Calculus Of Variations and Integral Equation

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Module # 01
Lecture # 36

Welcome viewers once again to the NPTEL lecture series on integral equations. In today’s lecture, we are going to discuss about the Fredholm theory to obtain resolvent kernel. Such that, we can find the solution of inhomogeneous Fredholm integral equation of the second kind.

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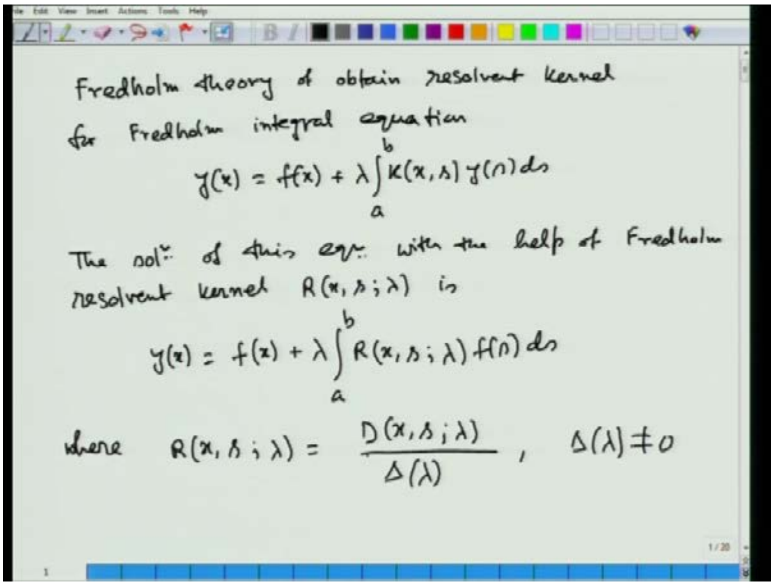

So, we are going to discuss that Fredholm theory to obtain resolvent kernel for Fredholm integral equation, which is given by y x equal to f x plus lamda integral a to b K (x, s) y s d s.

Now, you can recall that, we have this already discussed other possible methods to solve this kind of integral equation. And also, we have obtained resolvent kernel to find out solution of this non homogeneous Fredholm integral equation of the second kind. But here, we are going to discuss the method introduced by Fredholm, which is comes out to be ratio of two infinite series, that actually constitute the resolvent kernel. And as per the Fredholm theory the solution of this equation is given by solution of this equation with the help of Fredholm resolvent kernel. (No Audio From: 02:38 to 02:47) R x s lamda is y x is equal to f x plus lamda integral a to b R (x, s) lamda f s d s.

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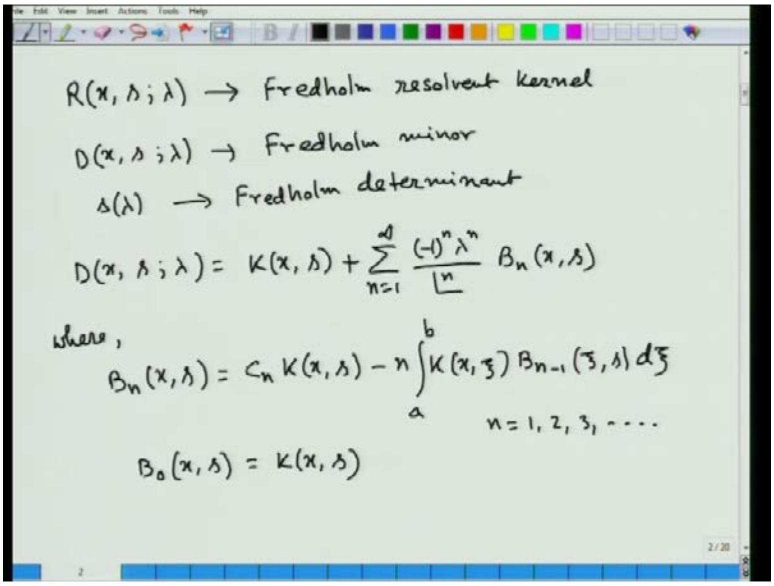

Where this resolvent kernel R x, s is a ratio of d x s lamda and delta lamda, where we assume that this delta lamda is not equal to 0. And this three quantities R x s lamda, this is known as Fredholm resolvent kernel. This D x s lamda this is known as Fredholm minor and delta lamda this is called Fredholm determinant. (No Audio From: 04:16 to 04:28) And now, we have to find out or define this Fredholm minor and Fredholm determinant. Such that, we can evaluate Fredholm resolvent kernel and which in turn gives us the solution for the Fredholm integral equation. This Fredholm minor D x s lamda, this is defined by K x, s plus sigma n running from 1 to infinity minus 1 whole to the power n lamda to the power n divided by factorial n B n x, s.

