Calculus of Several Real Variables
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology – Kanpur

Lecture - 20
Vector Fields

So, the last talk of the fourth week. We are going to talk about vector fields, a term which is so ubiquitous that a physicist for example takes it for granted. The idea of field that something is there, at every point some kind of an arrow is there, some kind of vector pointing somewhere, is there kind of flow is there was envisaged by Michael Faraday when he was giving a lecture in London about his experiments and his discoveries and he realized there was some time left to you know say something.

So, he started thinking on this idea that how could okay there is a charge and there is another charge, this charge would affect this charge, there will be a force field kind of thing, electrical field. So, this idea of an electrical field emerged from that lecture. So, it is very important to understand the notion of a field what we would now has been now remodeled and bought into an abstract format.

For a mathematician, field is a vector valued function whose every component is C1 in the sense that every component is continuously differentiable.

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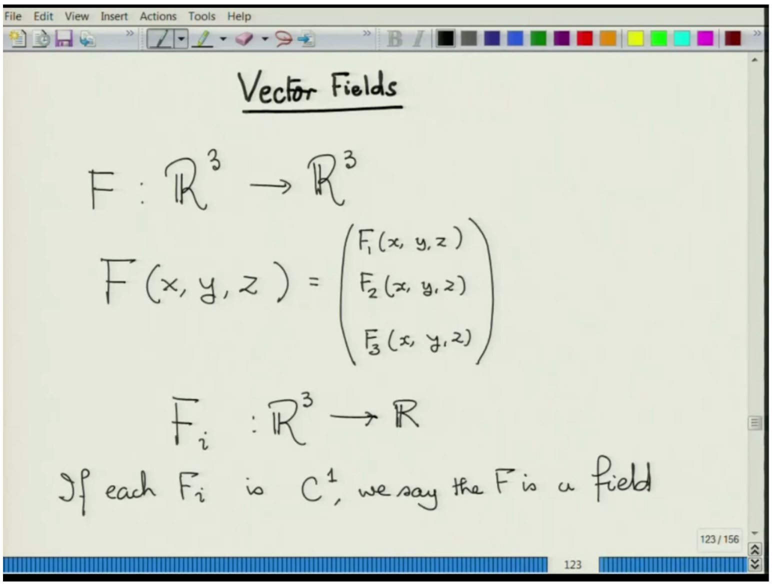

That is consider a vector valued function say F from R3 to say R 3 and F of these 3 variables x, y, z so that it gives me another 3 variables. It takes me to a point in R3. So, some point in a space is taken with some another point in a space but those points depend on this choice x, y, z. So, those points can we viewed, those 3 different coordinates itself depend on the choice of x, y and z can be viewed as functions from R3 to R.

So, these are the coordinates which are real numbers. So, each of Fi is now viewed as R3 to R. So, each of them are real valued functions right. Now, if each one of them if each Fi is C1, C1 is all the first partial derivatives exist and are continuous. Then, if each of the Fi is C1, we say that F is a field. So, this notion of a field we are expecting that for all our practical purposes, these functions value must have their first derivative and their first partial derivative and that first partial derivative must be continuous as a function of x, y, z.

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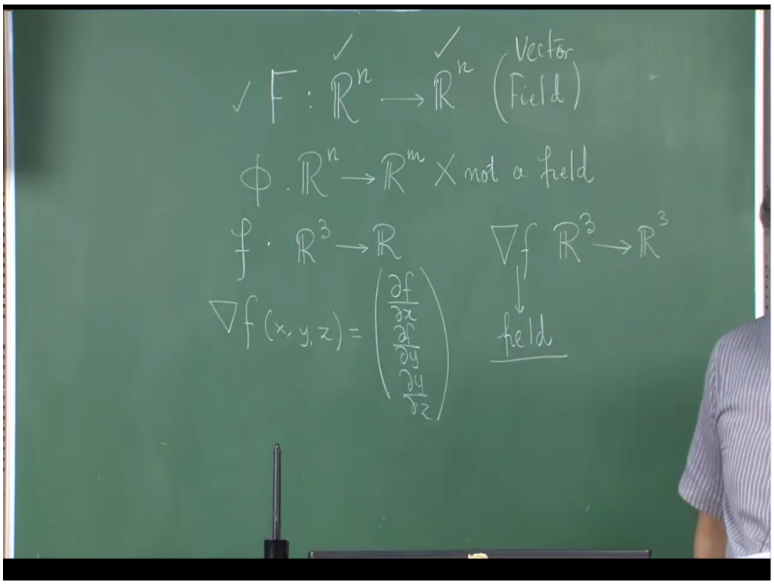

