Calculus of Several Real Variables
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Lecture - 21
Multiple Integration - I

So we start our discussion of the fifth week and we are going to speak about multiple integrals. Why you need multiple integrals. You know what a single, what is the meaning of an integral? Integral is always viewed as an area under a curve.

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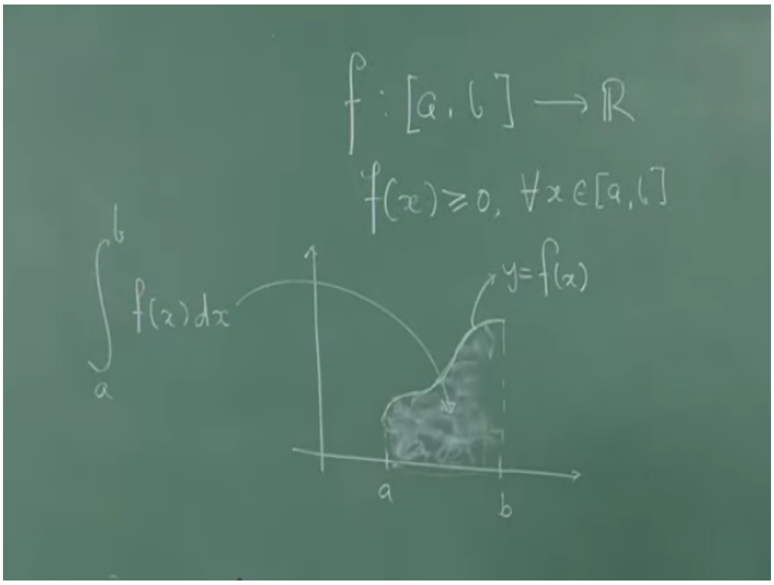

That is the first thing that we speak about that if you have a function f, a single valued function f from a closed interval a, b to R. Let us assume it is a nice continuous function and assume that f (x) is greater than equal to 0 for all x in this interval, then my integral of a to b f (x) dx we mean the following area, the area under the curve. So if this is a and this is b, so it is a non-negative function.

Now a and b could be, a could be negative, b could be negative also, does not matter. But the function itself is non-negative for example, no sorry it has to, for area it has to be non-negative. So this area is what is known as the integral. This is something already known to you. But when you want to find something more say a volume. So when you come to high dimension, three dimension, here this line this curve is representing the graph of the function y equal to fx.

But whenever you are in your handling function of two variables there are the graph would be in three dimensions. So if you try to look what is under that you basically want to figure out the volume. So for the volume, you need an additional dimension now, and that is why you need an additional integration and that is why you talk about multiple integrals more than one integration. So let us try to talk about in general what is the meaning of multiple integration.

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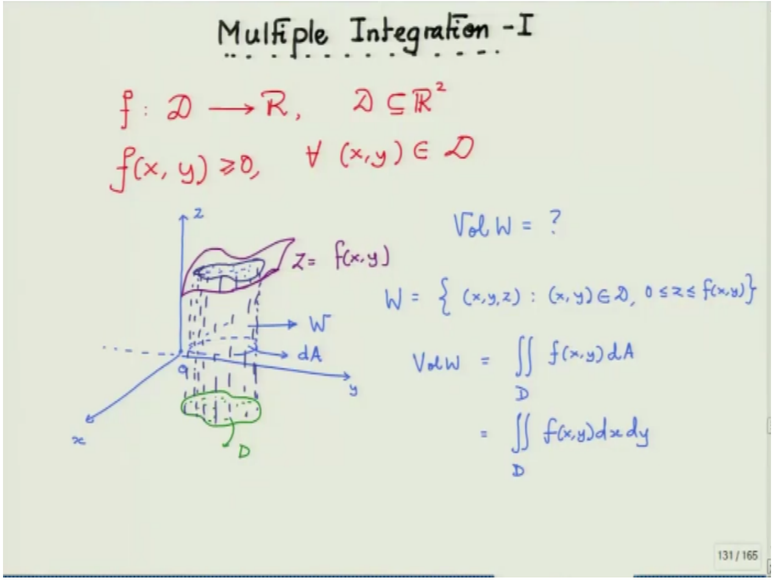

Let me just write down so that you remember what topic we are starting in. So this is the first talk on this. So now suppose you have a function f say defined over a domain D, which is a subset of R, to R where D is a subset of the two dimensional plane and f (x, y) is greater than equal to zero. That is it is non-negative for all x, y that belongs to D. So let me try to draw the diagram of such a thing and let us see what happens.

