Calculus of Several Real Variables
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Lecture – 10
Chain Rule for Partial Derivatives

So welcome to this last talk of the second week. So what are we going to learn today in this talk? So, we are going to learn today about Chain Rules. Chain Rules are important when you compose to functions. That is you take an element and get it to one set. And on that new set you put in another function you employ on the image, you employ another function to get another element so.

So you use two functions one after another, and that is called a composition of functions in a very abstract setting.

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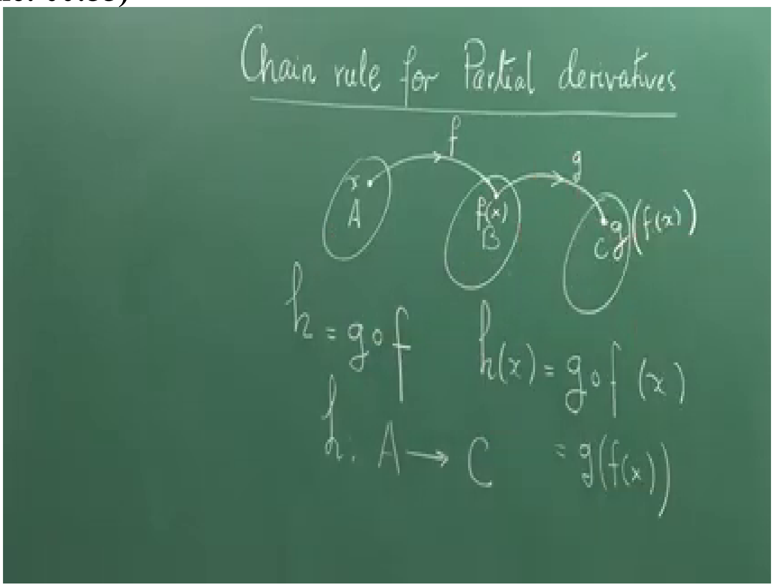

For example, if you have three sets, I assume that you guys have some idea what sets are we going I am using you have learned, you were we use sets trailing one variable. So here I have an x and our function f mapping meet to some point f of x here and then there is a function g. So, which takes meet a point and this point is g of f of x. So, if I considered right this function h as g composed of f that is first I apply f then I apply g.

So f followed by g. So this h of x which you can write g composed f of x is nothing but g of f of x. So, if f is cutting element from A to B and g is cutting element from B to C, then h which is g composed f is carrying an element from A to C. Now, how would you handle suppose we are talking these are sets in real line this could be a closed interval open interval and this is the real line and this is the real line, then we will be talking about sets will be talking about, so, you will be talking about functions on the real and competition of functions of the real and for example, if I talk about a function.

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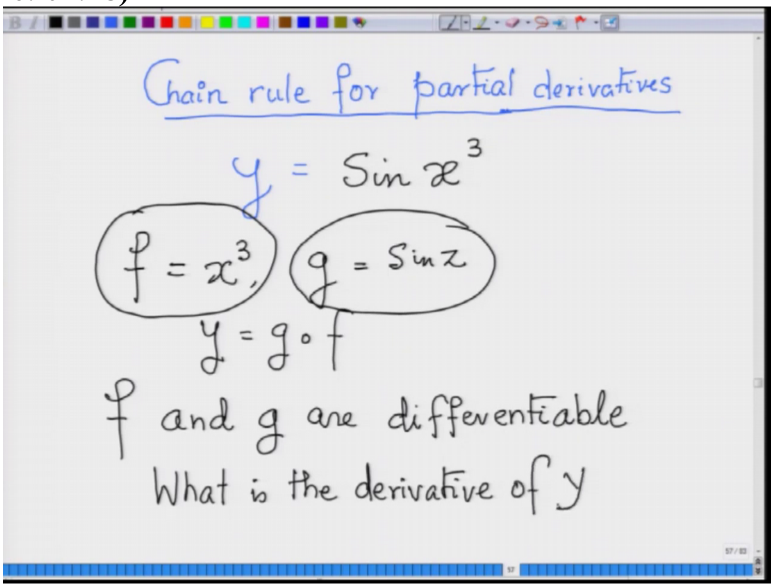

Like this y is equal to sin of x cube. So, if you observe, I can put f is equal to x cube and g is equal to sin of z. Then, what I am doing if f of x is giving me x cube. So if I actually x cube and then I am applying g of x if so, y can be actually done g composed of f. If you will look at this function f equal to y equal to sin of x cube then y equal to sign of x cube this function is differentiable y equal to sign of x cube the sin function is differentiable, so, here are my f is differentiable and here this f this function is differentiable and this g is also differentiable.

