Calculus of Several Real Variables
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Lecture - 28
Line Integral - II

So you must have understood that when I define line integrals, I define I depend heavily on the concept of parametrization.

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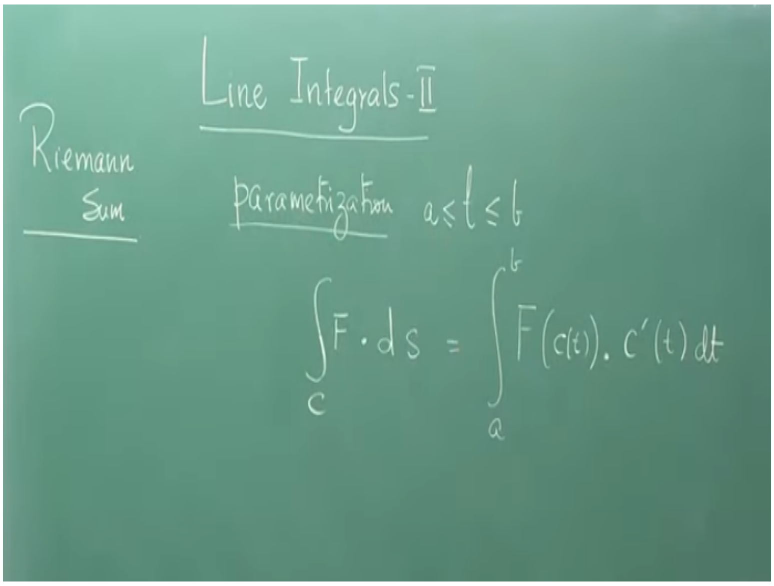

And this parameter is a real variable. So t between a and b. Once I know this fact, my attempt of writing a line integral further becomes like this. So if you know the way I have defined line integral, you know that the line integral for a vector field F has been defined as through the parametrization, right, a to b F (c(t)) dot c dash t dt. So what I have done is I have converted this so called integration of a vector function into integration of a scalar function involving t.

So that is the key idea behind this definition of the line integrals. So parametrization helps me to convert from a vector valued scenario to a scalar valued scenario involving one variable. Now the question is that can I write this, because all these integral definitions that we have learned has been done through Riemann sums, limit of Riemann sum. So we have to talk about Riemann integrals in various situations.

So can we have a Riemann sum representation here? Actually, if you look at it, if you look at the definition, maybe I will just do it here.

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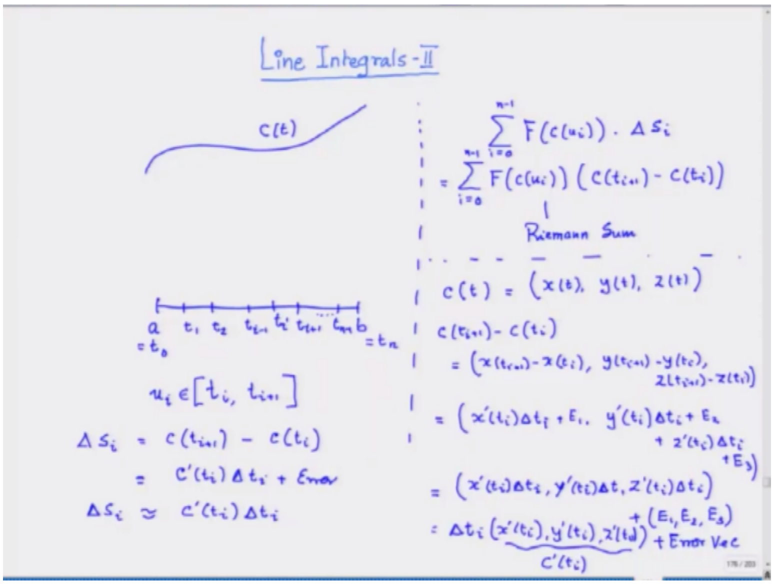

So here is my a to b and here is a curve which it parametrizes. So here is my curve c (t). So you do a partition and you know how to do it now, t 1, t 2, t i-1, t i, t i+1 and so and so forth t n-1. So the n+1 points giving n number of intervals. So take any interval t i to t i+1 which we will call the ith interval and choose a point u i, t i to t i+1 or whatever you want to say; u i as an element here.