So again as always we will understand that we have to draw three dimensional coordinate system because now we have two variables. So the graph would be in three dimensions. So let this be x, y axis with origin o and z axis. Here is my domain D. This is my domain D and the function on the domain D, is something like this for example, taking non-negative values. So let us see function.

So I do not know whether I have been able to just match the domain D. So assume that let me try now to try to figure out how much, so if I draw perpendicular lines from at every point of domain till it hits the surface and then trace out how much of how much of part of the function it comes under that. Then that is the functional value which domain D encompasses.

So let me just try to take a different color maybe this one and try to see, so maybe kind of this is the kind of area that domain D would encompass. So it will form a kind of a cylinder, not only a cylinder a kind of a, so we cannot form a cylinder but kind of an object, kind of an object which looks almost like a cylinder on one side but with very strange sides.

It is kind of, if you look at it, it is like a piece of wood and if you have seen when if you have seen any shop where they deal with heavy woods, heavy you know when you cut a tree for example and look at the tree, just cut from both sides, it looks like this. So how do you find its volume? That is the question. Here of course, you have z is equal to f (x, y).

And this region W and we want to find the volume of W and that is what we are trying to figure out. And so how do we do. The volume of W, so what is the volume of W or what is that region? So what is W essentially? If you look at W it consists of all x, y, z where x, y is in D and z is non-negative and less than f (x, y). So that is exactly the region. So all the regions.

So if you are here on the surface then z is equal to f (x, y) and if it is below it is z is less than f (x, y) and that is what you are trying to find the volume of. The volume of this is like, you know you take the function value at any point, the function values are like the length and this area, this elemental area, dA or any elemental area I would like to say not really this one. So take a cross section and get an area out of dA.

Then the volume of W is defined as the double integral over the region D because that is what you are integrating on; f (x, y) is the length. So we want to find the area, volume of a cylinder what do you do? You take you know the radius of the both sides. So you find the area of the circle pi r square circular side into the length of the cylinder. So that is exactly what we are doing.

We are taking a elemental area and then we are matching it up by the length multiplying it with the length that is it. Corresponding to this we are matching it up, right? So this is exactly so for a we are doing it for every piece. It will become very clear. We will give you a demonstration of what really is this. So this is also written as because dA area is length into breath.

So it is also written as or dy into dx whatever you want write. Sometimes it becomes much more easy when D is a rectangle. D could easily be thought as a rectangle and you can consider D to be a Cartesian product of two closed intervals in R.

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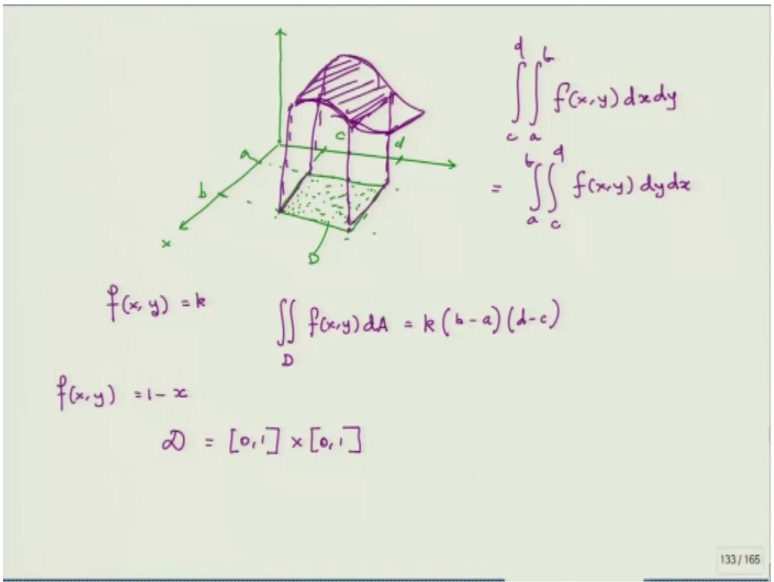

So like this is a, b on x axis and you have c, d on y axis. And as a result, if you take the Cartesian product, I am sure all of you know how to take Cartesian product, you get your domain D as this region, where regions or region where I am doing putting the green dots. That your x is varying from a to b and y is varying from c to d. So this is your domain D. And now of course you can have some function value, function whose you know and you want to get the volume under that.