So, both f and g are differentiable, Okay what is the derivative of age? That is the question we should ask. This is the word called Chain Rules. Chain Rule means taking the derivative in a chain one after another. What is the derivative of y? Nobody will have to function y in this case.

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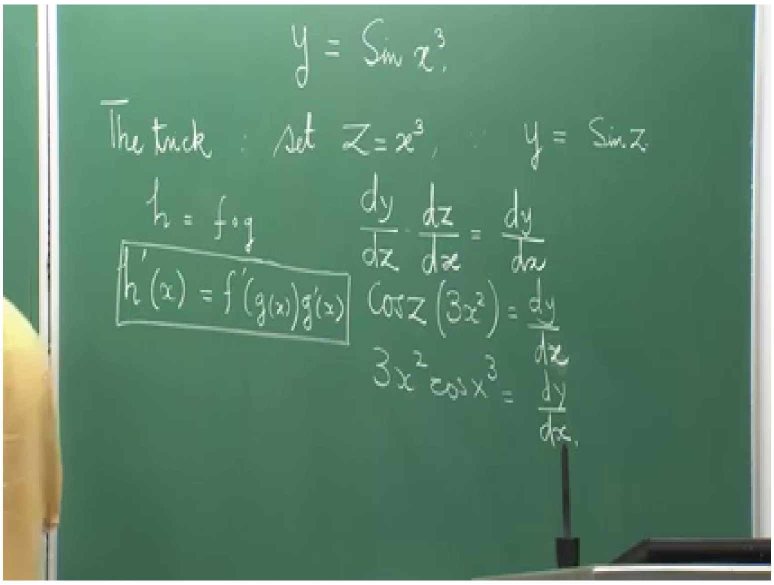

So, the trick is the following. So let us follow the trick. Set z equal to x cube. Therefore, y equal to sin of z so it is a functional z right? So if I now take the derivative, you know the Chain Rule. So what do you Do you do so, now you know what how to take a derivative of with respect to z and then if you take a derivative of z with respect to x, you get a derivative of y with respect to x, this is something known to you, when you had started talking about derivative of one variable. So, just the rules of differentiation that we have learned in calculus one variable, this is known to you.

So what we do, we first take the derivative of y with respect to dz and multiply with a derivative of z with respect to x. And that gives me the derivative y with respect to x. So in this case, dy dz is cosine of z and dz dx is 3x square nobody you know that z is nothing but x cube So then you put in place of z x cube it will be 3x square into cosine of x cube and that is giving me the sorry, dy of dx.

This is the standard Chain Rule, and you know this standard Chain Rule. But what happens if I have a scenario like this?

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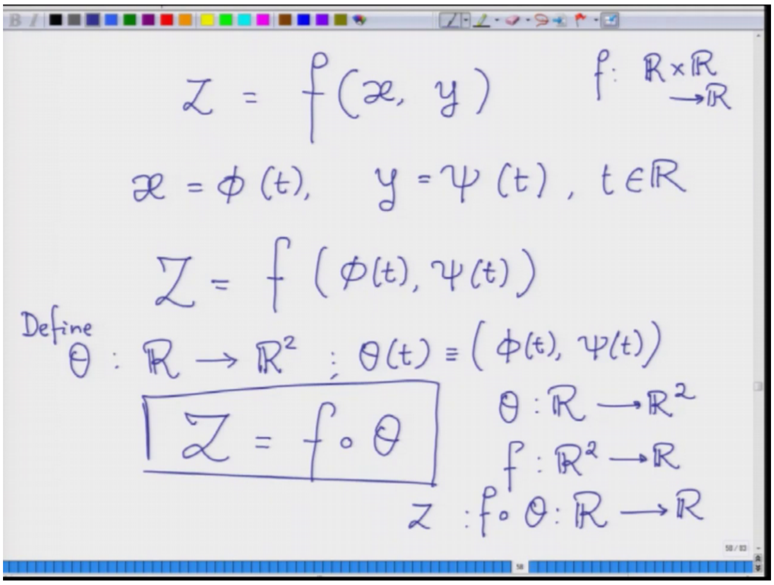

