Calculus of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Module No. # 01
Lecture No. # 40

Welcome viewers to the last lecture of the lecture series on integral differential equation under the NPTEL courses. In all previous lectures, we have considered linear integral equations only. In this lecture, I am going to discuss about some solutions and properties of non-linear integral equations. Of course, there are a wide variety of different types of non-linear integral equations, and it is not possible to discuss all those type of equations in this single lecture.

But I just want to give you some essence of these non-linear integral equations, because they produce two interesting properties of the solutions. Number one is the bifurcation of solution, and with help of this type of non-linear integral equation we can explain the concept of singular solution. And another important property that we can explain that some of the non-linear integral equations or you can say most of them do not have unique solutions with illustrating example, I will try to illustrate this phenomena that how non-uniqueness of the solutions comes into the picture.

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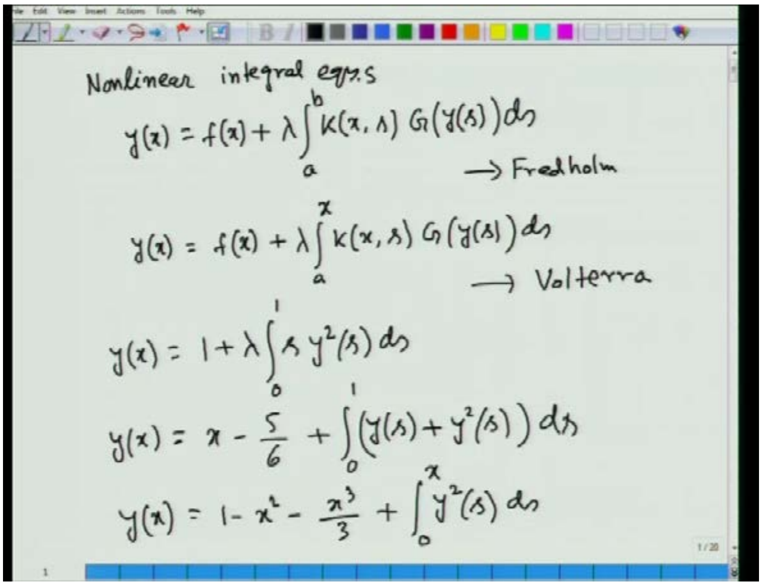

So, we will be considering two types of non-linear integral equations that is non-linear integral equations. We may consider this y(x) equal to f(x) plus lambda integral a to b k(x,s) G of y(s) ds. This is a non-linear integral equation of Fredholm type. And non-linear Volterra integral equation can be written as y(x) equal to f(x) plus lambda integral a to x k of (x,s) G of y(s) ds. This is of Volterra type where this G - capital G is a function of y which is a non-linear function of y. And some example of this type of equations are y(x) is equal to 1 plus lambda integral 0 to 1 s of y square s ds, in this example G is actually G(y) is actually y square. Similarly, another example is y(x) equal to x minus 5 by 6 plus integral 0 to 1 y(s) plus y square s ds. This is an Fredholm type integral equations.

And one Volterra type non-linear integral equation is y(x) equal to 1 minus x square minus x cube by 3 plus integral 0 to x y square s ds. These are some non-linear Fredholm and Volterra type integral equations. And of course, this G may be some function like e to the power x sin x and some other combinations such that they are non-linear functions. Once this G(x) is a non-linear function of y then equations of this type are actually known as non-linear integral equations. So, first of all we discuss the direct computation method to solve non-linear Fredholm integral equation.

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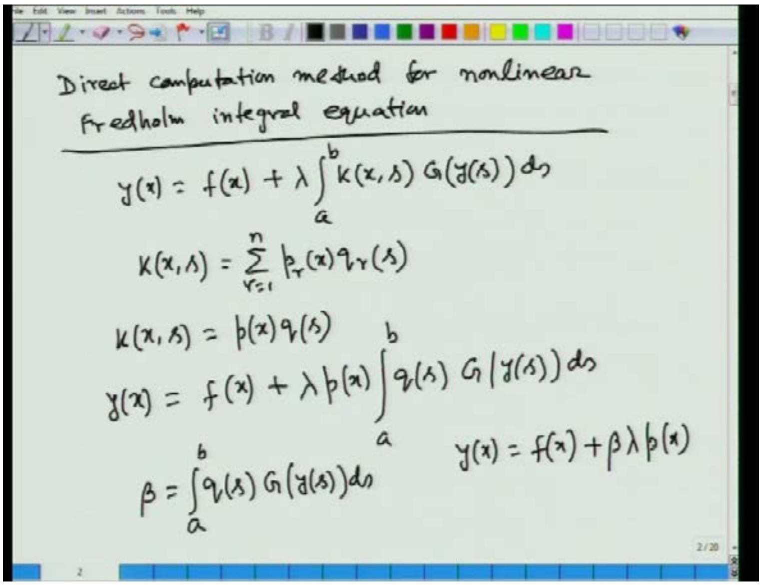

So, we are going to discuss direct computation method for non-linear Fredholm integral equation. So, equation is y(x) equal to f(x) plus lambda integral a to b k of (x,s) G of y(s) ds. And this direct computation method to solve non-linear Fredholm integral equation is applicable whenever this kernel is a separable kernel. So that means we are assuming the kernel k(x,s) is of the form sigma r runnings from 1 to n p r (x) q r (s). This is the kernel.

Now, for simplicity of understanding of the method by which we can solve this equation, we assume here k(x,s) is only product of two functions p(x) and q(s). Assuming this particular format, we are going to discuss about the direct computation method to solve this Fredholm integral equation which is a non-linear equation, and with illustrative example will be considering that kernel is not a just product of two functions, one is a function of x and other is a function of s, but it is a sum of two this type of separable kernels.