So you can obviously calculate the volume of this function. So you know in the same spirit what we had done earlier, we are not going to share or do anything like that. You know the volume of everything under this surface is given as integral a to b, c to d, f (x, y), dxdy. In fact, we will show very soon that it can you can also write this as f (x, y) you can change the switch c to d, a to b; you can do the switching.

So this is actually the volume. So this volume that you would have here within this zone, the volume under this part, so under this part only. The volume under this part of the curve. So that part, this volume is given by this integral. Sometimes computing this is not very difficult because if you already know the nature of the thing that you are talking about, for example, if this f (x, y) is a constant function, whatever your x, y this is the it cuts, there is a fixed value of z, it is k.

Then you know that this double integral this volume is nothing but or so I am writing this in short k times maybe in this case, I should just make a change. It should be b and a; b is bigger than a and that is however I have written here in the integration limits. So it becomes b - a into d – c. So there are certain cases you can immediately figure out what would be the volume.

But it is not true for every possible case. That that is really not possible. But for example, if you take an example which is here in the book. It is for example, if you take z or f (x, y) = 1 – x. So it means this is independent of y. So basically, first I should know how do I look at this problem? I should look at this problem in the following way.

So what this kind of problem you want to figure out what is the volume under such a curve and maybe I will give you the zone. So where my domain in this case, my domain D is the Cartesian product of 0,1 into 0,1. So it is the unit square with one vertex at zero and vertex at 1, 0 1,1 01. 101101. So what I am trying to do, first I have to observe there is a relation between z and x. There is no relation between x, y, and z.

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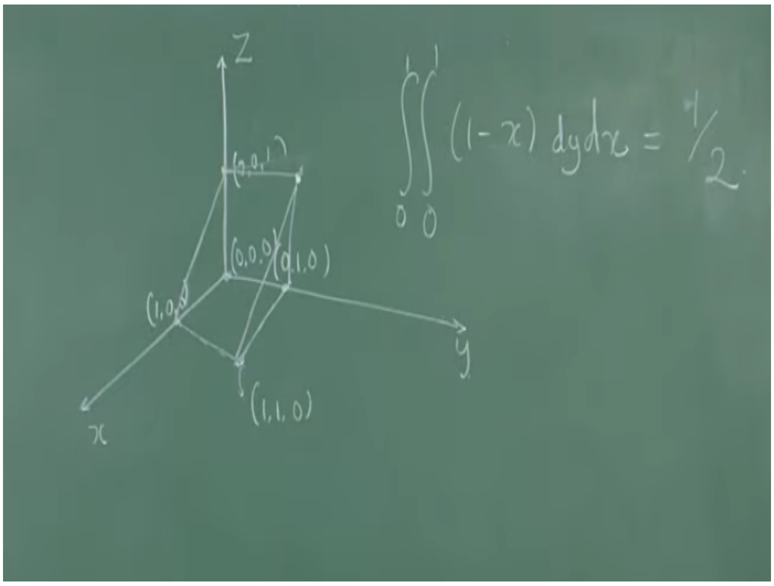

So if I draw the z axis here and so first I should know that my domain is, so this is 1. My domain is here 1, 0, 0 in 3D. If I go by the 3D coordinates, I can write down the 3D coordinates or I can write down the 2D coordinates also if I want but I am writing it as 3D coordinate because I am representing everything in terms of 3D. So it is better to write it down in terms of 3D coordinates.

So x is varying from zero to 1 and y is also varying from zero to 1. That is the whole point. So here, this is also to 1. So here, it is 0, 1, 0. And you take the Cartesian product you have a point here. This is your domain. And this whole, everything inside this is the domain. And this point is 1, 1, 0. And this point is of course 0, 0, 0. Now observe there is a relationship between z and x.