That z is a function of 2 variables x and y x and y a real variables while x is a function of Psi of t and y is Psi of t of the same function. Now if you look at this, you ultimately know that z can be expressed as a function of one real variable because I can write z as a function of Phi t and Psi t. This is finally a function of t how because defining a function Theta define the function Theta from R to R 2 to such that Theta t is by definition of Phi t Psi t. So I have taken a variable t is in R of course, these are real variable.

So I have taken a variable t in R and mapped it to 2 numbers which are given by Phi t and Psi t I take t and put them in one function Phi and another function Psi and get the 2 numbers and put them as a coordinate and that I call it a function Theta t z actually can be expressed as a composition of f with the function Theta so Z is a f compose Theta. But here what happens Theta is carrying R to R 2 to right and f is carrying R 2 back to R so f when f is the function obviously x and y are real variables R cross R to R.

So ultimately, f compose Theta which is Z it is carrying R back to R. So now we can now ask the question what is dz dt? This is the function of t.

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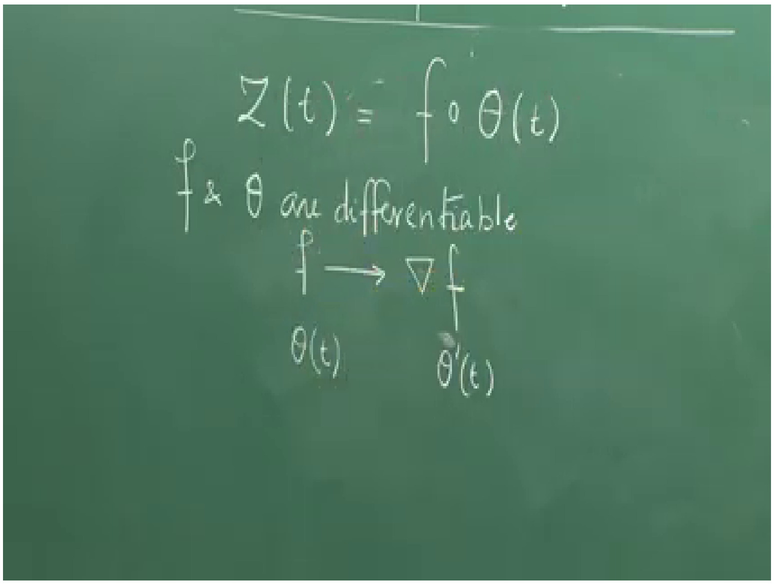

So, in that sense z of t is f composed Theta of t so if I want to play with this Chain Rule here so if I write for example, say h is so what did I do if I write for example h is f compose g. So what I actually do when f and g are functions for real if functions from R to R h dash of x is nothing but f dash of gx z is a gx here x cube is a gx into g dash of x. This is actually the Chain Rule if I write it in this dash format of derivative or is this the same as this is nothing but dy dz evaluated at z and z is nothing but your gx here and g dash x which is x cube I can write gx.

Instead of Z and then this is what you do dz dx is nothing but g dash x. So, we can read it like this. So, I can immediate this in this case, now, I have f and I am assuming that f is differentiable f is a differential function, so, it has the gradient is the derivative. So, let me assume that f and theta both are the differentiable functions f and Theta are differentiable. So, for f I have the gradient and for Theta t gradient f for dash Theta t and u know what Theta dash t.

Theta dash t is nothing but the derivative of Phi t and Psi t this way of taking derivative of vectors that we had learned when we had shown how we can study mechanics using function of more than one variables. When we had seen a little bit about computing derivatives and velocities of particles and mechanics, this is what we finally had now what to do. So this is what I have and then once I have this. So, go back and try to write a go back and try to write it in this format.

