Basic Calculus for Engineers, Scientists and Economists
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Lecture – 10
Maxima and Minima

Today, we start talking about maximization, minimization. In fact, the great mathematician Euler had ones remarked that nothing in this universe goes on without something we maximized or minimized.

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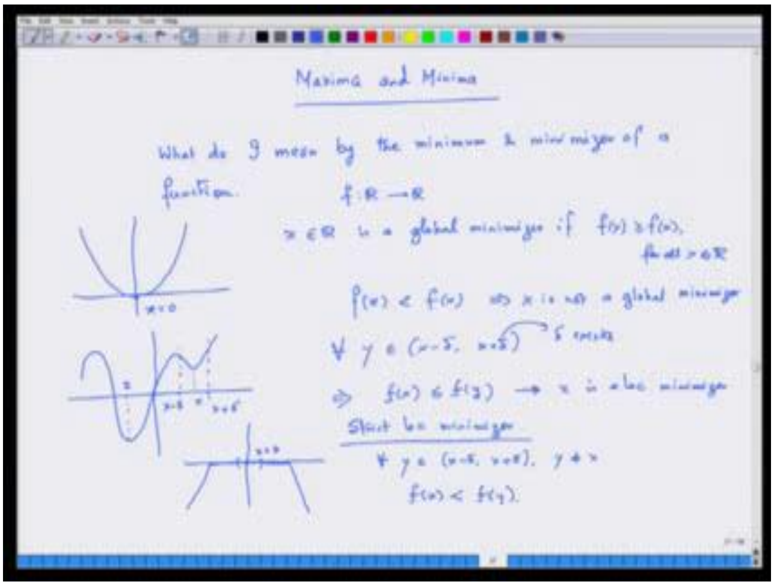

These are very, very important topic or subject which is also close to my heart and so. We are going to really talk about what do I mean by a minimum, minimum and minimize. Now the two types of minima, minimum concepts; one is a global minimizer one a local minimizer.

For example, if we take function y equal x is square, so this function y equal to x is square which is basically well known function, and all of you have studied it very well during your undergraduate or your high school days. So, at x equal to 0, the function value is 0, and rest it is nonnegative. So, x element of R is the global mean or global minimum or global minimizer I you would say. If f(y) is greater than equal to f(x) for all x in R, we are not talking of functions from R to R. And we will soon be talking about those functions which are also differentiable.

Now, what is the local minimizer means the minimizer which need not be satisfying this definition for all y, but for some y, which are very near x. Let me have a look at this one. Now what happens is that look at this function. Now here look at this particular point, see let us call this as the point x. If look at this point which we call this as the point z, the function value at the point z is obviously less than the function value at x. So, what we have is f of z is strictly less then f of x, so which means that f of x is not minimizing the function over whole of R.

So, what does it do what it there seems to you have valley. So, valley talks about a minimize or a mountain hills, some it talks about a maximizer which you can form a definition maximizer exactly in the similar way on you changing the signs, so which implies that x is not a global minimizer. So, we call these points minimizer; and the functional value at that point as a minimum value. For a global minimum, we say the function f(x), value at f(x) is the global minimum value of the function. We do not really talk about local minimum values, so we just talk about local minimizer. So, x naught is not a global minimizer. So, when we are going to talk about some points near y, let us look at a neighborhood around x.

This neighborhood say I say x plus delta and x minus delta, then what happens if you observe that for all y that is lying in this neighborhood is open set, we have that f of x is less than equal to f of y. This is called a global a local minimizer. So, x is a local minimizer, but there has been a interesting concept called strict local minimizer, which actually tends to alleviates situations where a local maximizer, you can look at a you kind of problem whose global maximizer may appear as a local minimizer to you.

For example, let us see a function of this form. Now, you take point x equal to 0, then you can easily sees, it is the global maximizer of this function, but if you take a small neighborhood around it then the function value is equal everything is equal to 0. It does satisfy this definition of a local minimizer. In this case, some people can mistake x equal to 0 as a local minimizer, which it is by definition, but it is more important and intrinsic property is that it is the global maximizer. It to alleviate this sort of problem, there is this notion of strict local minimizer, loc means local. It says that for all there would exist some delta.

For these delta is you have to shows such existence of a delta. There even const put find some delta and you can const of the (Refer Time: 06:16). So, you can find again a delta. So, says that for all y look x is also in this interval; all y in this but y not equal to x, f of x is strictly less than f of y. Then we call a strict local minimizer. For example, here x equal to 0 is actually a strict global minimizer. There is no chance of this sort of (Refer Time: 06:41), so when you have this concept.

Now, how do I characterize, minimize a point if the function is differentiable that is the next question, because we tend to work more with differentiable functions. But we have not shown you, but we will show in the exercise that every differentiable function is continuous, but every continuous function did not be differentiable. This is something you keep in you are head, you keep on repeating in your head, and you should be able to tell this to anyone even if you are blind folder under a shower.