So, if we have this format for k(x,s) then the given non-linear Fredholm integral equation can be written as y(x) equal to f(x) plus lambda p(x) integral a to b q(s) G of y(s) ds. And you can recall that linear Fredholm integral equation with separable kernel, we have considered and solved by using the direct computation method where instead of G(y(s)), here the terms appear to be a to b q r (s) y(s) ds, and we have assume each of these integrals a to b q r (s) y(s) ds those are constants. So, in the same manner here we assume beta equal to integral a to b q(s) G of y(s) ds and therefore, we have y(x) is equal to f(x) plus beta lambda p(x). So, this is a tentative expression for y(x).

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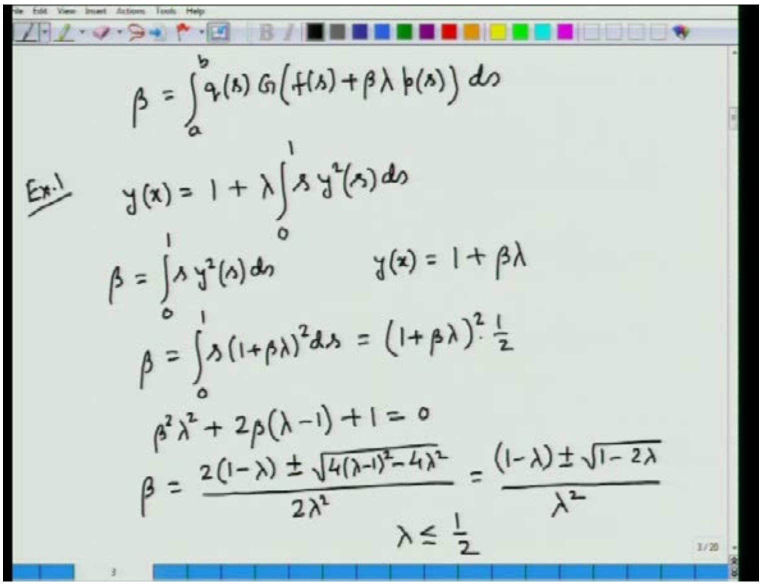

Now, if we substitute this expression for y(x) equal to that is f(x) plus beta lambda p(x) into the definition for beta, we find that beta equal to integral a to b q(s) G of f(s) plus beta lambda p(s) ds. Depending upon the nature of this non-linear function G and explicitly solvability of this integral, we can find an equation involving beta. And we can solve this equation for beta to get values or expression for beta in terms of the parameter lambda, and this lambda plays a crucial role in order to existence of solutions or existence of values or expression for beta which you can see from the forth coming examples. And depending upon the nature of the solution, if we substitute the expression for beta into the pervious result that is y(x) equal to f(x) plus beta lambda p(x), we can find the solution to the given problem. So, this is actually the method of direct computation to solve non-linear Volterra integral equation.

So, first we consider one example; y(x) is equal to 1 plus lambda integral 0 to 1 s y square s ds. This is the given problem. So, we assume beta equal to integral 0 to 1 s y square s ds. With this assumption from the given equation we find y(x), this is equal to 1 plus beta lambda. Now, if we substitute this expression for y(x) in the expression for beta, then we can find beta equal to integral 0 to 1 s 1 plus beta lambda whole square ds, and evaluating this integral we find this is equal to 1 plus beta lambda whole square multiplied by half. And rearranging the terms of beta from both sides, we can find a quadratic equation for beta as beta square lambda square plus 2 beta multiplied with lambda minus 1 plus 1, this is equal to 0. So, this is a quadratic equation in beta whose coefficients are functions of lambda, and solving for beta we can find beta equal to 2 into 1 minus lambda plus minus root over 4 lambda minus 1 whole square minus 4 lambda square divided by 2 lambda square, and after simplification this is comes out to be 1 minus lambda plus minus root over 1 minus 2 lambda divided by lambda square. So, these are actually two values of beta.

Now, in this case if we assume that y(x) is a real valued function then integral 0 to 1 s y square s ds should be a real constant. And therefore, for real values of beta from this expression we are getting a restriction on lambda that is 1 minus 2 lambda should be greater than equal to 0. So, real values of beta can be obtained for lambda, this is less than equal to half.

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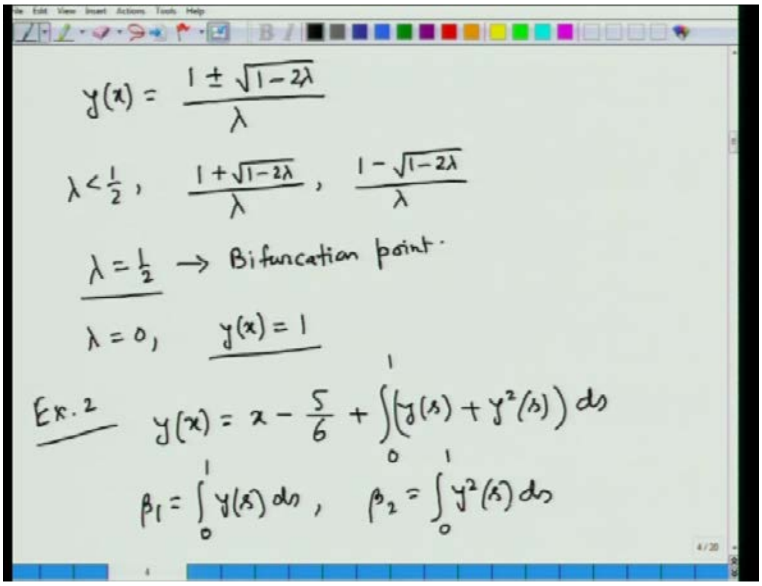

And substituting this expressions for beta into 1 plus beta lambda, we can find that two solutions are given by y(x) is equal to 1 plus minus root over 1 minus 2 lambda divided by lambda. Now, this expression is little bit interesting, little bit interesting in the sense first of all you can observe that whenever lambda less than half, we find two solutions; one is 1 plus root over 1 minus 2 lambda divide by lambda and other is 1 minus root over 1 minus 2 lambda divided by lambda. So, this clearly shows that the problem has two solutions. So, therefore we are not getting unique solution for the given problem.