Where each B n x, s can be evaluated using this formula B n x, s that is equal to C n k x, s minus n integral a to b K (x, psi) B n minus 1 (psi, s) d psi. This formula is valid for n equal to 1 2 3 and so on. And in particular B 0 (x, s) is defined by K (x, s). So, using this definition B 0 (x, s) equal to K (x, s) of course, the formula for Fredholm minor can be written as B 0 (x, s) plus sigma n running from 1 to infinity and so on.

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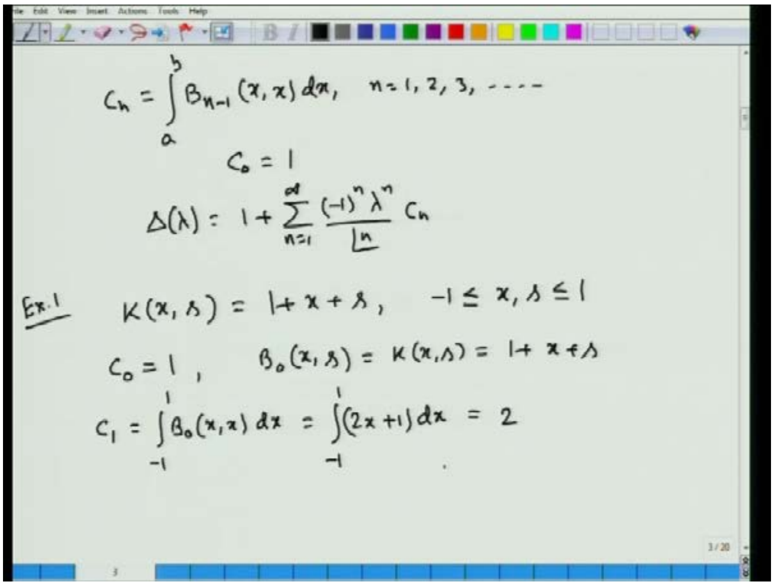

So, we can use this definition. Now, in order to evaluate this B n (x, s), we need the expression or the values of C n. This quantities C n these are defined by C n equal to integral a to b B n minus 1 (x, x) d x, where n ranging from 1 2 3 and so on. And in particular C 0 is defined as exactly equal to 1. So, with the these definitions for C 0 and C n. Now, we can define delta lamda this is equal to 1 plus sigma n running's from 1 to infinity minus 1 whole to the power n lamda to the power n by factorial n times C n. So, this is the definition for delta lamda, we have already defined D x s lamda. So, if we able to evaluate each B n (x, s) and this C n, then we can find the resolvent kernel, Fredholm resolvent kernel R (x, s).

Now, in some forth coming examples, you can see that most of the time, we will find that this C n and B n are 0 for values of n starting from say two and onwards. So, in most of the cases, you can find first few values C j's those are non zero and accordingly first few expressions for B n those are non zero and rest of the quantities are exactly equal to 0. And this happens most of the times, if we use the kernel's those are separable kernel that is called degenerate kernel. Then most of the cases you can find after some C j's first few non zero C j's, rest of the C j's will be equal to 0.And accordingly we can find those B n's are also equal to 0.

And therefore, instead of this infinite series for delta lamda and D x s lamda, most of the time, we will be having a polynomial containing a finite number of terms, because rest of the terms are identically equal to 0. So, first we consider one example, where we are just defining the kernel which is of course, separable kernel and for that, we construct the Fredholm resolvent kernel using this formula.

So, first example that we can consider K (x, s) this is given by 1 plus x plus s, where minus 1 less than equal to x, s less than equal to 1. So, you can recall as per definition C 0 equal to 1, B 0 (x, s) this is equal to K (x, s). So, this is equal to 1 plus x plus s. Next with this B 0, we can calculate C 1, you have to keep in mind, that formula for B n involve C n and B n minus 1 and formula for C n involve B n minus 1, we already have C 0 and B 0 as per definition B 0 is k (x, s).