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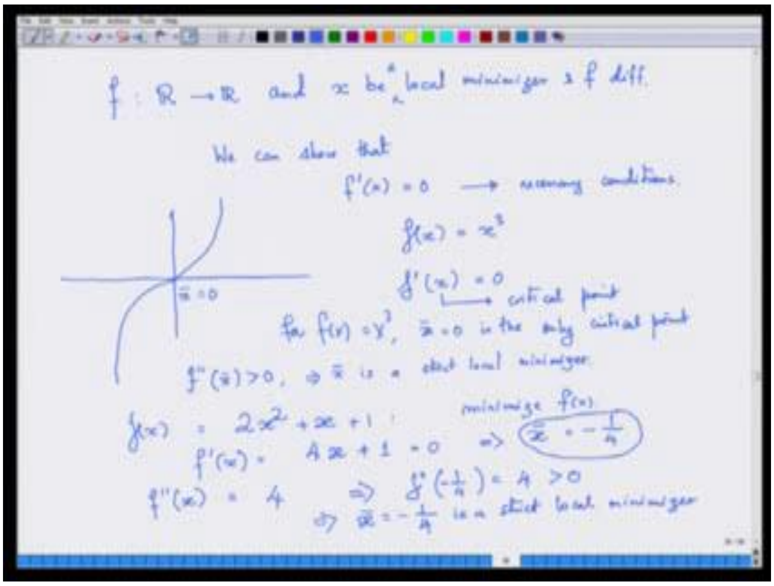

If I have f from R to R, and x is a local minimizer, and f differentiable, and I am writing the short diff, does x satisfy certain character, a certain property? We can show that that f dash x would always be equal to 0. But f dash x equal to 0, so this is, so if you have a minimizer, if you guarantee to me that this to given point x is the minimizer local minimizer then that we can always guarantee that this will happen. But we cannot make the reverse guarantee that if this sort of thing happen that f dash x equal to 0, I will get back a local minimizer.

For example, you take the function f x equal to x cube, so find, so first two so to check whether a point really is the minimizer, let me first try to find out which point satisfy this. So, such points, any point which satisfies this is called a critical point. And in the case of f x equal to x cube, x bar equal to 0 is the only critical point. So, for f x equal to x cube, x bar equal to 0 is the only critical point, but you see x bar equal to 0 is not a minimizer not the maximizer, it is just a critical point.

You do not have really bother about the fact that is the point of inflection, and the curve then they shape from convex down to convex up and all those things forget about it. So, here you have the case that f dash x is equal to 0 does not translate into a fact that x is the local minimizer .So, when does it translating to the fact that x is the local minimizer. So, we have to now talk about sufficiency conditions. This condition that you have seen is necessary conditions, if necessary.

Now, what does this mean, let us see. What I will show is that you mean additional information about the derivative of f in order to ascertain whether this given x which you have found to satisfy f dash x equal to 0 is a local minimize or not, you can tell us similar story for local maximizer. What I am just keeping myself local minimizer possibly I am comfortable with it, but even can also economics for example, who try to think in terms of maximizers, so here it is very important for people in economics this part and of course, in an engineer.

Here let us look at this, this give very well. Now, I calculate at the critical point, the second order derivative, the second derivative. And if this is greater than 0, then we say this implies that x bar is a strict local minimizer. This is the very interesting and wonderful fact which seems to you have huge amount of information. Let us just take up an example, very simple one – a quadratic function, but it might be a (Refer Time: 11:51) thing 2000 or convex analysis of convexity, but those do not know about it, it something interesting and will follow this method. So, we want to find, so my question is to minimize f x.

So, how do you minimize it? So my first step is will be to know are there any critical point, because every minimizer has to be a critical point. So, how was there is no critical point, there is no question of minimizer. Every minimizer is a local minimizer is a critical point. If there is nothing, no point is a critical point then they cannot be minimizer. If a point is not a critical point, then it cannot be a local minimizer.

Let us find the critical point, by taking the derivative of this. So, derivative of this is 4 x plus 1 and that is equal to 0, so that would imply that x bar for example, is equal to minus 1 of fourth. I would tend to the only possible critical point is this one. Now let us see what is f double dash x, f double dash x here is 4. It does not matter whatever be a choice of x, f double dash is 4, so which means f double dash minus one-fourth is also 4 and that is strictly greater than 0 that would imply that x bar equal to minus one-fourth is a strict local minimize.

In this case, it is global, but I would not go to be the explanation of those things. It is a strict local minimizer. Does every function have only one local minimizer or one global minimizer and only one point, which can satisfy the global minimizer, no.