Although both this solutions are actually constant; so that means y(x) equal to constant, this is a solution to the non-linear integral equation, but we are getting two solutions.

Secondly, both the solution coincides to the one solution whenever lambda equal to half. For lambda equal to half, we can find that both the solution coincides to two only. So now, this is the most important point that lambda equal to half, this is actually called bifurcation point. And the phenomena that for lambda equal to half we have only one solution, for lambda less than half we are getting two solutions that is 1 plus root over 1 minus 2 lambda by lambda and 1 minus root over 1 minus 2 lambda by lambda. These two solutions are actually bifurcating from the unique solutions obtained at lambda equal to half. So, this phenomena is called bifurcation of solutions. And this type of bifurcation in actually appears in most of the discussions on non-linear dynamics involve with non-linear equations. So, this is one simple example of non-linear integral equation where bifurcation of solutions may comes into the picture.

And another important aspect is that if we consider that lambda equal to 0. If we consider lambda equal to 0, from the given problem we find y(x) equal to 1 is a solution, and this particular solution y(x) equal to 1 cannot be obtain from the solution that we have derived that is y(x) equal to 1 plus minus root over 1 minus 2 lambda divided by lambda, and this particular solution y(x) equal to 1 is actually called the singular solution. And another important observation is that as y(x) equal to 1 plus minus root over 1 minus 2 lambda divided by lambda, so there does not exist any real solution whenever lambda greater than half.

So, with help of this particular simple example, we have explained three important points; number 1 - the concept of bifurcation of solution, number 2 - the existence of singular solution and number 3 - that solution depends upon the parameter for certain range of values of lambda we are getting two solutions, and for certain range of values of lambda they are does not exist any real solution for the given problem. So, all this situation most of the time come up with the case of non-linear integral equation.

Now, we consider one more example of a non-linear Fredholm integral equation where lambda is exactly equal to 1. So that means there is no parameter is involved; equation is free from the parameter. With these example that just we have consider it appears to our mind that whenever lambda is involved then we can find two solutions. But next example will explain that instead of for a particular value of lambda still for non-linear integral equation we can have more than one solution. So, in that case also there is no question of uniqueness of the solutions.

Second example is y(x) this is equal to x minus 5 by 6 plus integral 0 to 1 y(s) plus y square s ds. So, this y(s) plus y square s in this case is actually our G(s). Although this y(s) separately is a linear function, but y(s) plus y squared s is a non-linear function. Now, we define two quantities; beta 1 and beta 2, they are define respectively by 0 to 1 y(s) ds and beta 2 this is equal to integral 0 to 1 y square s ds.

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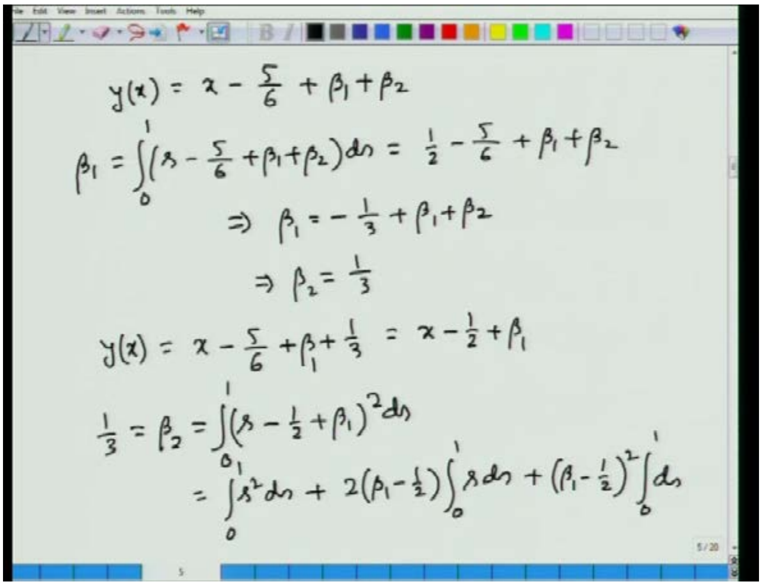

So, with these definitions we can write the given equation is y(x) is equal to x minus 5 by 6 plus beta 1 plus beta 2. So, using this tentative expressions for y(x) that is equal to x minus 5 by 6 plus beta 1 plus beta 2. In the formula for beta 1, we can find beta 1 that is equal to integral 0 to 1 s minus 5 by 6 plus beta 1 plus beta 2 ds. This is equal to half minus 5 by 6 plus beta 1 plus beta 2. This implies beta 1 equal to minus one-third plus beta 1 plus beta 2, and from here we find beta 2, this is equal to one-third. So, just substituting y(x) into the expression for beta 1, immediately we have obtained beta 2 equal to one- third, and hence y(x) is modified to x minus 5 by 6 plus beta 1 plus one-third, so that is equal to x minus half plus beta 1. This is the expression for y(x). And substituting this expression into the expression for beta 2, we find one-third equal to beta 2 equal to integral 0 to 1 s minus half plus beta 1 whole square ds. So, this is equal to integral 0 to 1 s square ds plus 2 into beta 1 minus half integral 0 to 1 s ds plus beta 1 minus half whole square integral 0 to 1 ds.