So if I just look at the whole problem as a relationship between z and x and it is a function of one variable x and it gives me z + x = 1. So if I put here the point 0, 0, 1 then first the relationship is actually the first let me just look at the relationship. This is the line which passes through this. So we are bothered only about this part because x is varying between 0 and 1.

Now here you have to observe that this is what would always remain the relation between x and z. It does not matter what y you keep on choosing. You can keep on choosing y. So it will simply be a kind of a wedge. I am making a mistake. Maybe the drawing is not so good. So is a kind of wedge. And if you look at this from symmetry, you can immediately show that it is half of the unit q.

So for this particular case, it is very simple to write this double integral value is actually half. There was a unit q as your unit volume. So this is a very simple case, but that is not the way we will do each and everything. What we are going to do today now we are going to use an idea from single variable and an idea which is due to a Italian mathematician called Bonaventura Cavalieri.

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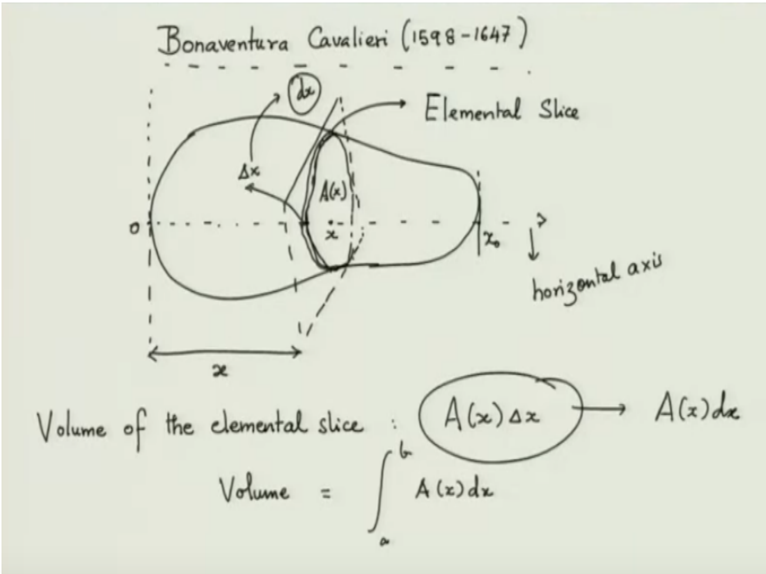

And it is good that this book actually gives a little bit of history, which is always a very good guide to pedagogy. So his idea was like this. Suppose you have a solid, kind of round solid and we will give you a demonstration. What he is doing is that, for example, this is your origin. He is keeping chronology next. And he is drawing a kind of line, which is tangential to this one part, and pushing a line perpendicular to this. And now he is chopping off.

So this volume, so he is cutting it off through planes which are perpendicular to this horizontal axis, means this axis, the one which I am pointing out now. So this is the horizontal axis. So when you cut out, you basically cut out a section from the, so in some sense you have cut off kind of you have made a cut and the cut is something like this.

Now this is at a point say x, where you have made the cut; x from the origin. This point we will call the origin and this is x from the origin. Then around x you make a slightly smaller cut like this. So you made a very small slice, you can get out of, it is like cutting potato, we will demonstrate. This small distance is what is called delta x. But when x is very small, when the difference delta x is very small, we often write it as through the symbol of the differential called dx.

Obviously, I am not going to deal with the meaning of this. I am trying to intuitively grasp it. Now what is the volume of this? This slice is called an elemental slice, or elementary slice, I am just writing elemental. That is the language I have learned from my childhood days, elemental slice. Because it is x, the x is a position. The area depends on this x. So area of this slice is Ax and suppose I can find this area.

And volume of this elemental slice, volume of this elemental slice is nothing but Ax into dx. Now what is this whole business whole how do I get back the whole object, I just keep on changing my x and keep on computing the volume of this, computing this elemental volumes. And when I sum up those volumes, I am making this delta x go small and small and small. This whole thing goes small and small and small.

So then basically you can say if you are not comfortable with writing dx I can call this as delta x. And I can write this elemental volume as Ax into delta x, does not matter. Now for every x you do it and you make a sum and you what you do, you then make delta x go smaller and smaller and smaller. So the volume finally is a sum of all such elemental slice, and that is nothing but integral a to b Ax dx.