So, using this expression for B 0 (x, s). Now, we can calculate C 1. So, C 1 this is equal to minus 1 to plus 1 B 0 (x, x) d x. So, this is equal to integral minus 1 to plus 1 2 x plus 1 d x. And after evaluating this integral, you can calculate this C 1 is equal to 2, because the integral minus 1 to plus 1 2 x d x this is equal to 0 and rest of the term will results in 2.

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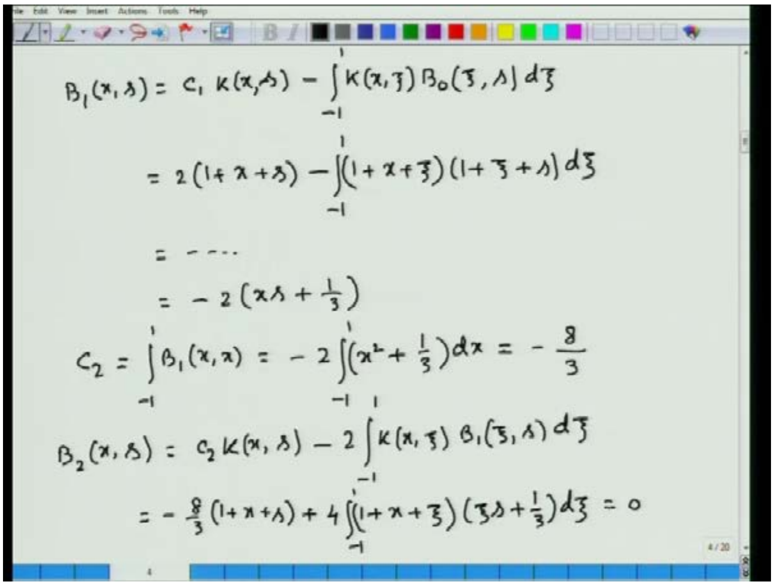

Now, for B 1 (x, s) as per the given formula, this is C 1 multiplied by K (x, s), then minus integral minus 1 to plus 1 K (x, psi) multiplied with B 0 (psi, s) d psi. And this will be equal to C 1, we have already obtained this is equal to 2. So, 2 into 1 plus x plus s minus integral minus 1 to plus 1 1 plus x plus psi this multiplied with 1 plus psi plus s d psi, because k x psi is nothing, but 1 plus x plus psi and B 0 psi s is actually K psi s. So, K psi s equal to 1 plus psi plus s and if you evaluate this integral, then it will results in minus 2 into x s plus one-third this will be the expression for B 1 x s. So, we have obtained B 1 x s, with this B 1 x s we can calculate C 2, C 2 is equal to integral minus 1 to plus 1 B 1 (x, x).

So, this will be equal to minus 2 integral minus 1 to plus 1 x square plus one-third d x and after evaluating this integral, you can find this will be equal to minus 8 by 3. So, now, we have obtained C 2 already we have in our hand B 1 (x, s). So, now, we can calculate B 2 (x, s), B 2 (x, s) is equal to C 2 K (x, s) minus 2 integral minus 1 to plus 1 K (x, psi) multiplied with B 1 (psi, s) d psi. This is the expression; you can recall already we have defined this formula for B n. So, therefore, substituting this expression, we can find this will be minus 8 by 3 multiplied with 1 plus x plus s.

And when will be substituting for B 1 (psi, s). So, this minus 2 will be clubbed with this minus 2. So, we will be having plus 4 integral minus 1 to plus 1, 1 plus x plus psi and from here, we will having psi s plus one-third d psi. And if you evaluate this integral, then you can find this integral is exactly equal to 0. Now, once you have B 2 (x, s) equal to 0.

