial z}{\partial x} dx dt$  is now common it is can be taken out  $dx dt + F_2 + F_3 \frac{\partial z}{\partial y} dy dt$ .

This should ring a bell immediately in your mind. Because now I can take the integral separately but what would be this? This is now by if I now I can write them again back as a I can take the  $t$  off by multiplying these 2 and write it as if I am doing a line integral on that domain  $D$  because now I am in 2 variables case  $x$  and  $y$ . I am in the 2 variable domain. I no longer have so I can now look at this whole thing as a function  $P$  and  $Q$  like I are the one I did for Greens theorem.

So the function  $F \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y}$ . So basically this is now written as  $F_1 + F_3 \frac{\partial z}{\partial x} dx + F_2 + F_3 \frac{\partial z}{\partial y} dy$ . Now this is the form of the Greens theorem  $P dx + Q dy$  sorry not  $dz dy$   $P dx + Q dy$  this is the form of the Greens theorem. So now we have to apply the greens theorem. So let me just again write it down.

**(Refer Slide Time: 39:22)**

By Green's Theorem

$$\int_{\partial D} \left[ \underbrace{(F_1 + F_3 \frac{\partial z}{\partial x})}_{P} dx + \underbrace{(F_2 + F_3 \frac{\partial z}{\partial x})}_{Q} dy \right] = \iint_D \left[ \frac{\partial}{\partial x} (F_2 + F_3 \frac{\partial z}{\partial y}) - \frac{\partial}{\partial y} (F_1 + F_3 \frac{\partial z}{\partial x}) \right] dx dy$$

Applying chain rule

$$\iint_D \left[ \frac{\partial}{\partial x} (F_2 + F_3 \frac{\partial z}{\partial y}) \frac{\partial x}{\partial x} + \frac{\partial}{\partial y} (F_2 + F_3 \frac{\partial z}{\partial y}) \frac{\partial y}{\partial x} + \frac{\partial}{\partial z} (F_2 + F_3 \frac{\partial z}{\partial y}) \frac{\partial z}{\partial x} \right] - \left[ \frac{\partial}{\partial y} (F_1 + F_3 \frac{\partial z}{\partial x}) \frac{\partial y}{\partial y} + \frac{\partial}{\partial x} (F_1 + F_3 \frac{\partial z}{\partial x}) \frac{\partial x}{\partial y} + \frac{\partial}{\partial z} (F_1 + F_3 \frac{\partial z}{\partial x}) \frac{\partial z}{\partial y} \right] dx dy$$

$$\iint_D \left[ \frac{\partial F_2}{\partial x} + \frac{\partial}{\partial z} (F_3 \frac{\partial z}{\partial y}) + \frac{\partial F_3}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial}{\partial z} (F_3 \frac{\partial z}{\partial y}) \frac{\partial z}{\partial x} \right] - \left[ \frac{\partial F_1}{\partial y} + \frac{\partial}{\partial z} (F_3 \frac{\partial z}{\partial x}) + \frac{\partial F_3}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial}{\partial z} (F_3 \frac{\partial z}{\partial x}) \frac{\partial z}{\partial y} \right] dx dy$$

So let me write down the Greens theorem. So by Greens theorem integral del D F1 + F3 del z del x is P dx + F2 + F3 del z del x dy. This is equal to integral over the domain itself right. So in 2 dimension your domain is the surface is a double integral. Now we are converting this computing this line integral in terms of this double integral over the domain of this is my Q this is my P in the Greens theorem, this is my Q in the greens theorem.

So del Q del x so del del x of F2 + F3 del z del y - del P del y del del y of F1 + F3 del z del x, z itself is a function of x, y and everything F1, F2, F3 is a function of x, y, z please understand that. So chain rule is now required to make an into dx dy. Chain rule is now required to make any further progress. So applying chain rule let me see how applying chain rule. Please understand that there is a product also here so we have to apply the product rule also.

While applying the chain rule do not forget to apply the product rule. So let us now integral D. So this whole thing is now integral D let us see how do I apply one chain rule and then another. First look into this. So all of these thing is in terms of x, y, z right. So I can first separate it out del del x of so here is a function of x, y, z. So I will write it in terms of chain rule. So I will first I will do for F2 and then I will write it out.

But remember my parameterization here and you have to understand z these F2s are functions of xy. F1, F2, F3 are functions of x, y, z and z is a function of x, y. So you have to do it when you

write  $\frac{\partial}{\partial x}$  of  $F_2$  you really have to do the chain rule right. So if I apply the chain rule here so first I will do  $\frac{\partial}{\partial x}$  of  $F_2 \frac{\partial x}{\partial x}$  which is  $1 + \frac{\partial}{\partial x}$  of  $F_2$  of  $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial x}$  because I am doing it with respect to  $x$ .