So the Riemann sum that you would create in this case, your delta s i arc length is actually c(t i+1) minus this is your change in the length of, change of the arc length as you move from t i to t i+1. These are delta s i. So the Riemann sum okay, so the Riemann sum actually, so in this case our Riemann sum, let us write down the Riemann sum here. So you choose any u i which is arbitrary.

So the Riemann sum would be f (c(u i)) into dot product of delta s i is a vector. So these are all vectors. So what we are doing, we are writing individual Riemann sums basically. This is dot s i is c (t i) minus this c is a vector. So we are writing for a given partition point we are writing this as a dot product. Because here multiplication means dot product. So here so this is written as f (c (u i)) into c (t i+ 1) – c (t i).

That is what I write. So now what I do to get the Riemann sum I need to sum this whole thing. From i equal to 0 to i equal to n -1. I need to sum up. So this is the

Riemann sum. Now the delta s i, now if I write down in terms of the Taylor series right? c dash t i, if I write down a Taylor series linear approximation of it is delta t i which is t i + 1 minus t i plus some error.

That is what a first order Taylor expansion we give me. Please understand t i is a real variable. You can say okay the function is R 3. But what we are doing, for each component of c (t i), c t i+1 or c t i, this has three components, c 1, t i+1 minus c 1 t i, c 2, c 3. Just three components. What are the three components?

The x, y, z basically. So on the x part I am applying the Taylor's expansion first order, y part I am writing the Taylor's expansion, z part I am writing the Taylor's expansion and putting them as a vector. So what I am doing here is like this. So your c (t) is expressed as, I am just writing in terms of rho vector (x (t), y (t), z (t)) points on the curve. So c (t i +1) – c(t i) is a vector.

Which is given as (x(t i+1) – x(t i), y(t i+1) now these are functions from r to r y t i; z(t i+1) – z(t i). So what you will do, you will write here the first order Taylor's expansion that is x dash t i into delta t i plus error which I am writing as E 1. Here what I will do, I will write y dash t i into delta t i plus error E 2 plus z dash t i delta t i plus error E 3. So that is what I am going to write.

So if you take them up, so it will what will happen you will have x dash t i into delta t i right? The vector y dash t i into delta t i and z dash t i into delta t i, that is added with this vector E 1, E 2, E 3 the errors. Now if I take the delta t i outside, scalar multiplication of vector, this is nothing but x dash t i, y dash t i, z dash t i plus error, plus the error vector. So what is this?

What is x dash t i plus z dash t i plus this damn thing sorry this thing is nothing but c dash t i. This thing is nothing but c dash t i. Sorry for the use of the word damn, it comes out of excitement. It is no hard feeling to anyone. So here what you have done, you have expressed these difference in terms of the derivative of c dash j. That is where the derivative actually appears in the first place. Now if the error is very small delta t i is very small, then you can write delta s i, you can approximate as c dash of t i delta t i.

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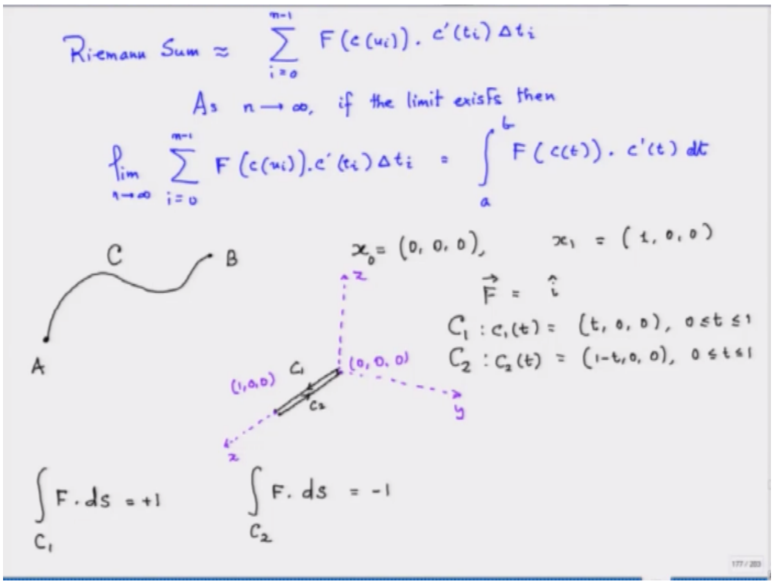