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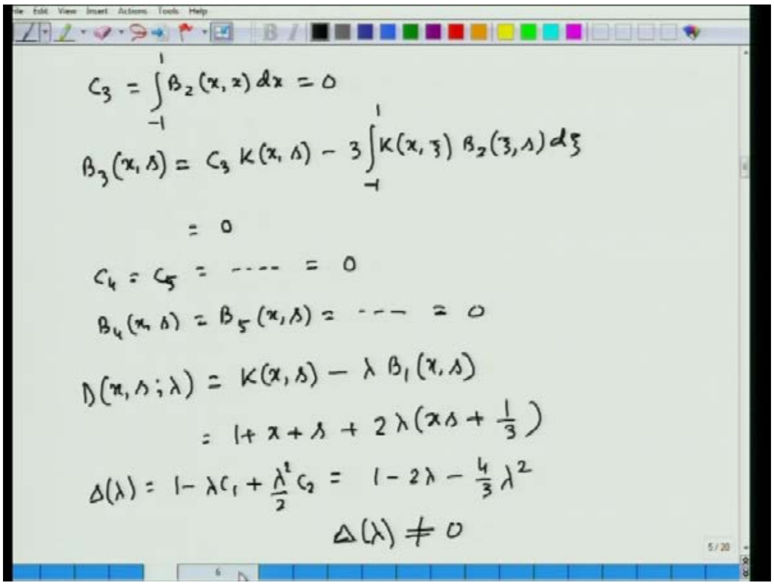

So, therefore, C 3 equal to integral minus 1 to plus 1 B 2 (x, x) d x this should be equal to 0. And once C 3 equal to 0, then immediately B 3 (x, s) according to the formula this is C 3 K (x, s) minus 3 integral minus 1 to plus 1 K (x, psi) B 2 (psi, s) d psi. Now, this C 3 equal to 0 and B 2 (psi, s) equal to 0. So, you will be having B 3 equal to 0 and therefore, you can easily understand that all C 4 C 5 and so on, this constant should be identically equal to 0. Accordingly B 4 (x, psi) B 5 (x, psi) and so on, all of them will be equal to 0. So, as C 3 and onwards are all equal to 0 and B 3 and onwards all this quantities are identically equal to 0.

So, therefore, we can find that D (x, s lamda) D (x, s lamda), this will be equal to K (x, s minus lamda) B 1 (x, s), because B 2 (x, s) equal to 0, B 3 (x, s) equal to 0. So, none of the term will come here and ultimately you will find this D (x, s lamda) is equal to 1 plus x plus s plus 2 lamda multiplied with x s plus one-third. So, this is actually your D (x, s colon lamda). And delta lamda this will be equal to 1 minus lamda C 1 plus lamda square by 2 C 2 and rest of the term C 3 C 4 are all identically equal to 0. So, therefore, will be having, 1 minus 2 lamda minus 4 by 3 lamda square. So, this is the expression for delta lamda. So, in case this delta lamda is not equal to 0 assuming this delta lamda not equal to 0.

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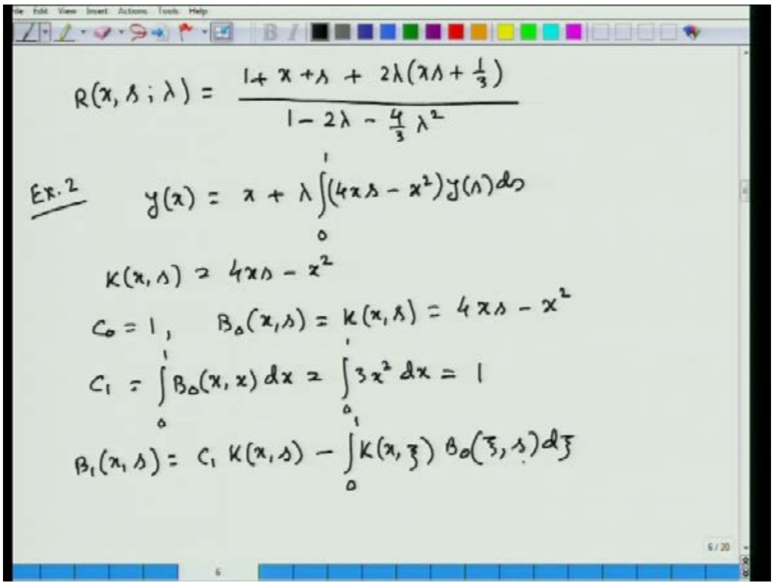

We can obtain the corresponding resolvent kernel, R x s lamda this is equal to 1 plus x plus s plus 2 lamda multiplied with x s plus one-third whole divided by 1 minus 2 lamda minus 4 by 3 lamda square. So, this is the resolvent kernel, Fredholm resolvent kernel obtained by Fredholm method.