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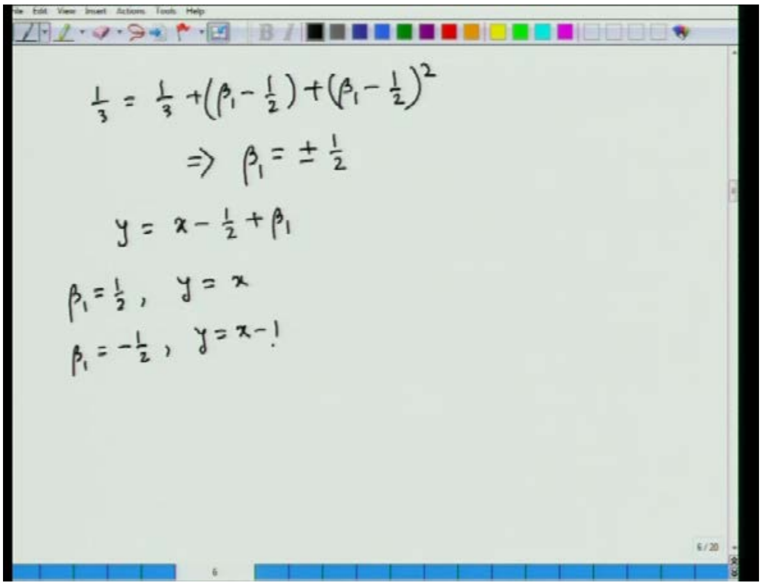

And after evaluating these integrals, we can find one-third that is equal to one-third plus beta 1 minus half plus beta 1 minus half this whole square. So, one-third cancels from both sides and solving for beta 1 we find two values of beta 1 is actually plus minus half. So, already we have obtained that y equal to x minus half plus beta 1. So therefore, substituting beta 1 equal to find equal to half, we find y equal to x is a solution and for beta 1 is equal to minus half, we find y equal to 1 sorry x minus 1 is a solution. So, although the given problem does not processes any particular parameter like lambda, but still the non-linear integral equation processes two solutions; one is y(x) equal to x and other one is y(x) is equal to x minus 1. So, this is actually the method of direct computation by which you can solve the Fredholm integral equation which are non-linear integral equations with separable kernels.

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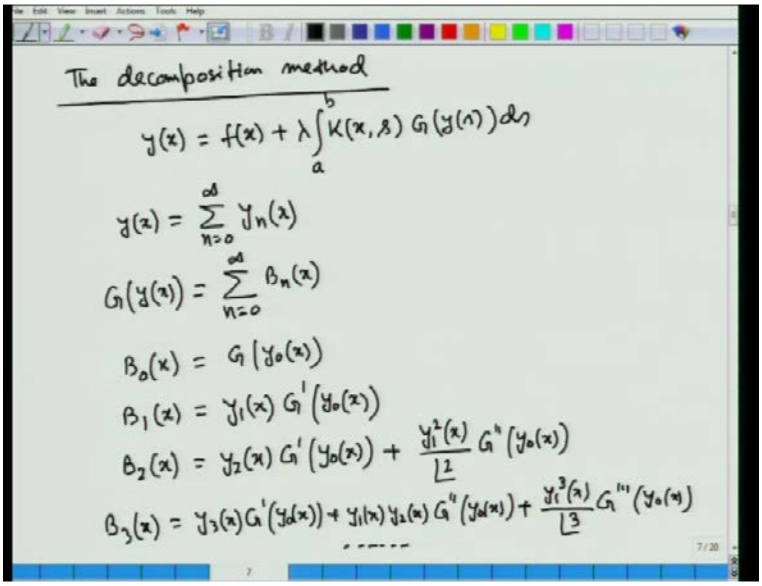

Now, we just look at the adomian decomposition method, how that particular method can be applicable to the case of solving non-linear Fredholm integral equation. So, we are going to use the decomposition method. To solve non-linear Fredholm integral equation is given by y(x) equal to f(x) plus lambda integral a to b k of (x,s) G of y(s) ds. So, first of all we can assume the form of y(x) as sigma n runnings from 0 to infinity y n (x). And this adomian decomposition method says we have to assume a form for G y(x) also, where G y(x) this is assume to be summation n runnings from 0 to infinity B n (x).

Now, you just try to understand in the previous examples, for example, where G(y) (( )) equal to y square. Then, if we assume y(x) is equal to n runnings from 0 to infinity y n (x), and we assume G(y(x)) equal to sigma n runnings from 0 to infinity B n (x). For existence of the solution and validity of this technique we need some relation should be satisfied by y (n) and beta n(s). I am not going to prove that result from where we can get this relation. But for general non-linear function G, we can define first few eta rates which shows the relation between B 0, B 1, B 2, B 3 and so on with y 0, y 1 y 2 and so on.

So, this relations are given by B 0 (x), this is equal to G of y 0 (x) - this one. Then B 1 (x), this is equal to y 1 (x) multiplied with G dot y 0 (x), here G dot y 0 (x) means we have to differentiate G(y) with respect to y and then substituting y equal to y 0 (x) we can find G dot y 0 (x). Then B 2 (x), this is equal to y 2 (x) G dot y 0 (x) plus y 1 square x by 2 or factorial 2 G double dot y 0 (x). Then B 3 (x), this is equal to y 3 (x) G dot y 0 (x) plus y 1 (x) y 2 (x) G double dot y 0 (x) plus y 1 cube x divided by factorial 3 G triple dot y 0 (x) and so on. So, in this way we can find the relations between B 0, B 1, B 2, B 3 and so on with y 0, y 1, y 2, y 3. And expressions for B 0, B 1, B 2, B 3 in terms of y 0, y 1, y 2 and so on is determine by the nature of the non-linear functions. So, there is no unique expressions for B 0, B 1, B 2 in terms of explicitly y 0, y 1, y 2, but those are determine by the nature of the function G.

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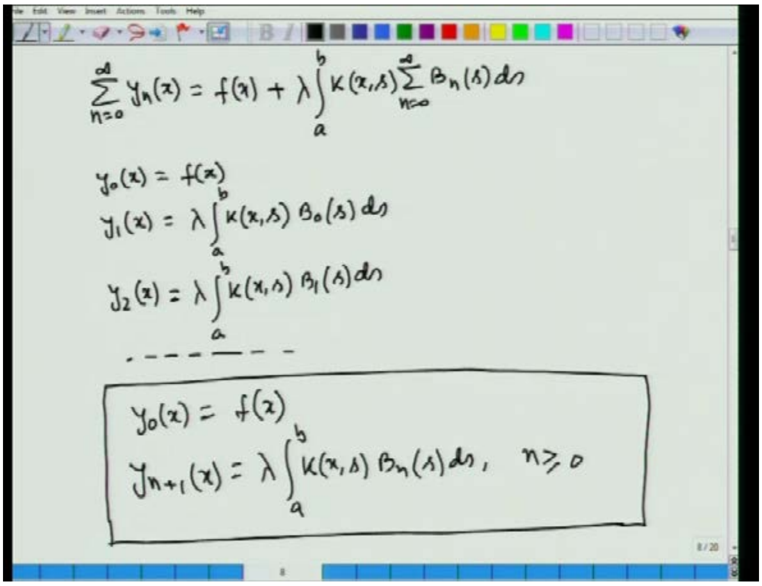