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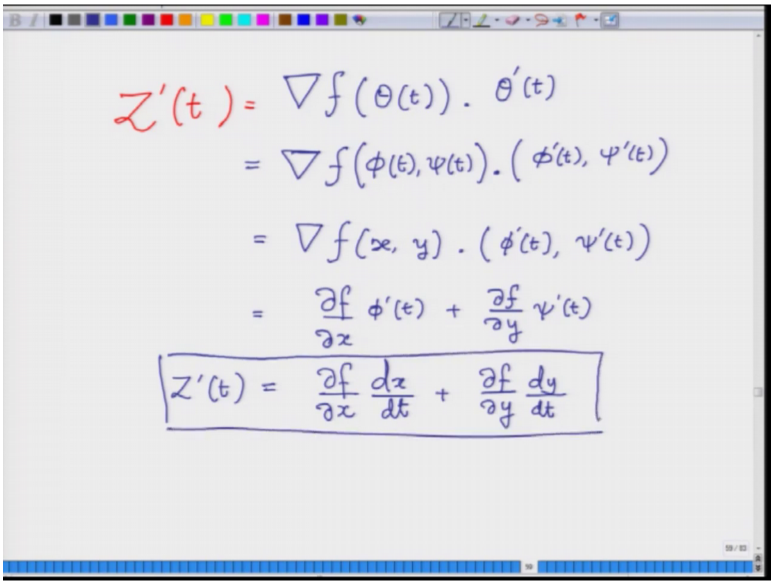

So, I will write Z dash t is gradient of f at this point Theta that is gradient of f at Phi and Psi which is x and y basically this dot product so, instead of multiplication allow a dot product. So, basically I can write if you want immediate this, the one here if you want to immediate this and put it in higher dimensions, so, we are not doing a direct, detailed proof. But we are using our intuitive to bring what we know in one dimension to higher dimensions.

That is exactly what we did for the definition of the derivative here I will do the grad f of Theta t instead of multiplication allow dot product into Theta dash t. So I know that grad f of Theta t. Theta t can be written as Phi t Psi t dot Phi dash t and Psi dash t you write dot also t often refers time, but in this case we are not mentioning. So we are put in a dash instead of so what is this? This is x and this is y so it is.

So do instead of multiplication I am using the dot product or the vectors grad f xy dot Phi t and Psi t so, what do I have Del f Del x right, Del f Del x Phi dash t starting the dot product Del f Del y the gradient these are the gradient Del f Del x Del y into Psi dash t. But what is Psi dash t. Psi dash t is actually x so I can write this as Del f Del x dx dt plus Del f Del dy the diverse Psi y basically. So this is your Z dash t.

And voila just using our basic idea is this fact this idea we had got by just by replacing multiplication without product, we have a Chain Rule for functions of this form. Okay, now we will go into some other type of composition. This is not the only type of composition available for functions are more than one variable. So if I am talking about some other kind of composition literacy, but the same idea, would be repeated the same idea.

Will intuitively put up and do the operations. It could be a Jacobian instead of gradient and then we have Jacobian matrix multiplication instead of dot product, of all kinds of such things. But once you get you can pull in your intuition is very important in mathematics to look at things. And intuitively, in many cases happen every time intuition does not work. Please understand, in every part of mathematics in intuition does not work.

For example, there is a part of important modern mathematics called typology where your intuition will fail. You must many of the times, but here in calculus each subject has itself has been developed as an intuitive one do not think that everything in calculus has proved in day one. For example, if you when you learn about negative binomial in school, you are to look at within this range, this negative binomial I should put x is in this range.

Then 1 plus x to the power minus 1 it only makes sense when say x is between minus one and plus one and all this kind of stuff. But when Newton wrote down the negative binomial, he just wrote down the coefficients. Absolutely from intuition there was a proof and most of the proofs in Newton's principle comes you comes not using calculus, but it comes using Synthetic Geometry Euclid Geometry.

So that point has to be kept in mind that in calculus, we can achieve a lot of things. We can make a lot of advancement by very careful intuitive thinking can use it okay. In intuitive thinking careful, of course, introducing, keys and make very big leaps in the imagination. But that does not mean that leap can be thought out carefully. So here we are doing that kind of leap to give you the Chain Rule.

When I was a student, I can tell my own experience is that these kind of leaves that you can make in the manufacturer are not told we are told these all formula memorize it and that is a thing that I want to take you out of once you understand what is the meaning of the Chain Rule in 1 dimension which is exact this is exactly this. For anyone here that can be it as gx if you are not comfortable I am writing f of gx.