So in that case, the Riemann sum can be approximately written as i equal to 0 to n – 1 F (c (u i)) dot c dash t i into delta t i. Now as the whole thing goes to zero, you can understand this is now as limit as n tends to infinity if this limit exists, if the limit exists then limit n tends to infinity, you can easily now figure out what will happen.

This will become d t this will become c dash t, this will become F (c (t)) because these things are t i and t i+ 1 are coming closer and closer as n goes to infinity. This limit if it exists is mocked by the integral where t is varying from a to b F (c (t)) dot sorry dot c dash t dt. That is the game right. So it has a expression through Riemann integral.

There are few technical things we should remember about this thing is that when you are taking a line integral suppose you have a point a in space. So let me have a path A of a point A and a point B in space. So there are there is a I say that it I move between A to B along a path along a path C. Then the path C actually determines how you are moving. Either you are moving from A to B or you are moving from B to A.

The nature of C will determine that. There is a beautiful example in the book and the value of the line integral can change depending on which way you are moved that whether you have gone along C from A to B or whether you have come from B to A. That orientation, that change of direction can change the value of the line integral. So there is a nice example and which I think is very useful.

So here, they give you two points in three dimension, x is 0, 0, 0, some point x naught. Another x 1 point which is the final point is 1, 0, 0. See if I draw the axis. So this is my x axis, this is the z axis and this is the y axis. So essentially I am looking to move from this point to this point along. So here is the line segment. So let me remove this line segment and oh my God.

I should take the black and join the line segments. The remaining part can be in this color. Which color I take, oh my God, okay? So now I will describe two paths. One path will take me from 0, 0, 0 to 1, 0, 0. Another part will take me from 1, 0, 0 to 0, 0, 0 and on these two different parts, the value of the line integral would be different. Let me consider the vector field F given as i. So it is 1 into i, it is constant.

That f position is always constant. And there are two curves. One is c 1 t which is t, 0, 0. So when it is 0, 0, 0 when t is 0 it is t is between when t is 0 we are at when c 1 0 is 0, 0, 0. When t is 1 that is c 1, 1 we are at 1, 0, 0. So c 1 comes along this direction. And then I define another path c 2, parametrized path c 2. So this is my curve c 1 and this is my curve path parametrized by c 2, 1 – t 0, 0, 0 small tweaking you see.

So when I put t equal to 0 it is giving me this point. When I put t equal to 1, it is giving me 0. So there is another path c 2 is taking me from this point to this point, the reverse. C 2 is taking me on the reverse direction. So let us now calculate the line integral. The line integral I will not calculate, I will ask you to calculate. Take it as a homework and calculate.

The line interval calculation is very F is fixed here, it does not matter, whatever c t put it is the value is the vectorial component is just 1. And c dash t is you know what to do. It is a 1, 0, 0. And of course, is not a very big idea, very big thing to do. And c dash 1 t is nothing but the i vector, i vector plus zero j vector zero k vector right. So you know how to do the thing. So it is i dot i basically.

So it is 1 essentially 0 to 1 dt. So if I go by the curve c 1. So let me call this a c 1 the big curve. C 1 is the big curve c 1 is parametrized by this. Another big curve c 2 is parametrized by this. Both are the line segments. So by c 1 if I take my line integral, that value is +1. But I come by c 2 so there is a physical meaning to this. This F is a force acting on the particle moving along this path.

Then if the particle goes from 0 to 1, 1, 0 and comes back from 1, 1, 0 to 0, 0, 0 along c 2 then the total work done by the particle is 0. So you see this orientation has a important thing to do about the value of the line integral. So this is something that one needs to keep in mind while talking about line integrals. Another thing I do not want to get into the issue is of reparametrization.