So, observe one very important thing. When you have a field F, you expect Rn and Rn, so a vector valued map whose domain and codomain are the same for a field and for all practical purposes we would need F to be in C1. So, that class of vector valued maps we will call fields. A vector valued map in general, any vector valued map say phi can be from say Rn to Rm. We need not call this a field.

Please understand a field has to have an important property not only that it is C1, the field as I have given a description here, field has to have its domain and codomain same. So, this is an important characteristic of field, same domain same codomain. While such a requirement is not there if you take a general vector valued function, this is not a field, I will not call it a field. Gradient mapping of a function can be viewed as a field.

For example, if you have a function f from R3 to R. So, what does the gradient mapping does? The gradient map f of x, y, z is the vector del f del x del f del y del y del z. So, grad f if these are continuous then grad f is from R3 to R3, so grad f is a field, it is called the gradient field, it is called a field. This is a field, the gradient is a field. So, this is an example of a field. So, field is ubiquitous and very helpful in understanding many physical phenomena.

And that is why fields are so important. For example, of course some people talk about a scalar field. That is suppose there is a temperature, take any point in this room, at any point if you put a thermometer, there will be a temperature showing different temperature. May be it would not vary much but okay there will be a distribution of temperatures. So, all these are for every point in space here x, y, z there is a temperature.

So, that is the mapping from R3 to R that is sometimes called a scalar field but in general when I talk about a field, when I am talking about a vector field, I just have to mention that this not just field I should call it a vector field. When I am really talking about a vector field and not a scalar field, the domain and the codomain must be of the same dimension. That has to be an important fact to remember. So, how do you talk about of field? So, let us draw it diagrammatically.

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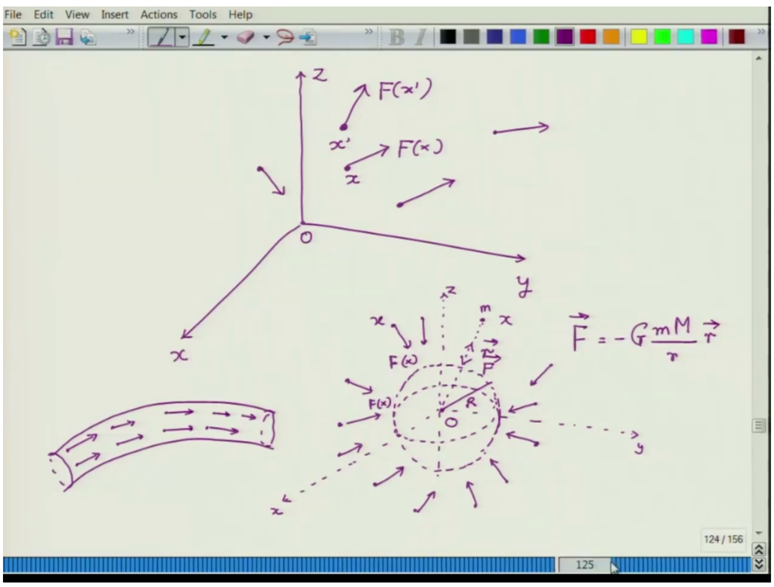

So, here is the origin, x-axis, y-axis and z-axis and here is your x which is x, y, z in this case and here there is a field Fx and that field has a particular direction. So, if you think the field could be in this direction, the field value F could be in some direction, could be in this direction, this direction. So, another point the field could be in this direction, another point so this is the F, F of x.