Once with this particular expressions, now if we substitute the expression for y(x) and G(y(x)) into the given integral equation, then we find sigma n runnings from 0 to infinity y n (x), this is equal to f(x) plus lambda integral 0 to infinity k of (x,s) sigma n runnings from 0 to infinity B n (s) ds. And assuming the inter changeability of the integral and the summation, and then equating the terms from both side following this rule that y 0 (x) equal to f(x), then y 1 (x) is equal to lambda integral a to b k of (x,s) B 0 (s) ds B 0 (s) ds, then y 2 (x) is equal to lambda integral a to b k of (x,s) B 1 (s) ds and so on. So, we can find out y 0, y 1, y 2 and so on.

Now, with this assumption if you just recall the expressions for B 0, B 1and so on, you can see that B 0 (x) is a function of y 0 only. So, first of all we are assuming that y 0 (x) equal to f(x) that means we are equating y 0 (x) from left side with f(x) on the right hand side. So, that means with this result y 0 (x) equal to f(x), now y 0 is known. Next we equate y 1 (x) from left side with the term lambda integral a to b k(x,s) B 0 (s) ds from the right. So now, B 0 (s) is known because B 0 is a function of y 0 only, and therefore, we can evaluate this y 1 (x). Once y 1 (x) is known then using the relation between B 0, B 1 and y 0, y 1; we can understand that now B 1 (x) is a known function. So, once B 1 (x) is a known function then equating y 2 (x) from the left with lambda integral a to b k(x,s) B 1 (s) ds, we can find y 2 (x). So, proceeding in this way, we can find the terms that is the adomian polynomials y 0, y 1, y 2, y 3 and so on, and summing up with all this terms we can find solution to the given problem.

So, in the compact form we can write that y 0 (x) is equal to f(x), and y n plus 1 (x); this is equal to lambda integral a to b k of (x,s) B n (s) ds, this is valid for n greater then equal to 0. So, using this formula we can calculate the eta rates are the you can say adomian polynomials y 0, y 1 and so on, and using the relation between B 0, B 1, B 2 with y 0, y 1, y 2 and so on.

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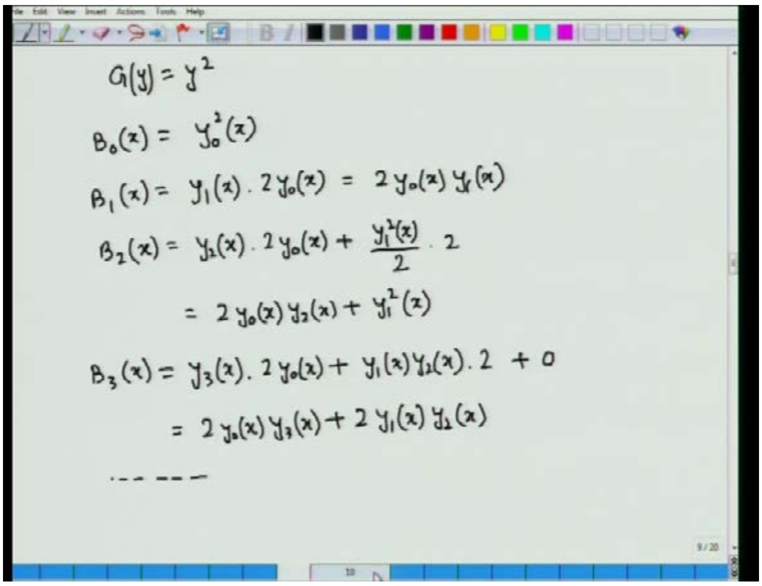

Now, we first consider the relation between B 0, B 1, B 2 with y 0, y 1, y 2 for the particular case that is G(y) is equal to y square. And then we use these expressions in order to find out solution of an non-linear Fredholm integral equation involving the term y square s. So, as per results that we have written B 0 (x) is equal to y 0 square (x), B 1 (x); this is equal to y 1 (x) into 2 y 0 (x). G(y) equal to y square, G dot y equal to 2 y, this 2 y evaluated at y 0 (x) gives 2 y 0 (x). So therefore, B 1 (x) will be 2 y 0 (x) into y 1 (x).

Then B 2 (x), B 2 (x) is equal to y 2 (x) multiplied with 2 y 0 (x) plus y 1 square x by 2 into 2, because second derivative of G with respect to y is 2 and therefore, G double dot y 0 is simply 2, and therefore, B 2 (x) is comes out to be 2 y 0 (x) multiplied with y 2 (x) plus y 1 square x. Then B 3 (x), this is equal to y 3 (x) multiplied with 2 y 0 (x) plus y 1 (x) y 2 (x) into G double dot y 0 (x), so this is 2, and last term is equal to 0, because third derivative of G with respective y is 0. So therefore, B 3 (x) is equal to 2 y 0 (x) y 3 (x) plus 2 y 1 (x) y 2 (x) and you can find next eta rates. Now, we use this results for B 0, B 1, B 2, B 3 in terms of y 0, y 1, y 2, y 3 for G(y) equal to y square in order to solve a non-linear Fredholm integral equation.