I do not want to confuse you at the moment with reparametrization. That is I can another I can take another parameter u and write t as a function of u and I change the parametrization of the curve again. But ultimately the line integral values would be same. That is what I just want to tell you. I do not want to get into the technicalities of reparametrization.

But now for example, we are going to talk about the physical aspect of this line integral. We will talk about the meaning of work done. Of course, there are certain things which you need to learn yourself.

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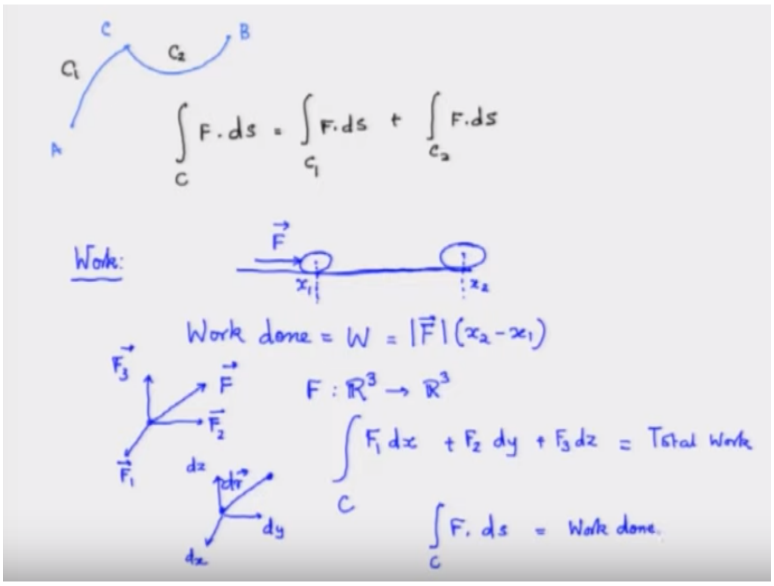

Which is for example, if you have from point A to point B, if I move in curves, A to C and then from C to B and I call this curve as C 1 and I call this curve as C 2 then the integral over the whole curve can be done separately. First over C 1 and then over C

2**.** I am not writing the vector sign when I am writing this just as a function on the integral, but it is actually a vector valued function, plus C 2.

So these are some things you can understand. Your intuition is absolutely fine, young people to understand this. Even if there are some senior people who are watching this, this is not a very big issue. So we will now talk about physical aspects of this notion of a line integral. One of the most fundamental physical aspect is that of work. Work has already been defined in physics as force into distance.

So work done, so if there is a, say a stone lying here and you are pushing it with a force F in this direction, and it starts from this position. I am just taking the position of the center of mass, and starts from a position x 1 and reaches a position x 2 along the real line, right? Then the work done is simply the magnitude of the force into the distance that is traveled, that is work done.

But when you come to higher dimensions when an object a particle is moving in a trajectory, then there this distance, how you compute the distance? The distance is a vector, the distance has to be given by a vector. And of course, the work done has to be expressed in terms of a scalar product. Work done cannot be, so basically what you are doing? You are looking at the work done along all the components.

And then summing them up and that is exactly the inner product. So basically if there is a particle here in three space and here is a force acting on it. And this force can be decomposed into three component forces. I am just talking about mechanics now. Those who know some mechanics will understand. This is F 1, F 2, F 3. And that is why, these are a component forces, right?

So that is why this F vector, F is viewed as a function from R 3 to R 3. So there are three components because you can decompose a force. If there is a particle and force is acting on it, I can decompose it along the horizontal direction, the what I call, horizontal direction, the vertical the height direction, along the length, along the breadth, along the height, three directions.

So now when you have decomposed it, now you assume that you are making a small movement along so, if you take a particle and if you move it very, along a very small movement, you make a very small movement from here to here. This is a displacement vector, which is dr vector can be expressed into three different components along, one along the length, breadth, and the height.