So, another x dash F of x dash, so we start at the x point and point of vector in the direction of Fx, could be this direction also, opposite direction also. So, this, that, this, this scattering of arrows, this picture diagram of arrows that forms what is called a field and that is a pictorial viewing of a field. For example, if you look at the velocity field of a fluid passing through a pipe. So, here is the velocity field you see.

So, magnitude of the field that is decreasing as you have, basically so here you see the magnitude changes, the velocity. So, here is the velocity field. Similarly, you can talk about the gravitational field. Now, you take the 3 axes, x-axis, y-axis and z-axis and let at the origin be the centre of the earth and we assume that earth is completely a spherical object though it is really not; it is slightly thrashed on the top.

But for all practical purposes we can consider it to be a spherical object. So, this is the origin, this is origin of this earth and now this earth so this is the radius of the earth and any point x which is whose distance from the origin is more than the magnitude of the radius, any point x, you can consider a field there. The field of gravitation and field of gravitation is attractive in a sense that it is always directing in the direction.

So, this is of any point x, for that point this is the R vector. Sorry this I can write as capital R, radius of the earth and for that point R vector is the position vector but gravity is always an attractive force. So, it will always act in the opposite direction, the direction of force will always be in this way F. So, at any other point x, you will always have a force, so the direction along R is taken to be positive direction, opposite to R is taken to be negative.

So, given in x this always is gravitation force F of x. Given any x, there is another gravitational force; x is of course the vector. I am not writing F of x, I am just writing F of x. So, there is gravitational field at any point, there is a gravitational field which is directed towards the center of the earth and this every gravitational field of course can be described mathematically.

Any point outside has and you know that this gravitational field F it has to be given a negative sign because it is in the direction opposite to R. So, m is the small m of the mass m here and m is the mass of the earth, distance of the mass m from the center of the earth and r vector. So, this is how the gravitational field is described right.

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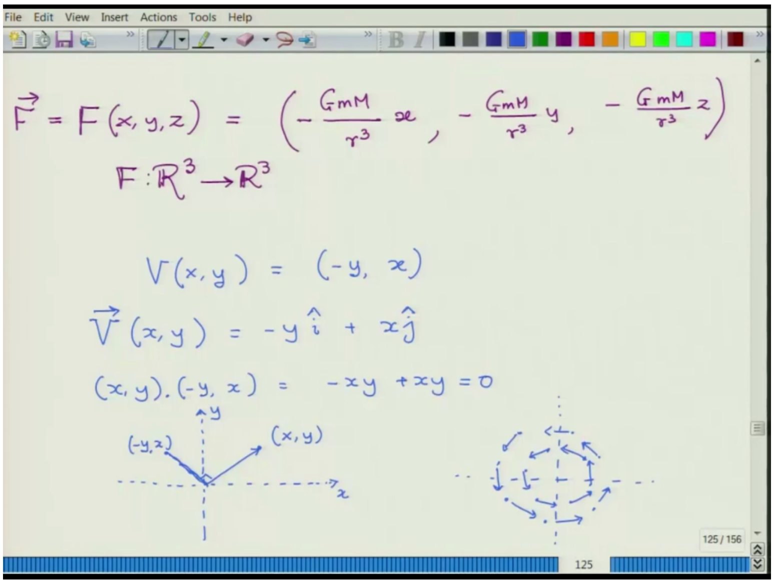

So, now you can in fact describe it as a vector valued function, the gravitational field function F x, y, z. When I am writing it in the form of a function, I am not putting the vector sign. So, F vector is actually F of x, y, z and this is given as minus GmM/r cube into x, minus GmM/r cube into y, minus Gm into M/r cube into z. So, this function describes the gravitational force field. So, this F of course is a function from R3 to R3.

See because of the fact that it is from the same, the domain and codomain and range are same, you can actually when you are in R3 at least you can draw the field diagram. The field diagram is possible because the range and domain are the same. If it was not there, the field diagram would not be possible. You cannot draw field diagram for anything. Whatever you get, you cannot draw the field diagram.