You can put dx here also, does not matter. I would like to put dx because that is the way I have learned nothing but if you want to keep it at del x, and then try to understand it in that form, that is all right for me because then you can say okay, I take various points x 1 and partition this whole say zero to this is the point up to which the body is zero to say x naught I partition this thing.

And then at every partition points I construct this elemental things and the width is the partition token or partition width and then that is also a way, this the same way. So you can put del x here dx here it is up to you. So I leave it, I leave it to you. You can put, you can put dx here also. So elemental volume sometimes you can write if even if you write it like this, this is absolutely an acceptable thing.

Physicists actually write in this way, but I do not mind if you are writing Ax into del x. So now this is the whole procedure to I will just my TA would demonstrate to you by cutting a potato. And this is exactly what we are going to do, what one does. So here is the potato. So here I want to find the volume of this potato. So here is the zero axis. So here is the zero point right? Or if I go as per you also.

Here is a zero point and then through this I am pushing in a vertical line along which this body can rotate. And then I am cutting it by planes which are perpendicular to this vertical line. And then I know the slice, so she will demonstrate to you how to cut this. She is a better expert in cutting it. I am sure you can, she can show you. So here is no here is at x or here is at x is one slice, A(x) and then I can got a very small slice.

And here is your elemental slice. Ax into dx is the volume. This is your elemental slice. You observe how small it is. You can keep on making thin and making thinner and thinner and thinner and thinner. So basically you will have very thin slices. So you take the area, all those things you have covered all the point. Basically you have computed the volume as you move from zero to x naught.

So this is what it is. So you see, you can give real life demonstration to things that you do in calculus. Mathematics actually works. Of course, you can ask me what is Ax here. I can actually compute this Ax for this potato by some kind of approximation with a circle. If I look at Ax very carefully, you do not bother much about this part. This Ax can be computed if this part was also there as a kind of circle.

So let us see how double integral arise. You said okay, you have bought the volume problem down to a single integral issue. So why double integrals. Double integrals makes life much more simpler. So every time you are not going to do this whole procedure. For this whole procedure, this procedure can be used to handle double integrals.

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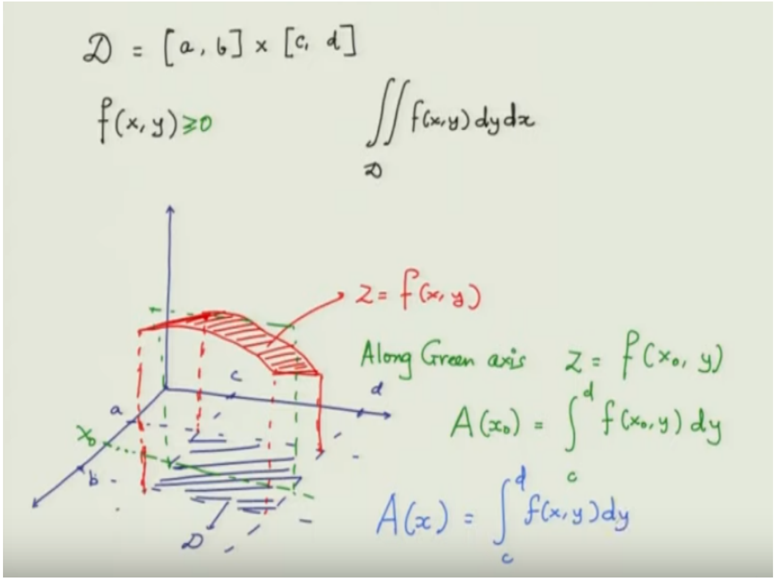

So suppose you are given to find, suppose you are given here R or your domain D as a cross b, c cross d. And you are given a function f (x, y) or 2 to R and you are asked to find the integral over these domain D f(x, y) dydx. Now there are two ways of doing this, computing this integral. Basically the volume under this graph over the zone D. So as before I draw the graph. This is b, this is a, this is c, and this is d.

So this is your domain D. Now you have your functional graph somewhere here. Let me just draw the functional graph. This is an arbitrary drawing, do not think that this is some kind of serious kind of drawing. This is your z equal to f(x, y). Now what I will do first, let me fix up an x naught; x naught is fixed. And then I actually cut this at x naught along the line x naught the line perpendicular to x naught crossing the domain D.