That is d x, d y and d z, you can decompose it along these three directions. So then along the direction of x the work done is F 1 into d x. Along the direction of y the work done is F 2 into d y. And along the direction of z the work done is F 3 into d z. Because we are, the components are decomposed along length, breadth and height. So the force acting along the x direction is F 1, so F 1 into d x.

Along sorry I have to make it F 2 here. This should be F 2, just to maintain a symbolical things. Now along the breadth via y direction it is F 2 and along the z direction it is F 3. So then the magnitude, work is a scalar. It we have to give the magnitude of the work. You do not talk about what direction of work. It is the magnitude, how much work you have done.

Energy is a capacity of doing work in some sense. So work, so because when you do work, you spend some energy. So F 1 dx is the work along the x direction, F 2 dy is the magnitude of the work along the y direction, F 3 dz is the magnitude of the work along the z direction. So total magnitude of the work is this along in this small, once you make this very small movement, this is the total magnitude of the work.

And so if you move along a curve C, this is the total work. So this one dimensional idea is again bought into the higher dimension and you know what is this. This is the differential form and you know how to do this thing. This is not the only case, this is not the only scenario where, this is not the only scenario where this line integral can be used. Let me talk about the notion of work done in a gravitational field, right?

So suppose, there is a three dimensional particle and at every point other than upon origin 0, 0, 0, 0, that the gravitational force field can be defined and that we have, we know how we can define it.

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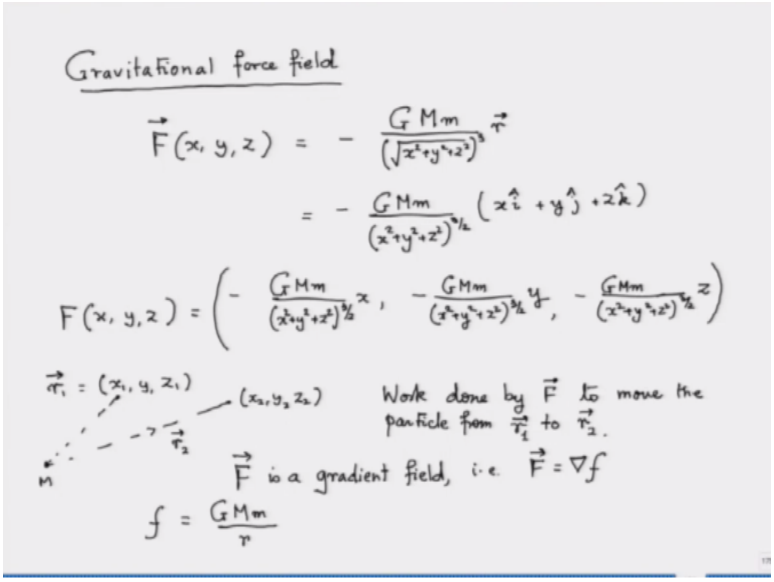

So the gravitational force field is defined in this way. So how is the gravitational force field defined?

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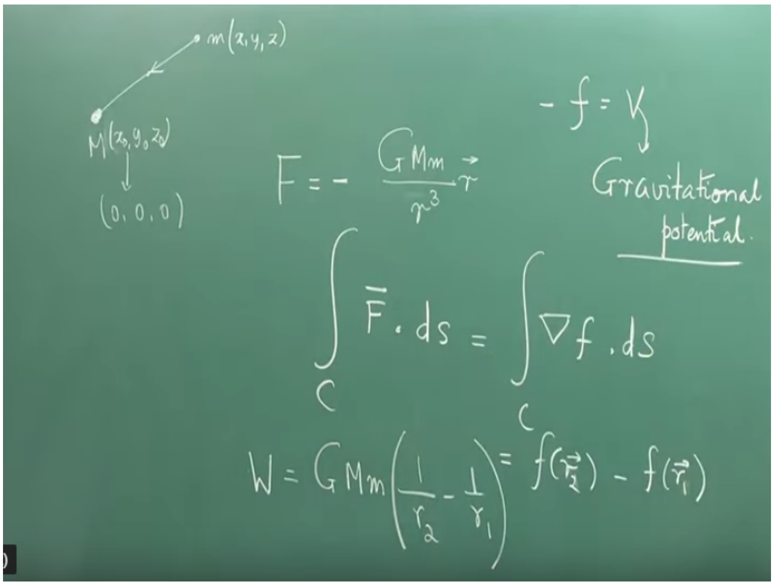

You know the gravitational force field F is an attractive force. So, if a bigger mass M is there and a smaller mass m gets attracted, right. Now I will put this bigger mass at the point is 0, 0, 0 or it does not matter. I will put it at some point say x 0, y 0, z 0. And the smaller mass at the point x, y, z and x, y, z is not equal to zero. Then along this line, there is a force which attracts m towards capital M, this mass.