So  $\frac{\partial}{\partial x}$  of  $F_2$  into  $\frac{\partial}{\partial x}$  I first separated  $\frac{\partial}{\partial x}$  this  $F_2$  and then I am doing separately this one. So I will have here I am just writing it and then I will rewrite is  $\frac{\partial}{\partial x}$  of  $F_2$  of  $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial x}$ . So if I write the chain rule for  $F_2$  when I with respect to  $x$  and this will be the chain rule. Because there it is a function of  $x, y$  please understand here  $z$  is a function of  $x, y$  and that is why I have to write down this whole chain rule.

So  $\frac{\partial}{\partial x}$  of  $F_2$  is coming through this way. You may do it like this or there is another method of doing it. I think I will try to make it much more simpler for you. So when I am computing this part plus this part so  $\frac{\partial}{\partial x}$  of  $F_2 + F_3 \frac{\partial z}{\partial y}$  into  $\frac{\partial}{\partial x}$  of  $F_2 + F_3 \frac{\partial z}{\partial y}$  of  $\frac{\partial y}{\partial x}$ . So these respect to  $x$  of  $\frac{\partial y}{\partial x}$ . So this whole chain rule is with respect to  $x$ . So I am just doing this knowing that  $z$  is connected through  $F(x, y)$ ,  $x$  is  $x$ ,  $y$  is  $y$ .

But  $z$  is connected through  $x$  and  $y$ . So basically I am doing the  $uv$  thing. This is in terms of  $uv$  basically but okay  $u$  is  $x$  and  $v$  is  $y$ . So now I will have  $\frac{\partial}{\partial x}$  of  $F_2 + F_3 \frac{\partial z}{\partial y}$  into  $\frac{\partial}{\partial x}$  of  $F_2 + F_3 \frac{\partial z}{\partial y}$  of  $\frac{\partial y}{\partial x}$  this thing minus the same story I can write in terms of  $y$   $\frac{\partial}{\partial y}$  of  $F_1 + F_3 \frac{\partial z}{\partial x}$  this is in terms of  $y$  now.

So  $\frac{\partial}{\partial y}$  of  $F_1$  sorry  $\frac{\partial}{\partial y}$  of  $F_1$  plus no I making a mistake. It should be  $F_1 + F_3 \frac{\partial z}{\partial x}$  it is right into  $\frac{\partial}{\partial y}$  of  $F_1 + F_3 \frac{\partial z}{\partial x}$  actually you can write everything in terms of  $uv$  you know and then put  $\frac{\partial z}{\partial x}$ ,  $x$  instead of  $u$  and  $v$  instead of  $y$  and  $z$  as  $z$ .  $F$  into  $\frac{\partial z}{\partial y}$  this is the chain rule because you have to take into account that  $z$  is a function of  $x, y$ .

So this is what I have written. How does this is into  $dx dy$ . So how does this expression pan out. Let me write it down. Let us write it down this expression. So  $\frac{\partial}{\partial x}$  of  $F_2 \frac{\partial x}{\partial x}$  into  $\frac{\partial}{\partial x}$  of  $F_2 \frac{\partial x}{\partial x}$ . So you have  $\frac{\partial}{\partial x}$  of  $F_2 \frac{\partial x}{\partial x} + \frac{\partial}{\partial x}$  of  $F_3 \frac{\partial z}{\partial y}$  into  $\frac{\partial}{\partial x}$  of  $F_2 \frac{\partial x}{\partial x} + \frac{\partial}{\partial x}$  of  $F_3 \frac{\partial z}{\partial y}$  so it is 1

plus this part  $\frac{\partial y}{\partial x}$  is 0,  $y$  is not represented by  $x$  and  $y$  and  $x$  are independent to each other. So now it remains  $\frac{\partial}{\partial z}$ .

So  $\frac{\partial}{\partial z}$  of  $F_2$  into  $\frac{\partial z}{\partial x} + \frac{\partial}{\partial z}$  of  $F_3 \frac{\partial z}{\partial y}$ . Now you have to apply in this cases some cases product rule minus so here in this case  $\frac{\partial y}{\partial x}$ , this part is 0 it goes away. So if the first part becomes  $\frac{\partial F_1}{\partial y} + \frac{\partial}{\partial z}$  of  $F_3 \frac{\partial z}{\partial x}$  plus finally this one becomes  $\frac{\partial F_1}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial}{\partial z}$  of  $F_3 \frac{\partial z}{\partial x}$  into  $\frac{\partial z}{\partial y}$ . Now I will expect that you should be able to write the chain a product rule here.

You have to write product rule here. You have to write the product rule in this case  $\frac{\partial}{\partial x}$  of  $F_3 \frac{\partial z}{\partial y}$  del  $\frac{\partial z}{\partial x}$  of  $F_3 \frac{\partial z}{\partial y}$  del  $\frac{\partial z}{\partial x}$  of  $F_3 \frac{\partial z}{\partial y}$  del  $\frac{\partial z}{\partial x}$  of  $F_3 \frac{\partial z}{\partial y}$ . So if I write down that product rule.