I just cross a plane which is perpendicular to the line which passes along the line perpendicular to perpendicular to the x axis passing through x naught. So there will be a slicing here. Now if I slice it here, then when I have fixed x naught along this axis, along this axis, this green axis, along the green axis, along the green axis z is equal to f (x naught, y). Now what I have to do first find the area of this.

So area at x naught is integral c to d f (x naught, y) dy. It is just applying the standard thing that you know in calculus one variable, area under a curve for a non-negative function. Here the function is non-negative. We will of course, we are assuming that the function is non-negative for the time being. Later on things can be made slightly more general. At least on the domain it has to be non-negative.

Now what is important here is to observe that this I can do with any point x naught that I choose between a and b. So in general I can always write, if I cut by planes along cut by planes which are cutting this xy plane perpendicular along the lines which are vertical to the x axis like planes like this. So I am slicing the cake like this, instead of slicing it along the y direction, I am slicing it I am slicing it along on the x direction.

So instead of slicing, I am slicing it along the so what I what I am doing so I am fixing my x, I am slicing it not along the x direction but along the y direction. Basically along this curve, along as y is moving from c to d, you see I have taken a slice and found the area. So now I am slicing it along sorry the x direction. So I can move from a to b now, right. So I found the area and this is a slice. I fixed my x.

Then I passed a vertical line through that x passing through the positive orthant here in this photograph in this diagram and then I passed a plane cutting through the volume which is perpendicular to this green line passing through x naught. And this green line is perpendicular to the x axis. Now I found the area, now so in general this is true for any x that I choose.

So in general I should have A (x) should be integral for a given x. So this finally becomes a function of x because I am integrating out y.

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$$\text{Volume} = \int_a^b A(x)\,dx$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$\text{Volume} = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

So now, my final volume, now my final volume should be as I move from a to b in x is integral a to b A x dx. While I know what Ax is, I know that Ax itself is an integral f(x, y) dy. So that gives us this integral. So this is called iterated integration. First integrate with respect to y get a function of x and then integrate it with respect to x.

This is how you compute, but you have to observe in this picture that here instead of doing these I could have fixed a y or say y naught. I go have fixed a y naught here and then the same procedure. By taking a plane which is cutting the diagram, cutting this curve as I move the y. So it is cutting the x axis, this xy plane vertically. But cutting, but passing along the, so it is passing along the line which is vertical to the y axis and passing through y naught. So that plane is passing along that line.

So in that case the whole thing can be changed instead of Ax I will have Ay. And here this will be a to b f (x, y) dx. So actually, it is not, the volume can also be written as, also be computed as first you compute, so you are finding the area along the y axis. And then you integrating to get the volume, you can have this. So we will show this through an example, a simple example so that you are not intimidated.

This is a very simple example. We will take see, this is such a simple thing that once you know what to do, you do not need when you have a rectangular thing, when you have rectangles when you have here we have taken rectangular scenario, rectangular domain, then you will always get this type of elementary computation, one integral after the other.

So I will give you now, what calculus does is that you do not have to draw pictures every time. You can draw a picture, but you need not. For example I am taking this example. Though you can draw the picture I will not draw the picture.

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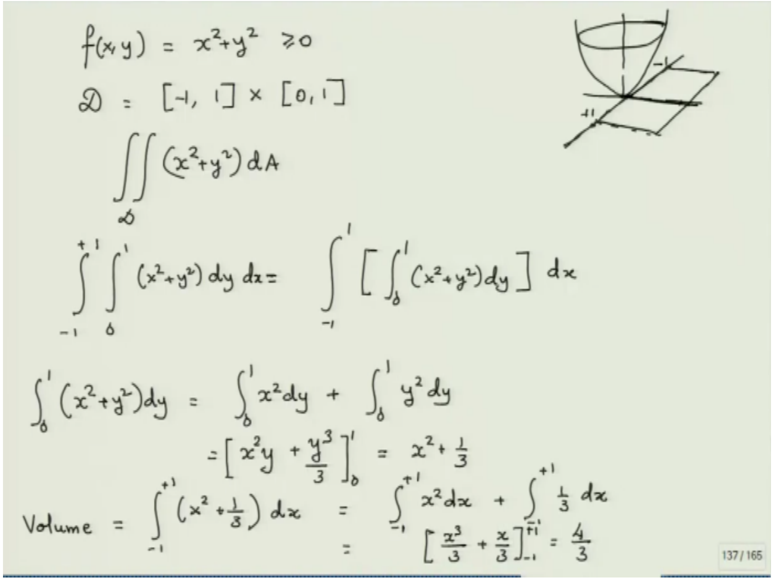