So this is an attractive force and it is along the direction negative to the radius vector by which I reference the small m with respect to the capital M. So the gravitational force is always written like this. If this is r for example, if this was 0, 0, 0 and it is

GMm by r vector the distance between them or r cube into r vector, with a minus sign because this minus sign is telling that this force is attractive.

Now what happens is that here, if I write it down, and I can write at any point x, y, z where a particle of mass m is there. I will write it down I will take this x 0, y 0, z 0 as 0. The bigger mass is at the origin, this is origin that is this is the point 0, 0, 0. And at any other x, y, z not equal to 0, 0, 0 the mass the whole thing is written as GMm by r cube. So basically r is root over x square plus y square plus z square cube of that into the r vector.

And what is r vector, x i vector plus y j vector plus z k vector, right? And this can be written as minus GMm x square plus y square plus z square to the power 3 by 2 into r vector is x i vector y j vector and z k vector. This completely defines the vector field f. If you want I will put a f symbolism here.

It completely defines, its components are, see if you want to write it as a function from R 3 to R 3 then this is given as I am writing as rho vector instead of a column vector for just for the convenience of the space. It is written as GMm s square plus y square plus z square to the power 3 by 2 into x minus these are the function. This is here F 1, capital F 1.

GMm by x square plus y square plus z square to the power 3 by 2 y minus GMm by x square plus y square plus z square to the power 3 by 2 into z. That is the way I write it. Now let me look at so let r 1, let r 1 vector be given by a point x 1, y 1, and z 1. So I am moving the mass in a gravitational field from x 1, y 1, z 1 to x 2, y 2, z 2. And r 2, so here is the big mass and here is first position and then the second position.

So r 1 is this and r 2 is x 2, y 2, z 2. Then I really have to find the work done by the gravitational force field, work done by the force F; to move from r 1 to r 2, to move the particle from r 1 to r 2. It is like a sun and the planet is moving. So there is always an inverse square law of force, the gravitational force towards the sun. Because when the earth is moving around the sun, the earth is constantly being pulled towards the sun by the sun's gravitational force field. Okay, now how do I calculate this?

It is very important to understand that the gravitational field is a gradient field, F is a gradient field. That is grad f itself is a gradient of some function f. So what is that function f? So f is actually GMm by r; f is actually GMm by r. Now the work done by the gravitational field to move from this point to that point is same as grad f dot s ds. And that is nothing because for a gradient field we just you know it does, it just depends on the value of the function f at the two points.

So it is f (x 1) or r 1 if you want to say, f (r 1) – f (r 2) sorry f (r 2) to r 1. So it went from r 1 to r 2. So what is that? So the work done finally is GMm into 1 by r 2 minus 1 by r 1. So it just depends on the radius vectors, value of the radius vectors. That is all. So f is the scalar field whose gradient is the vector field.

So the negative of f, the negative of f, if I write the negative of f as V then V is called the gravitational, by definition, by convention we do it. It is not that we really have to do it always, but it helps in many calculation. It is called the gravitational potential. So you see how this basic idea of line integrals can be helpful even in determining how a particle moves in a gravitational field and what is the work done on it.

So we are learning some mechanics and mechanics is always fun. So with this I close this talk and in the next one we will talk about surfaces. So if you can parametrize a curve by a single variable, can a surface be parametrized? I may not know whether it is a graph of any function, but if I have a surface, can I parametrize it by two variables, two real numbers. The answer is yes. And that is what we will discuss in the next class. So thank you. Thank you once again.