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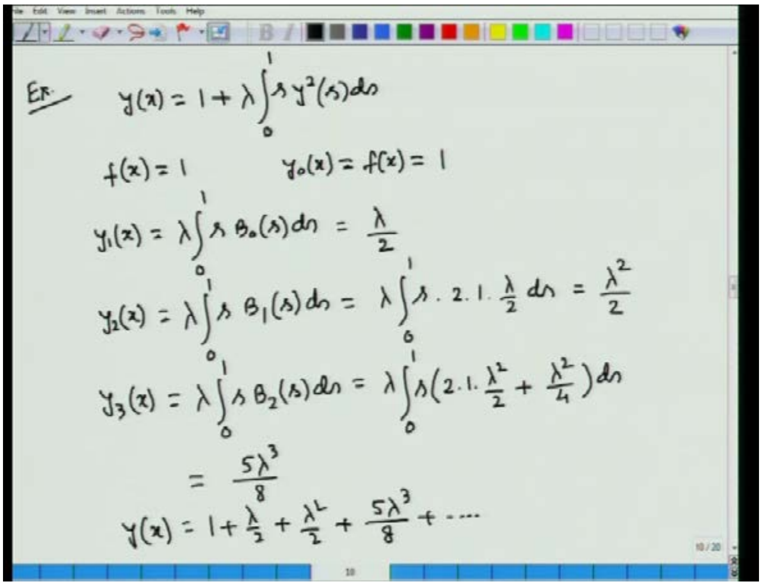

And here we are going to solve the same example that we first considered which we have solved using direct computation method, and that is the equation y(x) is equal to 1 plus lambda integral 0 to 1 s of y square s ds. So, here f(x) equal to 1, so using the formula y 0 (x) equal to f(x) we can find y 0 (x) equal to 1. Then y 1 (x) that is equal to lambda times integral 0 to 1 s B 0 (s) ds, now B 0 (s) is equal to y 0 (s), so this is 1, so this will comes out to be lambda by 2. Then y 2 (x) equal to lambda integral 0 to 1 s B 1 (s) ds, now B 1 (s) it was 2 y 0 y 1 (x); y 0 equal to 1, y 1 (x) equal to lambda by 2, so this will be equal to lambda integral 0 to 1 s into 2 into 1 into lambda by 2 ds and this will be equal to lambda square by 2.

And similarly, y 3 (x) is equal to lambda integral 0 to 1 s B 2 (s) ds, and this is equal to lambda integral 0 to 1 s 2 into 1 into lambda square by 2 plus lambda square by 4 ds. Because B 2 (x) was 2 y 0 (x) y 2 (x) plus y 1 squared x. So that means 2 into 1 into lambda square by 2 plus lambda by 2 whole squared, so this one, and after evaluating the integral and little bit simplification, we find this is equal to 5 lambda cube divided by 8. So therefore, adding this particular quantities 1 lambda by 2, lambda square by 2 and 5 lambda cube by 8, we find solution to the given problem is 1 plus lambda by 2 plus lambda square divided by 2 plus 5 lambda cube divided by 8 plus dot, dot up to infinity; this is the solution.

Now, in this case we have a non-linear equation. This equation to find we have solved with respect in using the direct computation method, then we have obtained two solutions, and here we are getting one solution. And now we have to verify whether this solution is matching with any one of them or not. So that means this gives us a new solution apart from what we have obtained earlier or not that we have to verify.

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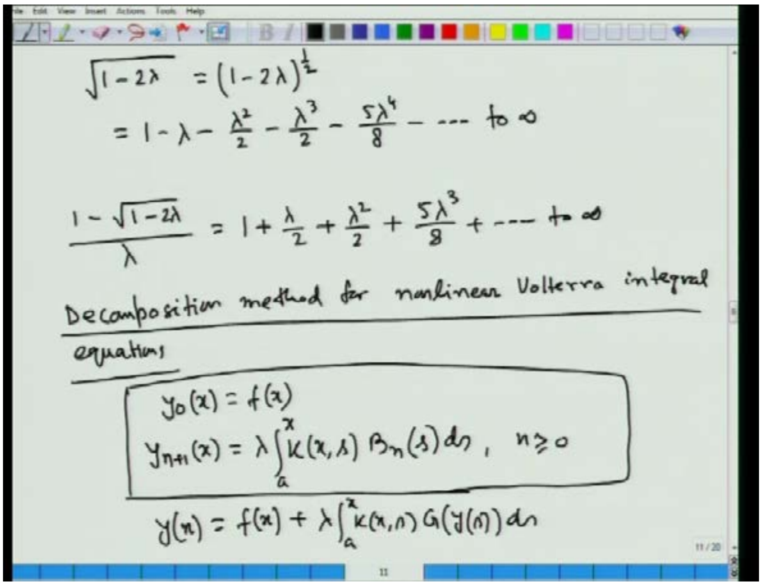

But if you calculate this expression that is 1 minus 2 lambda this square root, so that is actually 1 minus 2 lambda whole to the power half and expending this particular quantity in a negative binomial series, and assuming that 2 lambda less than 1. Therefore, you can find that lambda less than half restriction will come out from the condition for binominal series expansion of this particular expression. And after expanding in binominal series, you find this is equal to 1 minus lambda minus lambda square by 2 minus lambda cube divided by 2 minus 5 lambda to the power 4 divided by 8 minus dot, dot up to infinity, and with these expression for root over 1 minus 2 lambda you can find 1 minus root over 1 minus 2 lambda divided by lambda, this is equal to 1 plus lambda by 2 plus lambda square by 2 plus 5 lambda cube divided by 8 plus dot, dot up to infinity. So that means it is matching with the solution what we have obtained with adomian decomposition method.

So, by this you can verify that adomian decomposition method is not giving you a new solution which is not coincident with either of the solution what we have obtained from direct computation method. But this method has a limitation for this particular problem that it fails to capture or produce the other solution that is 1 plus root over 1 minus 2 lambda divided by lambda. So, with these example on one side we are try to understand how the method works in order to find out solution this problem, and also this method has a limitation that it is unable to capture or unable to produce both the solutions what we have obtained by direct computation method.