So Del z Del x is same as Del g Del x ok so Del y Del z is same as Del y Del g right now, once you know these form what do you have, once you know this form, you can use this form to get things for higher dimension. So once this intuition is clear, you do not have to do any rote learning. This is something so you can look at the problem you and easily create the Chain Rule sitting there. You do not have to do rote learning.

The whole purpose of this course. And also my course in functional one real very well has been to take you out of this rote learning of mathematics that you can with your intuition if you do not always want to get into the rigorous proof very, at the very first and I think it is not advisable to get into rigorous proof at the very first place was rigor sometimes causes rigor mortis, so I do not want that to be done.

It might kill the fun of the subject, once you can, from your intuition go from one concept to another, and you see that the concept actually works. That is a huge fun, and I think it is very important to have that fun. So, now let me go to something higher may, something different, some different kind of Chain Rule. Okay, so here my Chain Rule number one, it may be as you call it, Chain Rule 1. So next I will give Chain Rule 2, I will give you a problem I will set it up with a problem and let us see let us discuss how we go about it.

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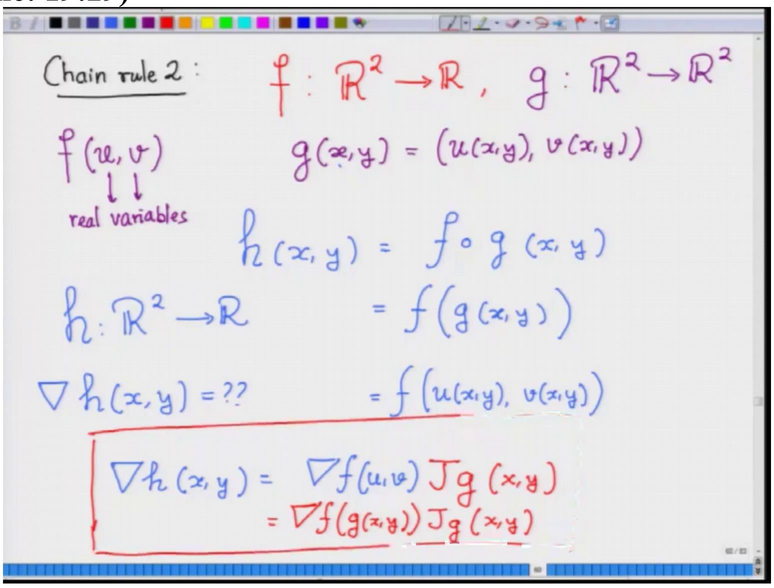

Chain Rule 2 so what is this Chain Rule 2 so here I have function f, which is from the R 2 to R see what I am doing from R 2 to R. I am keeping the whole thing within R 2 just for your comfort. You can actually take it to R, you can actually take it R 2 or R 3 but once you learned not to. You can actually extended to R 3 are not as you feel right we can go up to R 3 as an example.

So our function f is R 2 to R and another function g given from R 2 to R 2 f is expressed in terms of uv these are real variables and g of xy so xy is an element R 2 is expressed as u of xy and v of xy so now we construct the function h of xy so if f is carrying R 2 to R sorry g is Carrying R 2 to R2 and f is carrying R 2 to R then h which is a composition of f and g it carries R 2 to R.

So g of xy is first operated by g that is f composed g of xy. So, this is can be written as f of g of xy and can then be written as f of g of xy is u of xy v of xy my question now is h what is h h is a function from R 2 to R so, is a function in higher dimension not R 2 to R for R2 to R we need to compute the gradient for function R 2 to R we need to compute the gradient, assuming there h is differentiable.

Then there is a gradient of partial derivatives. So basically I am asking you what grad of h so, basically h is expressed in terms of f and g, which are also differentiable. So now I am asking you that can in terms of the derivatives off and g, you can compute the gradient of derivative of h, and what is the answer? So write again, the derivative of h is a derivative of f in terms of u and v.