Another interesting field for example if I give you a vector field like this, V x, y = - y, x. There is an interesting feature here. So, of course from a vectorial point of view from a physicist point of view, it is written like this minus y i vector plus x times j vector. What does this vector field represent? It represents actually a rotation of a point, a coordinate point x, y by 90 degrees. Why? Because you take x, y and take – y, x and take their dot product.

So, x, y dot product minus y, x is minus x, y plus x, y is equal to 0. Showing that x, y and minus y, z must be perpendicular to each other. So, basically what it is telling that okay this two-dimensional field, if you have the x-axis and y-axis, if you have x, y here, then you rotate it by 90 degrees. So, here you draw a vector, 90 degrees of the same length. So, it has to satisfy the orthogonality condition and this point is minus y, x.

So, how do you draw a field? How this field is depicted? The field is depicted like this. So, take any point, it is rotated 90 degree, so it is a kind of thing like this. So, even do it for other, so this is the field diagram. Another important kind of field which actually Faraday had envisaged is the notion of an electric field.

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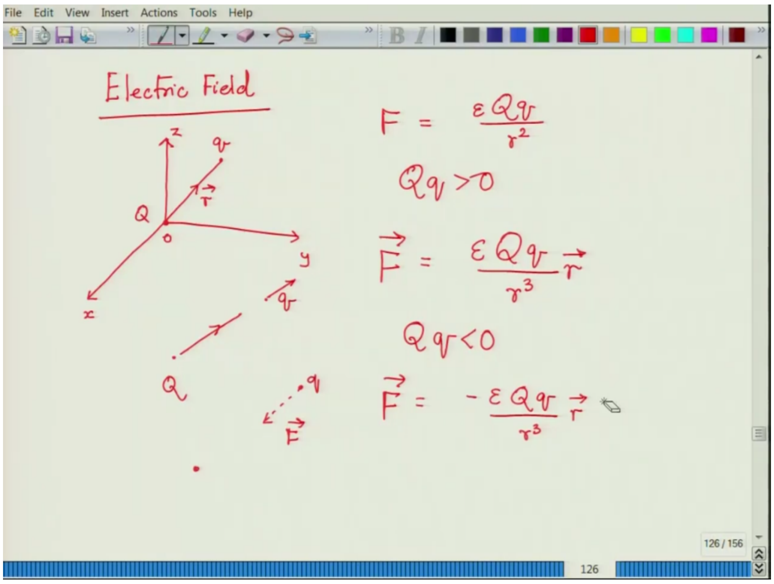

So, now think of a three-dimensional space where a charge Q could be a positive charge or negative charge you do not know, a charge of magnitude Q is at the origin O and then at any point in the space say which is at a distance R, at distance given by the radius vector R, there is a point with charge Q, which could be positive or negative. Now, Coulomb's law says that the electrical force has the magnitude F given by some constant epsilon.

So, we will not get into the details of the physics here, Qq/r square. So, it is quite like the gravitational law that its product of the two charges divided by square of the distance that it is inversely proportional to the square of the distance and directly proportional to the product of the charge. It says that if you take the two charges far and far apart, if you take this charge Q far and far apart, the force, electrical force on the charge q because of the charge Q decreases.

Now, how do you depict it as a field? How does a field look like? That is the question that we are going to ask. Observe, suppose Q and q are like charges, say both are positive charge or both are negative charge. We already know that like charges repel each other and unlike charges attract each other. So, if they are like charges, if both are positive and negative, we have Qq.

So, if it is so that both are like charges, obviously charges are not 0, both are like charges so Qq is strictly be greater than 0. So, which means it is a repelling charge. So, repelling means when it is repelling if this is Q and if this is q here, the repelling force is always in this direction, from Q the force is away from q right. It is always away from q. So, the small charge q is shown away from the big charge Q.

So, in that case, the vectorial representation would be Qq because it is along the direction of the radius vector now r cube r vector. So, this constant we are not going to discuss, this is a constant which comes from physics. Now, suppose the 2 charges are different, Q is positive and q is negative where Q is negative and q is positive that is 2 charges are different. So, they will attract each other.