Next, we solve an Fredholm integral equation using the same method. The given problem is y x this is equal to x plus lamda integral 0 to 1 4x s minus x square y s d s. So, this is our problem. So, in this case K (x, s) is equal to 4x s minus x square this is our actually kernel. So, our first task is using the Fredholm method, we have to find out the Fredholm resolvent kernel. And for that purpose, we have C 0 equal to 1 B 0 (x, s) that is equal to K (x, s) that is equal to 4x s minus x square.

Now, we can calculate C 1, C 1 is equal to integral 0 to 1 B 0 (x, x) d x. Now, B 0 (x, x) this will be equal to 4 x square minus x square. So, that means 3 x square. So, integral 0 to 1 3x square d x and after evaluating this integral, we can find this is equal to 1. Next we can calculate B 1 (x, s) B 1 (x, s), as per definition C 1 K (x, s) minus integral 0 to 1 K (x, s), it will be (x, psi) multiplied with B 0 (psi, s) d psi.

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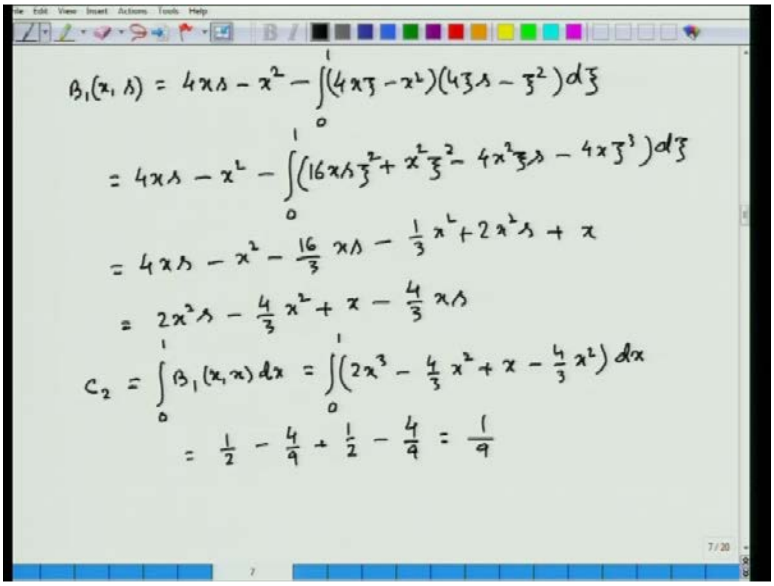

So, substituting this expression, we can find this B 1 (x, s), this will be equal to 4x s minus x square because C 1 equal to 1. So, C 1 K x s is nothing, but K x s minus integral 0 to 1 4x psi minus x square this multiplied with 4 psi s minus psi square d psi. This will be equal to 4x s minus x square minus integral 0 to 1 16x s psi plus x square psi square minus 4x square psi s minus 4x psi cube d psi. And evaluating these integral you can find 4x s minus x square minus from first one, you can find 16 by 3x s plus one-third x square minus. It will be plus 2 x square s and then plus x, this will be actually minus, because integral of psi square probably 0 to 1 is one-third. So, minus 16 by 3 x s, then psi square d psi again one-third. So, minus one-third x square and from here half into 4x square s so, minus if this minus combine will give you plus 2 square s.

And finally, 4x by 4 so, this will be plus x. And after rearranging the terms under simplification, you will be having 2x square s minus 4 by 3 x square plus x minus 4 by 3 x s. So, this is our B 1 (x, s) that, we have calculated and as we know the B value of B x expression for B (x, s). So, we can calculate C 2, C 2 equal to integral 0 to 1 B 1 (x, x) d x. So, this is equal to integral 0 to 1 2 x cube minus 4 by 3 x square plus x minus 4 by 3 x square d x. And after evaluating this integral, this will be half minus 4 by 9 plus half minus 4 by 9 and this will be equal to 1 by 9.