That is a gradient of f in terms of u and v because f is from u to v multiplied in by now g is a function from R 2 to R 2 so it is a function from R n to R n times function, Vector function of a vector has been carrying to the vector in that case we compute Jacobian matrix. So here we are to be very careful here we computer Jacobian matrix this into Jacobian of g computed at xy. So what is this grad uv this is nothing but g grad f of g computed at xy into the multiplied with the Jacobian matrix of xy this is the Chain Rule.

Now elaborate on the Chain Rule and the next page will elaborate on the Chain Rule and we will write down the Chain Rule in details. So you see we are taking the same law same chain rule for derivatives and by intuition we are just coming up to this the same story we are repeating but we are putting in the correct form of the derivative as the conscience are and this is very important that we are absolutely using intuition.

We have not gone into rigorous proven in this case, I will not go into any rigorous proof. Of course, everything and we prove a rigorous using the limits and all those things fully using derivative definitions, but we are not going to get into rigorous proof under any in any circumstances here in this case will not get into rigorous proof will get into intuition. That is what I want to teach in this course, and had been trying to teach in my course on calculus one real variables.

I never thought that I will do courses in calculus, but I am really doing it but when I am doing to try it again and again stress that you can use your intuition to move from one dimension to the other by really filling up the same story with the appropriate objects. That is it.

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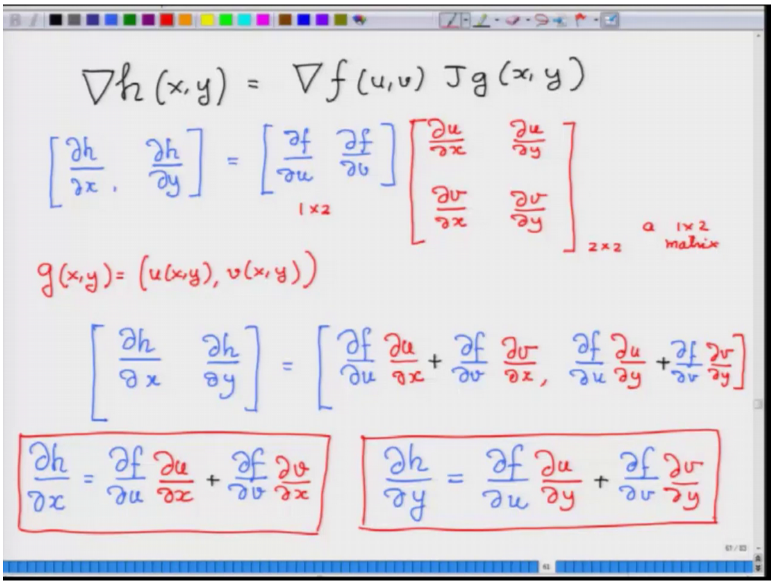

So we had written letters again write grad of h xy is grad of f uv into Jacobian of g of xy. So, let me do the computation here. So, grad of h xy is Del h Del x, all evaluated at a given point where your computing of course, Del h Del y there are 2 components. Here I have Del f Del u Del f Del v and now I am multiplying by the Jacobian matrix. So, what is the Jacobian matrix of g, the Jacobian matrix is a function from R 2 to R R2 to R 2. So it is a 2 cross 2 matrix. So we will take the gradient of f In this case u and v. So g is from g xy is u xy v xy. So the Jacobian matrix first I will take the gradient of u and put it in the first row.

I will write the gradient of the row vector. So I will do Del u Del x and Del u Del y. And then I will do the second thing. I will take the second thing I take the second function v and compute this gradient and put it as a row vector del v Del x Del v Del y ok. Now, I will do matrix multiplication. So here it is a 1 cross 2 matrix which is a vector row vector. Here it is a 2 cross 2 matrix. And that will give me finally a 1 cross 2 matrix which is a vector. And that is exactly same as Del h Del x Del h Del y.

So let we do the multiplication. So how do I do the multiplication, take this and multiply with the first row. Right? And take this multiply by the second row. So I will have Del h Del x into Del h Del y. So how do I get my first up one row? So I will take the first row of this matrix, it has one row, multiply with the product with the first column of this matrix, the Jacobian matrix, and then I will take the for the first entry in the first row. For the second entry.