**(Refer Slide Time: 49:43)**

$$\int_{\mathcal{V}} \vec{F} \cdot d\vec{s} = \iiint_D \left[ \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial F_2}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial F_3}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + F_3 \frac{\partial^2 z}{\partial x \partial y} \right] - \left[ \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial F_2}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial F_2}{\partial x} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + F_3 \frac{\partial^2 z}{\partial y \partial x} \right] d\vec{s}$$

$$= \iiint_S (\nabla \cdot \vec{F}) \cdot d\vec{s}$$

Then finally my integral  $\int_S \vec{F} \cdot d\vec{s}$  will take an interesting thing. So if I write down that whole thing so it will become  $\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial F_2}{\partial x} \frac{\partial z}{\partial y} + F_3 \frac{\partial^2 z}{\partial x \partial y}$ . So here you have to see so here it will be  $\frac{\partial F_3}{\partial x} \frac{\partial z}{\partial y} + F_3 \frac{\partial^2 z}{\partial x \partial y}$ . So that I will put in the end the second order thing into  $\frac{\partial z}{\partial y}$ . Second order there is another second order term will come here.

We put it at the end and will do the other things and I am just rearranging this thing  $\text{div } \mathbf{F} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{x}$   $\text{div } \mathbf{z} \cdot \mathbf{y} + \text{div } \mathbf{F} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{x}$   $\text{div } \mathbf{z} \cdot \mathbf{y}$  here what would happen  $\text{div } \mathbf{z}$  of  $\text{div } \mathbf{F} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{y}$ . So here if I look at it carefully  $\text{div } \mathbf{F} \cdot \mathbf{x}$  into  $\text{div } \mathbf{z} \cdot \mathbf{y}$   $\text{div } \mathbf{F} \cdot \mathbf{x}$  into  $\text{div } \mathbf{z} \cdot \mathbf{y}$  which also come here  $\text{div } \mathbf{F} \cdot \mathbf{x}$  into  $\text{div } \mathbf{z} \cdot \mathbf{y}$ . So  $\text{div } \mathbf{F} \cdot \mathbf{z}$  into  $\text{div } \mathbf{z} \cdot \mathbf{y}$  into  $\text{div } \mathbf{z} \cdot \mathbf{x}$  and you also have to do so  $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{y}$ .

When you do  $\text{div } \mathbf{z} \cdot \mathbf{y}$  and then if you take  $\text{div}$  so here you will have  $\text{div } \mathbf{F} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{y} + \text{div } \mathbf{F} \cdot \mathbf{z}$  into  $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{y}$ . So  $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{y}$  is 0 right that is a function of  $x, y$ . So you write this part minus I will just write it in a fast way  $\text{div } \mathbf{F} \cdot \mathbf{y} + \text{div } \mathbf{F} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{y}$  plus I am running out of time  $\text{div } \mathbf{F} \cdot \mathbf{y}$   $\text{div } \mathbf{z} \cdot \mathbf{x} + \text{div } \mathbf{F} \cdot \mathbf{z}$   $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{y}$   $\text{div } \mathbf{z} \cdot \mathbf{x} + \text{div } \mathbf{F} \cdot \mathbf{z}$  into  $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{y}$   $\text{div } \mathbf{x}$  into  $dx dy$ .

You know if you rearrange this whole thing you know what you will get? You will get back what we got. If you rearrange this whole thing you will get back this. If you rearrange what we wrote at the end you will get back this that is the whole key. So this is nothing but  $\text{div } \mathbf{F} \cdot \mathbf{z}$  into  $\text{div } \mathbf{z} \cdot \mathbf{z}$   $\text{div } \mathbf{y}$  and voila we have proved the Stokes theorem and that is the end. So I wrote something very fast because I expect you to do it because I explained you how the chain rule is working.

But we have proved if you do this computation if you check it out because you see the second order term will cancel. This term will cancel with this term. So we will be left with first order terms and if I go back to this you will exactly get back this. Because this  $\text{div } \mathbf{F} \cdot \mathbf{z} = \text{div } \mathbf{z} \cdot \mathbf{z}$  that is it. And we have proved the Stokes theorem voila as a French would say. In the next class we are going to talk about examples of Stokes theorem.

So we have exceeded our time quite high okay it does not matter. We will go along a bit thank you maybe we will not do the proof of the Gauss theorem but so if we have got 1 hour another 40 minutes. So we will go for a single full class today we will have 3 classes where 1 and 2 and there will be a single full class tomorrow and that will be the end and I hope because there is no just breaking up things. Now we will do the problems for Stokes theorem is very important to know at least one proof. Thank you.