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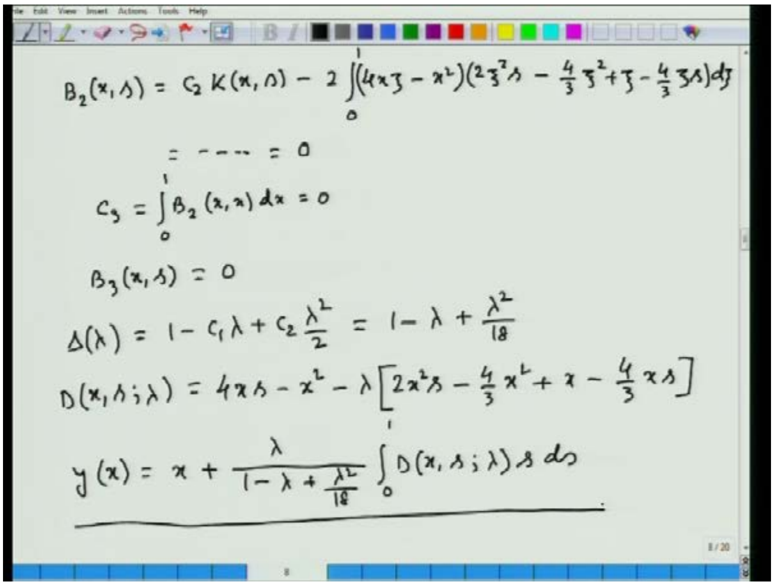

So, this is the value for C 2, with this value for C 2. If we calculate B 2 (x, s) this will be equal to C 2 K (x, s) minus 2 integral 0 to 1 (4 x, psi minus x square). This multiplied with we have to write the expression for B 1 (psi, s). Now, B 1 (psi, s) is 2 psi square s minus 4 by 3 psi square plus psi minus 4 by 3 psi s d psi. And if you evaluate this integral and after simplification it will comes out to be exactly equal to 0.

And once you have B 2 (x, s) this is equal to 0. So, immediately C 3 equal to integral 0 to 1 B 2 (x, x) d x this is equal to 0. And using the formula for B 3 (x, s), you can calculate this is equal to 0. So, C 3 equal to 0 and B 2 (x, s) equal to 0 will give you B 3 (x, s) equal to 0 and therefore, C 4 should be equal to 0. According to the formula C 4 equal to integral 0 to 1 B 3 (x, x) d x and therefore, B 4 (x, s) equal to 0. And hence we have all the constants C 3 C 4 C 5 and so on; those are identically equal to 0. And the functions B 2 (x, s), B 3 (x, s) all this functions are also equal to 0; that means, B 4 B 5 all of them are identically equal to 0.

And therefore, delta lamda that is Fredholm determinant is given by 1 minus C 1 lamda plus C 2 lamda square by 2 that is equal to 1 minus lamda plus lamda square divided by 18. And D x s lamda this is equal to 4 x s minus x square minus lamda into D 1 x that is 2 s square s minus 4 by 3 x square plus x minus 4 by 3 x s. This is the expression for D 1 (x, s). And therefore, solution to the given problem is y x equal to x plus lamda divided by 1 minus lamda plus lamda square by 18, then integral 0 to 1 D of x s lamda s d s because f x equal to x. So, f s is equal to s.

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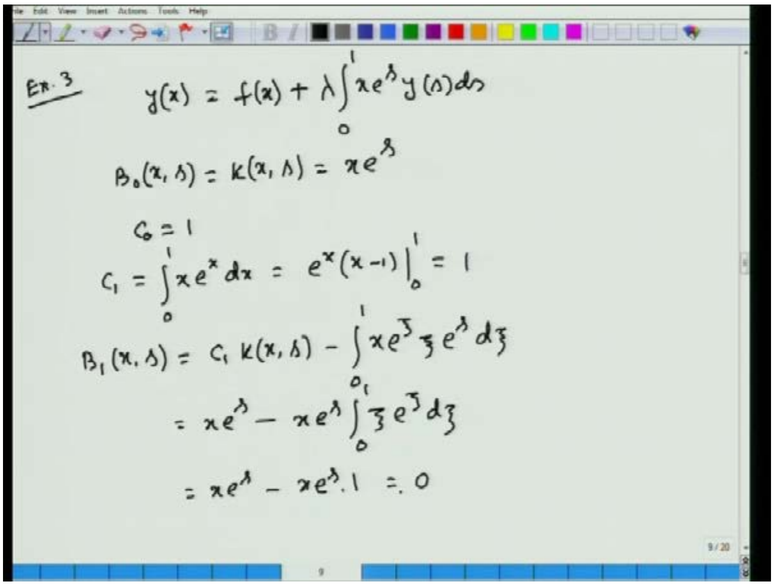

So, this is the solution to the given integral equation. Next we consider one more example, this example is taken from Allan Jerry's book on integral equation, where first of all will find out the solution of the given Fredholm integral equation by this method. And then, we will discuss another formula by which you can evaluate the quantities B 1 x s B 2 x s and so on. And a constants C 1 C 2 and so on in terms of repeated integrals. And where integrants are actually determinants of order, n plus 1 and of order in respective way.