I will take the first row and multiply with the second column of the Jacobian matrix that is matrix multiplication. So, I will have Del f Del u Del u Del x + Del f Del v into Del v Del x and the second entry is again to begin with is Del f Del u into Del u Del y plus Del f Del v into Del v Del y. So, which means Del h Del x is this part. Though, this Del h Del x is this part. So, this part that we have obtained the first component of the right hand side is Del h Del x second component of the right hand side is Del h Del y.

So, I can now write Del h Del x is now Del f Del u into Del u Del x plus Del f Del v into Del v Del x this is what I have as Del h Del x. Similarly, I can have I can now right down Del h Del y is equal to Del f Del u into Del u Del y + Del f Del v into Del v Del y. So, here we clearly were able to compute the partial derivative of h which is giving it is which is giving us a gradient or the derivative in this case. So you see how beautiful and simply, this rule has been about over to this case, to the higher case, and that is what I want to stress on that you can actually do the thing in such a neat and simple way.

And that is what we really have to look into the whole thing that just by institutional pickup; we can get such neat, clean results. I will leave you to have fun. What is the fun is I will not do anything. I will give you two situations and in these two situations, you have to figure out how to write the general this would be you can write down you can check it up with the forum or you can ask us will give the answer is them as homework.

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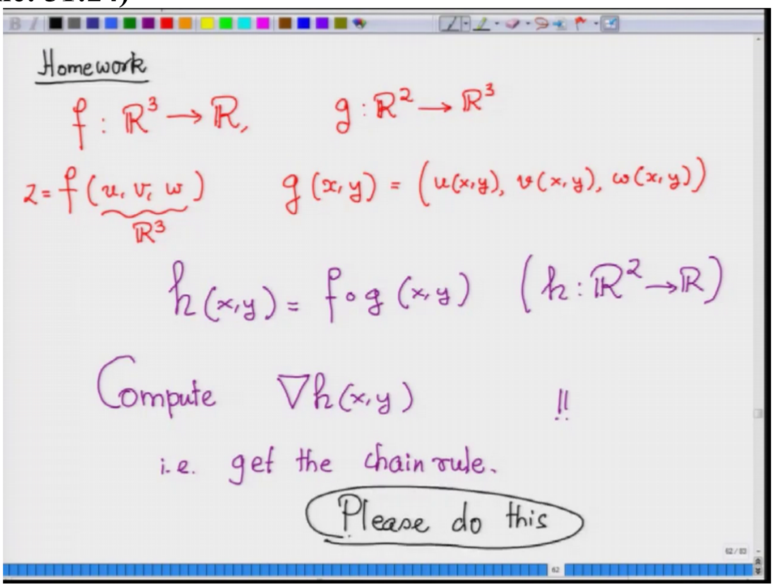

So this is what you have to do before you check on with anything else. So f is now from R 3 to R we have just made it look a little more fearful for you but I am going to tell you just increasing one more component nothing as it is just fun. You can check it out if it is from R 3 to R and g is from R 2 to R 3 is a R 2. So, f is written as a function of variables Such that z equal to u v w. So, this constitute element in R 3 and g is written as follows g xy picks up an element in R 2.

And puts it in R 3 which is u of xy, v of xy and w of xy h xy is equal to f composed g of xy. So what is happening here? Here h is been carried from R 2 to R. So, g is caring from R 2 to

R 3 and f is carrying one from R 3 to R so h has been carried from R 2 sorry h has been carried from R 2 to R. So, h is finally back at a function from R 2 to R my question of course is compute grad h xy basically come get the Chain Rule that is get the Chain Rule that is compute grad h of xy in terms of the derivatives of f and g get the Chain Rule.

I can make the problem little more harder by writing f is from R 3 to R and g is from R 3 to R 3 okay. So, f is from R 3 to R but g is from R 3 to R 3. So, g will become xy function of xyz. So, g will be g of xyz u of xyz v of xyz w of xyz and grad would be h would be h of xyz. So, it will be R 3 to R h would be a functional reward, then you can get the general you can keep and make it much more complex and complex and complex.

But just for homework, do this and I am requesting you please try this just in the way I just worked all these things out just the one I worked out before, please do this. So with this, I end my discussion today. I hope you enjoyed it. I still want to show you this neat, simple work. And I would rather appreciate your comments on this lecture. There is anything you can definitely ask us. And I closed the class today. Thank you and have a nice day.