And of course, you can apply this type of adomian decomposition method for Volterra integral equations also. So, in that case decomposition method for non-linear Volterra integral equations, we have only one difference that is with the evaluation of the eta rates. Here y 0 (x) will be equal to f(x) and y n plus 1 x this will be equal to lambda integral a to x k of (x,s) B n (s) ds. So, these are the formula for evaluating the eta rates y 0, y 1 and so on for Volterra integral equation that is non-linear Volterra integral equation, which is given by that is y(x) is equal to f(x) plus lambda integral a to x k of (x,s) G of y (s) ds, and of course in this formula B n (s) these are defined as earlier. I am not going to consider any particular example here; you can find some examples in several places and try to solve it at your own.

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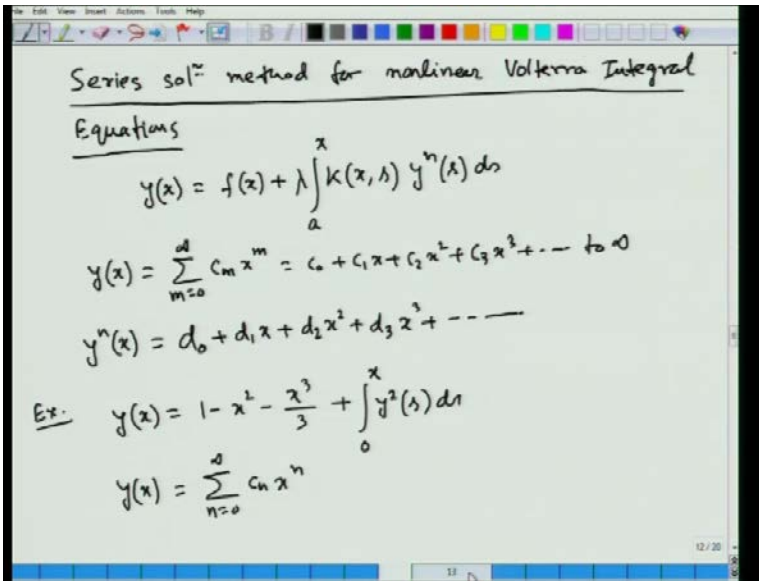

Next we come to the series solution method for non-linear Volterra integral equations. Although I have written here non-linear Volterra integral equation, but method that we are going to discuss that is for a specific type of non-linear Volterra integral equations. Those are given by y(x) is equal to f(x) plus lambda integral a to x k of (x,s) y n (s) ds. Once we will be discussing the method then you can understand why I am considering here the particular form y to the power n here, and that is just for simplicity and understanding of the method.

The method is that we assume that this equation processes a series solution of the form sigma m runnings from 0 to infinity c m x to the power m. So that means we are intended to find a solution in this particular format c 0 plus c 1 x plus c 2 x square plus c 3 x cube plus dot, dot up to infinity. Now, in case of y to the power n, we can calculate fast few terms of y n (x) and for example, say these are denoted by d 0 plus d 1 x plus d 2 x square plus d 3 x cube and so on. Then substituting these two expressions into the given equation and after evaluating the integrals we can equate light powers of x from both sides. And here this d 0, d 1, d 2, d 3 and so on, all of them are actually functions of c 0, c 1, c 2 and so on. So, depending upon the evaluation of this d 0, d 1, d 2 in terms of c 0, c 1, c 2, and then equating the light powers of x from the both sides, we have some non-linear system of equations for this unknown quantities c 0, c 1, c 2 and so on.

So, if we are able to solve those non-linear algebraic equations involving c 0, c 1, c 2 and so on, we will be able to find out solution of this particular problem. And we can explain this idea with help of one example. We consider this example that is y(x) is equal to 1 minus x square minus x cube by 3 plus integral 0 to x y square s ds. Here we are assuming y(x) is equal to sigma n runnings from 0 to infinity c n x to the power n.

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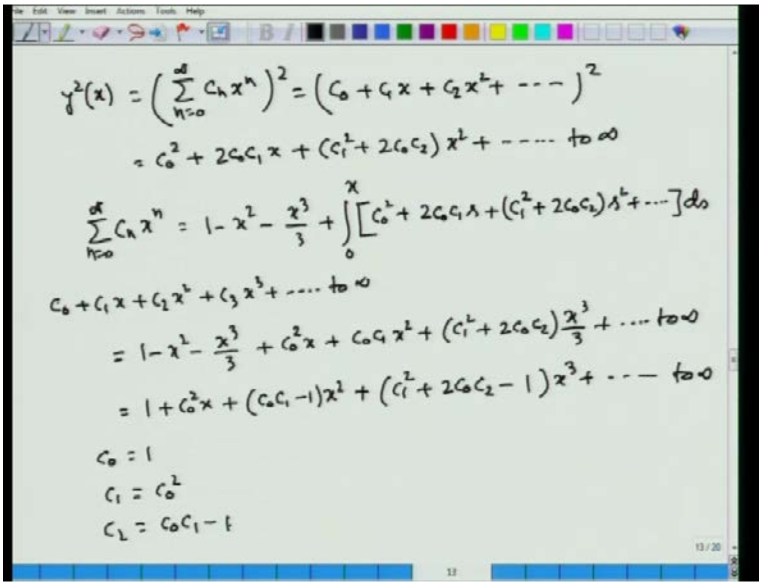

So now, if we calculate y square x, so that is equal to sigma n runnings from 0 to infinity c n x n whole square, and looking at first few terms of this particular series, we can find this is equal to c 0 square plus 2 c 0 c 1 x plus c 1 square plus 2 c 0 c 2 x square plus dot, dot up to infinity. So, if you just compare with the pervious discussion that d 0 is now equal to c 0 square, d 1 is equal to 2 c 0 c 1, d 2 equal to c 1 square plus 2 c 0 c 2 and so on. So, this relation between d 0, d 1, d 2, d 3 and so on can be obtained in terms of c 0, c 1, c 2, in the cases where non-linear functions involved with the non-linear integral equation is of the form y to the power n x. In other cases, it is very difficult to find out this type of relation between d j(s) with c i(s).