So I am giving you f (x, y) is equal to x square plus y square, which is anyway non-negative function. It is always greater than equal to zero. And the domain D is minus 1 to plus 1 which is the x axis x domain and 0 to 1. And I if one wants to evaluate the integral. So here this is a parabola. It is like this, x square plus y square. If you draw it the parabola it is like this.

So you know that the x domain is -1 to +1 and 0 to 1, -1 to +1 -1 +1 0 1 okay. This is your x domain. So basically your domain now becomes like this. So this becomes your domain. No I am making a kind of mistake. Minus 1 is far off compared to plus 1, symmetry is getting lost. So here is a domain and then you are trying to find an area under this curve here the volume.

A kind of a very strange looking shape, of course and you are trying to find the volume. You might be intimidated, because you might think, oh gosh, what is happening because this x square y square business is greater than equal to zero and because of because it is always non-negative, I really have to see because it just keeps on moving towards infinity right?

So but it, once it gets out of your domain function values finally gets out of this domain, this x, y domain. Then you do not have to bother about that function, you have to see only up to that part where it goes to functional value at this boundary. And from there you have to drop the perpendiculars to create that volume. But it is very difficult to geometrically see. Let us let me admit to you one thing.

If any people who say that they can think fairly, very well in three dimensions they are top class mathematicians. People who say that they can think very well in two dimensions, they are pretty well to do mathematicians. People who will say that they can think even in four dimensions are lying. It is very difficult. I strongly believe that up to three dimension you can make up you can visualize because we are ourselves in three dimension.

It is very difficult for us to visualize three dimensional objects clearly. We can visualize two dimensional objects clearly. For example, for ant, for ant it is very difficult to make a guess of the height of a person. But not but for me, I can easily understand what that ant is doing. So that is, ant for us is almost like a two dimensional animal.

So please understand that this so what calculus does, is that this geometry that it is so difficult to visualize in three dimension, it gives you a tech mechanical tool to compute that. So you can talk about geometrical concepts like area, volume, all these things even without looking at the geometrical picture. And that is so important, that is the power of calculus that we need to appreciate.

So we really have to find this domain D x square plus y square dA. This is what we have to evaluate, dA or dxdy whatever. So now let us evaluate 0, 1. First we evaluate with respect to y and -1, +1. So dx. So first we evaluate with respect to only y. So let us separately work this out 0,1 x square plus y square dy is equal to integral 0 to 1 x square dy plus 0 to 1 y square dy.

X square cannot be integrated, it is taken as a constant. So you take this out, okay. So you will have here is x square y plus y cube by 3 standard integration evaluated from 0 to 1. 0 you do not have to bother. So this gives you a function of x, x square plus one-third. Now you what you have to do now the remaining part what is left, so the volume is -1 to +1 x square plus 1/3 dx which is -1 to +1 x square dx plus -1 to +1 one-third dx.

So here if I integrate I will have here x cube by 3 plus x by 3. -1 to +1. So if I put 1 I will get one-third plus one-third two- third. If I put -1, I will get what? If I put 1 I will get one-third plus one-third, it is two-third. And the next part also I will get -1. Here x cube will be -1, minus one-third plus -1 minus two-third. So two-third minus – two-thirds. So two-third plus two-third. So it will become four-third.

So volume is four-third into whatever you need centimeter, meter whatever be the units. Units, I am not writing volume is four-third. You can I would now ask you to go back to your house and do a reverse operation. So you do first -1 to +1 x square plus y square dx, evaluate that and then do, that will become a function of y and then do evaluate 0 to 1 that function of dy. You will see you will get the same answer.

So here is a demonstration of how volume can be found when we have a rectangular domain using the double integral and for functions which are non-negative. In the next class, we will continue to discuss double integral over a rectangle. And here we will be talking about bounded real valued functions. We are not going to fix our mind on just a function which has non-negative values.