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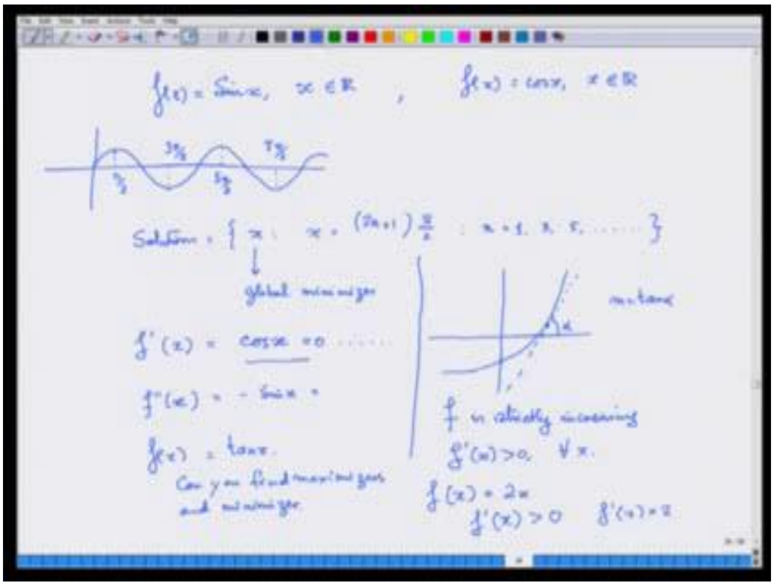

For example, if you take f x is equal to sin x. Now I want to find at which points these are the minimizer local, global whatever. If function values are of sin is (Refer Time: 14:23) x in radian then x it is varying between plus 1 and minus 1. So, what happens so and I know what are the points where you have plus 1, and what are the points where you have minus 1. So, what are the points or you can also which you try out this example later on f x is equal to cos x. So, what are the points where you have f to be plus 1 and f to be minus 1. This pi by 2, this pi, this is 3 pi by 2, (Refer Time: 15:28) this is again come back to 0 in 2 pi ,and then 2 pi plus another phase of pi by 2, so you know what is it 5 pi by 2.

If you know and similarly, so the points so what are the points of maximizer and what are the points of minimizer. Here, it will be again 3 pi, 3 pi plus pi by 2. So, 3 by plus pi by 2 would be, so pi (Refer Time: 16:12) 0, pi, 2 pi, 3 pi, and 3 pi plus pi by 2 would be 7 pi by 2. Let us look at the position. So, global maximizing value is 1; and global minimum value is minus 1. Now if I take the set of solutions, there is the minimizer and that exactly equals to what, what would be the minimizer, they would be all the form x, so as that x is equal to.

Of course, so you have infinite such minimizer, now countable in fact, n equal to 0 that gives you 3 pi by 2; n equal to 1 that gives you 5 pi by 2, so n equal to, so if you take the next minimizer, it has 7 pi by 2, so see how you do the math. If I do I have to do put 5 here, so we have doing some experiment, suppose we have 2 n plus now ok, we say 2 n plus 5 let us, will this work. I put n equal to 1, I put n equal to 0, and then I have of course 5 pi by 2. You see this thing is not working. So, what should I do, let me just try out to n plus 1; and to try out this I have n equal to 1, I take the odd numbers n equal to 3, n equal to 5 and so on.

If I put n equal to 1, I am getting 3 pi by 2; n equal to 3, I am getting 7 pi by 2 and so on. These are the points of global minimizer. These are global minimizing points. Now but now if you want f double dash x sorry f dash x of this, you have to have here function cos of x. So, you have to find cos of x equal to 0. So, you have to find a solution to this equation, but for cos of x equal to 0 does not mean that each and every point would be a minimizer.

If you then have to take f double dash x and that will give you minus sin of x. So, you have to see for what x among the solutions of this equation among for what x, these value would become strictly positive then you take it has a minimal maximizer, what x is if this value at f double dash x becomes strictly less than 0 then it becomes a maximizer. It has to be very careful that there could be many, many points which have local minimizer and global minimizer, so that is the very, very important idea.

But it is not at every function that I give you will have certain minimizer or maximizer; for you, just for f(x) equal to x cube. So, you take for example of a function which is increasing, so strictly increasing function, if you look at draw tangent to this curves, this function of curve of the graph of the function at any point, and if you look at the tangent of this, and it a slope of this tangent that where we calculate an alpha. And know that from a high school that this slope is nothing but that derivative. Then you would see that when f is strictly increasing.

So, we do not actually have a minimizer, in this case, it just goes up or goes down, for example. If you have a linear function, which is strictly increasing, let us see what happen. Suppose, I take of linear function f(x) is equal to 2 x, when you see f dash x is strictly bigger than 0, because f dash x is equal to 2. So, you cannot find a maximizer minimizer of linear functions. This is something which you also have to keep in mind. So, with this I am closing my discussion of maximization and minimization, this is the very short discussion. Now, for example, you can try out this f(x) equal to tan x, and can you find sets of maximizer and minimizer, can you find.

And the next talk, we want to talk about the magical thing called the mean value theorem; we are giving going to give a lot of examples of how powerful that notion is and that also comes by using the derivative. What we are trying to do once we know the function is differentiable, we are trying to approximate the function value at a given point using its derivative at some point nearby and that is the fundamental tool of calculus, which you really have to know.

The idea of the next topic is following, if you have information about the derivative and if you want to know something about the function, you have to use what is called a mean value theorem and that is what will (Refer Time: 22:39) in the next talk.

Thank you very much.