Unlike charges attract, that is the law of physics and that is the law of nature rather. So, in that case, the forces in this case this q the force would be attractive, this q and this force now is attractive, it is against the direction of the radius. So, if this happens, then the force is represented as, they all depend on the sign. Now, Qq is negative because this is the attractive force, I will write it like this initially when to start with.

But because Qq is negative, it will cut with this minus and this Qq would cancel minus of this. So, ultimately you will get a positive quantity here right. Now, there is an interesting class of force field, they are called conservative fields.

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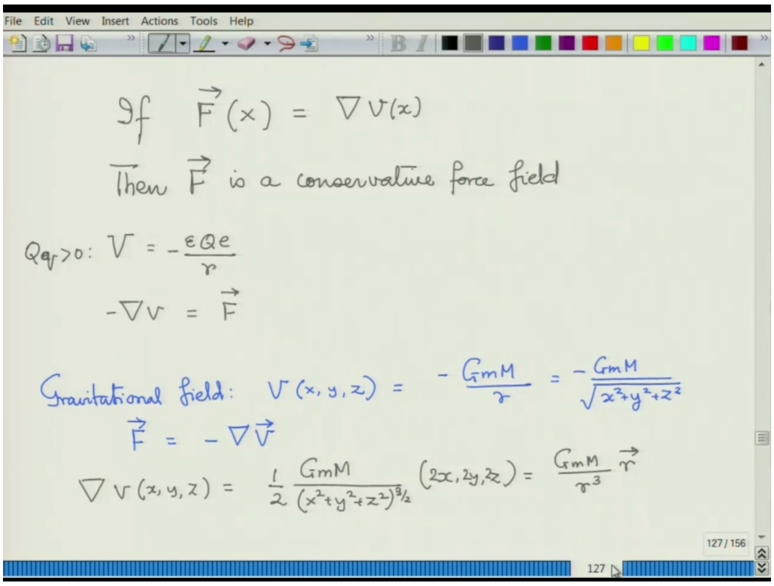

If there is a force field F sorry a vector field. If there is a vector field F, let me just change the color. Consider a vector field F say in Rn or R3 or whatever; say in Rn such that it is the gradient of some scalar valued function say V of x. Then, F is said to be a conservative force field if. Of course, if say if you take V is equal to suppose Qe in the last sorry Qq in the last is greater than 0 and you take V to be epsilon Qe/r.

Then, if you take V to be epsilon Qe/r and take a minus sign here, then gradient of V would exactly be equal to your F. Similarly, for the gravitational field, suppose I put V x, y, z as minus m sorry minus GmM/r that is minus GmM/root over x, y, z is the coordinate of that point. Then, F the vector field is negative of the gradient of V. Here also sorry there should be a negative sign.

So, instead of V you can take minus V as your V here. So, all these fields, the electrical fields, so what do we conclude, so let us see, let us differentiate this V. So, if you differentiate the gradient of V, so this is my 1/2 say go as minus 1/2, minus of minus 1, so it will be minus 3/2, so it will be, so grad V x, y, z if I want to differentiate, first I will differentiate this and then first I will differentiate this and then take one with x, the derivative with respect to x.

So, minus of that is this is to the power 1/2 that will become upper minus 1/2 so that minus 1/2 will come up so we just do this V is this minus 1/2 is 1/2 again, 1/2 into root over or rather x square plus y square plus z square power 3/2, this into the vector x, y, z. So, this is nothing but 2 we will cancel from here, it will have 2x, 2y and 2z because you differentiate x square, take the partial derivative.

So, it will become GmM this is nothing but 1/2 of x square plus y square plus z square to the power 1/2 which is the root is nothing but r which is r cube into here it is r vector, x, y, z, 2 gets cancel with this. So, now if you take a negative of this gradient, so F is nothing but the negative of this gradient if you go back and see the force field, this is nothing but negative of this gradient.

Two important but every field that you see is not a conservative force field, please understand. When you have a conservative field, the conservative force field, in a conservative force field, energy conservation laws hold, that is the very important idea in physics, you check, you go and read some classical mechanics; it will be good for you, for understanding all these things.