Now, if we substitute this expression into the given integral equation, then we actually find that sigma n runnings from 0 to infinity c n x to the power n that is equal to 1 minus x square minus x cube by 3 plus integral 0 to x c 0 square plus 2 c 0 c 1 s plus c 1 square plus 2 c 0 c 2 s square plus dot, dot up to infinity ds. And therefore, after evaluating this integral we find c 0 plus c 1 x plus c 2 x square plus c 3 x cube plus dot, dot up to infinity. This is equal to 1 minus x square minus x cube by 3 plus c 0 square x plus c 0 c

1 x square plus c 1 square 2 c 0 c 2 x cube divided by 3 plus dot, dot up to infinity. After rearranging this terms, we find this is equal to 1 plus c 0 square x plus c 0 c 1 minus 1 into x square plus c 1 square plus 2 c 0 c 2 minus 1 x cube plus dot, dot up to infinity. Then equating light powers from both sides we find c 0 equal to 1, then c 1 is equal to c 0 square, c 2 this is equal to c 0 c 1 minus 1.

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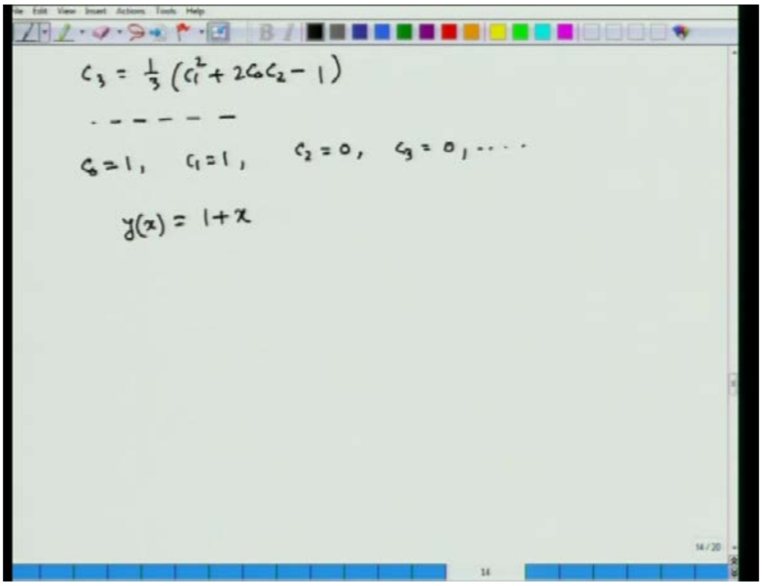

Then c 3, this is equal to one-third c 1 square plus 2 c 0 c 2 minus 1 and so on. So, from the first relation we find c 0 this is equal to 1, from the second that is c 1 equal to c 0 square we find c 1 equal to 1, then c 2 is equal to 0, because c 2 equal to c 0 c 1 minus 1, so 1 minus 1 equal to 0, then c 3 this is equal to c 1 square plus 2 c 0 c 2 minus one-third. So, substituting there you can find this is also equal to 0, and you can verify all other constant that is c 4 and onwards, if you solve this you will find all of them are exactly equal to 0. And therefore, the series for y(x) comes out to be that is 1 plus x, and you can verify this y(x) equal to 1 plus x is solution to the non-linear equation.

So, in case this non-linear function involved with the Volterra integral equation is of the form y n (x), then you can make a attempt to solve this type of non-linear Volterra integral equation with help of these series solution method. So, at this point I stopped the discussions on this non-linear integral equation. But before completing this lecture I like to mention that the materials I have covered in this lecture these are available in four books.

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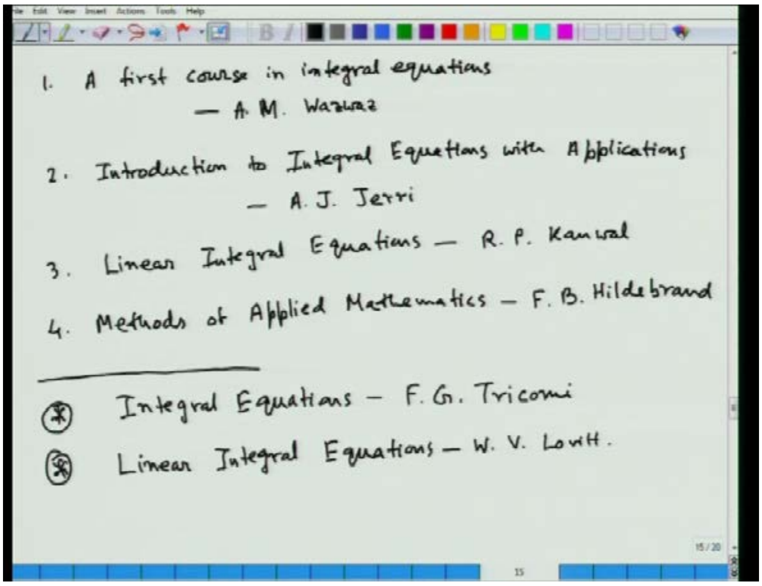

This four books are number 1 - A first course in integral equations by A. M. Wazwaz, this is the world scientific book, then number 2 - Introduction to Integral Equations with Applications, A. J. Jerri, number 3 - Linear Integral Equations by R. P. Kanwal and number 4 - The chapter on integral equation under this book that is Methods of Applied Mathematics by F. B. Hildebrand. And if you are interested to have a look to some advance level books, then this two books are very well written books. Number 1 that is Integral Equations by F. G. Tricormi, and the book that is Linear Integral Equations by W. V. Lovitt.

And before concluding I just like to remark that in all these books you find several discussions on in a integral equations is method of solution and other advance level treatment, but in the book by Wazwaz, the adomian decomposition method and modified adomian decomposition method are discussed and in that book you finds some (( )) differences where some advanced techniques for adomian decomposition method to solve some complicated linear and non-linear integral equations are also discussed. So, in this lecture series, we are mainly focused on linear integral equations, but in the last lecture I just consider two, three examples on non-linear integral equation just want to give you the essence of bifurcation concept of bifurcation involved with the non-linear integral equations. So, thank you for your attention.