So here what we are going to do is we are not going to bother about. We are not now, we are not thinking in terms of area. We are, we will be using the idea of the area computation, but using that for a general function, we will define an integral in the way one defines an integral and once that is done so that would be for any bounded real valued function, okay.

When though that bounded real valued function is non-negative over the domain then there is a physical meaning that there is a volume. If it is just a bounded real valued function, then we define something called an integral of the function. We do not attach a physical meaning to it. The physical meaning comes when the function is non-negative.

For example, if you take the function y equal to sin x and just take the function y equal to sin x and integrate it from -5 to +5. The answer is zero, because it is a odd function. But what about the area under the curve? Zero? No, you might think oh, it looks strange, but I can argue okay, it is not a bad idea because I can say okay, I can take the area vector.

Area can be viewed as a vector, as a vector perpendicular to that area. In when in one direction the area vector is in this side, or the other direction the area vector is in other side. So the vector their magnitudes are same behind symmetry and they cancel. Some people say no, I do not want to take area like this, area has a magnitude. So the magnitude of this area and magnitude of this area, mod of this plus absolute value of this.

So in minus infinity to sorry minus pi to plus pi sin x dx is an integral. It does not give you the physical meaning of any area, area under the curve.

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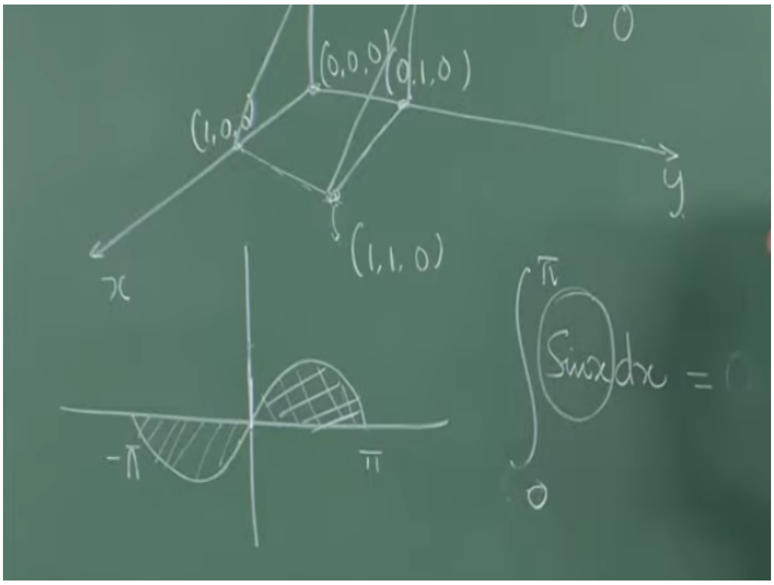

Because here because this is something one has to keep in mind while doing in math that – pi to + pi, the function is not non-negative. You just cannot say the area under the curve is zero, but I can say area under the curve by area under the curve means so here what is, the whole point again would be what is the meaning of area under the curve. Here I am looking at area above the curve.

What is area under the curve? So if I still want to talk about area under the curve or area above the curve, then it has to be something like this. Okay because it is negative I am just form a symmetry, I am taking this area plus this area. So if I am trying to talk about two regions of whose areas I have to really find I want to find the area of this whole thing and I take area of this plus area of this.

But when you integrate it, because this is not a non-negative function, because negative value is here, integral is zero. So you do not attach physical meaning to it. This does not tell you that this is area under the curve, this curve. It is simply an integral, right? Which, so it is a proper integral define accordingly. So please understand when we make such statement so area integral simply means area under the curve. Yeah, it does mean but for a particular class.

If you talk a non-negative, if I talk about this, then it has a meaning. Now it is area under the curve. If I am just talking about this spot, then this area under the curve. Because once you have negative oh, there is a whole idea of meaning of area under the curve vanishes. They will be, then you will be able to of course, you can say okay I can write down the whole theory for functions is a negative over the domain and I will say area above the curve, you can say.

But then if you have a function which is partly negative on the domain and partly positive, what do you mean by area under the curve. So physical significance can be only attached if there are certain behavior of the curve on the given integration domain and that has to be kept in mind. Thank you very much. We will continue discussing general notion of double integral in the next class.