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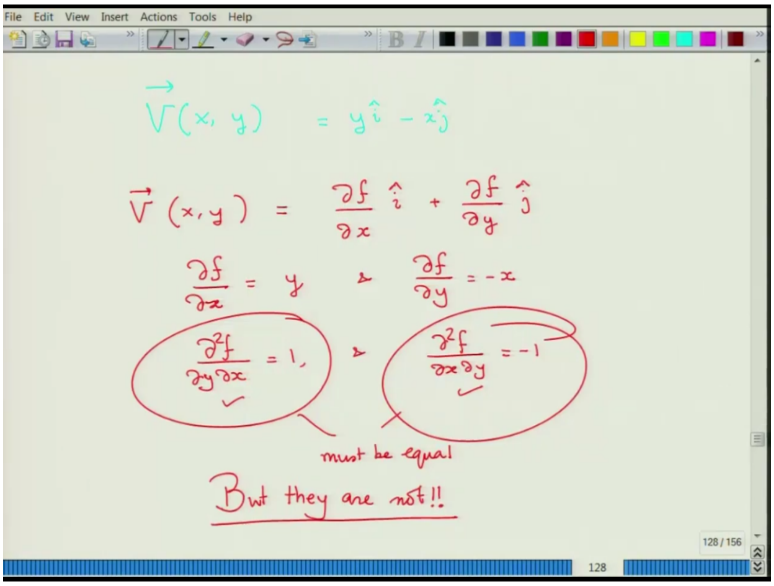

For example, if I take this V, I am just trying out nice colors, do not worry, V x, y is vector x, y is yi vector – xj vector okay. V x, y is yi vector and xj vector, is this a conservative force field? Is this a conservative force field means can I write it as a gradient of some function. Suppose, I can write it as a gradient of some function, suppose that V x, y can be written as del f of del x, f is that function of i vector and of course a function of two variables del y of j vector.

Of course, this must have del f but then if all the this del f del x is equal to y and del f del y is equal to minus x but here if you look at V, look at the components of V, x, y; y and minus x. It not only has first partial derivatives which are continuous, it has second partial derivatives which are continuous. So, we expect that F must also have its second partial derivatives continuous because a vector field F must have its first partial derivative continuous and if V is grad F, it must have its second partial derivative to be continuous if it has to match with V.

Because V has its first partial derivative continuous and V is grad of F, so grad of F must be having second partial derivatives to be continuous, so it is mixed partial derivatives must be equal, so which means let us see del 2 f del x del y del x let me see what happens, del y del x is 1 and del 2 f del x del y it is minus 1 but if really had mixed partial derivatives which are continuous, these two must be equal and these two quantities, these two must be equal.

But they are not and hence that brings us to the close of our discussion because it shows that every field but the best field that we know in physics is force field. Friction for example does not satisfy this conservative force field business and non-conservative force fields are not so interesting always. So, there are many things which I am doing many things which I am leaving for example proof of Taylor’s theorem for second order.

Such things would be actually the several things which there are which I am writing down to put it on your portal and which you will soon have. So, you do not have to worry, this promise would be kept. So, the several things I have promised which has to be kept and so the fourth week has gone. So, we are going to get in to integrals now. The next session starts with integrals, multiple integrals.

You have learnt to do single integrals and now you are going to talk about integrating on a surface where your elemental area is not a length but a surface. When you are talking about both length and breadth at the same time, we are talking about area, instead of length we are talking about area, surface area. So, we are talking about length and breadth. So, we have to integrate along the length and integrate along the breadth.

So, there will be two integrals repeatedly, iteration, so that is what, so that begins the idea of multiple integrals which will be a part of our discussion from the next class and hope that you are enjoying. If there are questions, please put it up in the portal, I would be very happy to answer with such you know it is not possible to cover each and everything but largely we are trying to give you the ideas that would be helpful keeping in view the type of audience that I might be addressing.

Of course, the audience is an audience I do not see. Without seeing you I am trying to view you that is the whole point of this teaching in MOOC courses. So, thank you very much, thank you once again and I would rather say bye for today. Have a good weekend.