So, first of all we consider the problem here, the problem is y x is equal to f x plus lamda integral 0 to 1 x e to the power s y s d s, this is the given problem. Clearly B 0 (x, s) is equal to kernel of the given problem k (x, s). So, this is equal to x e to the power s. So, this our kernel and C 0 by definition this is equal to 1. So, now, we can calculate C 1 this is equal to integral 0 to 1 x e to the x d x evaluating you can find e to the power x into x minus 1 limit 0 to 1 and this will results in this is equal to 1.

Now, if we calculate B 1 (x, s) this is equal to C 1 K (x, s) minus integral 0 to 1 x e to the power psi multiplied with psi e to the power s d psi. So, this will be equal to since C 1 equal to 1. So, x e to the power s minus, we can take x e to the power s outside the integral x e to the power s then integral 0 to 1 psi e to the power psi d psi. Here we have already evaluated integral 0 to 1 x e to the power x d x equal to 1. So, therefore, this will be equal to x e to the s minus x e to the power s into 1. So, that means, that this is identically equal to 0. So, B 1 (x, s) is equal to 0 and once B 1 (x, s) is equal to 0.

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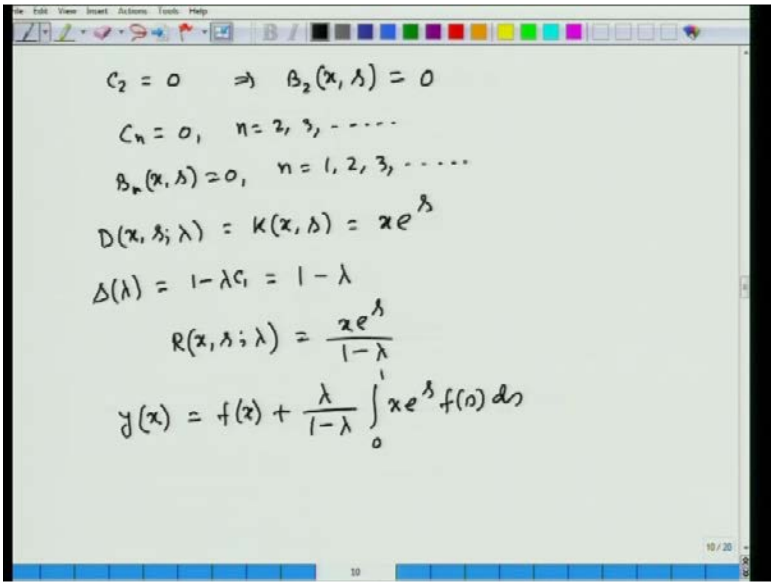

So, therefore, C 2 this is equal to 0 and C 2 equal to 0 this implies B 2 (x, s) this is also equal to 0. And therefore, we can write C n this is equal to 0 or n equal to 2 3 and so on. And B n (x, s) this is equal to 0 for n equal to 1 2 3 and so on. So, therefore, D x s lamda this is only K (x, s). So, that is equal to x e to the power s. And delta lamda this is equal to 1 minus lamda C 1 since, we have evaluated C 1 equal to 1. So, this will be equal to 1 minus lamda. And therefore, the resolvent kernel obtained using Fredholm method, that is R x s lamda this is equal to x e to the power s divided by 1 minus lamda this is the resolvent kernel.

And therefore, solution to the given problem is given by y x equal to f x plus lamda, whole divided by 1 minus lamda integral 0 to 1 x e to the power s f s d s, this is the solution to the given integral equation. Now, we define the alternative way by which, we can calculate this B n (x, s) and C n. Of course, we have to keep in mind that B 0 (x, s) is nothing, but K (x, s) and C 0 is also equal to 1.

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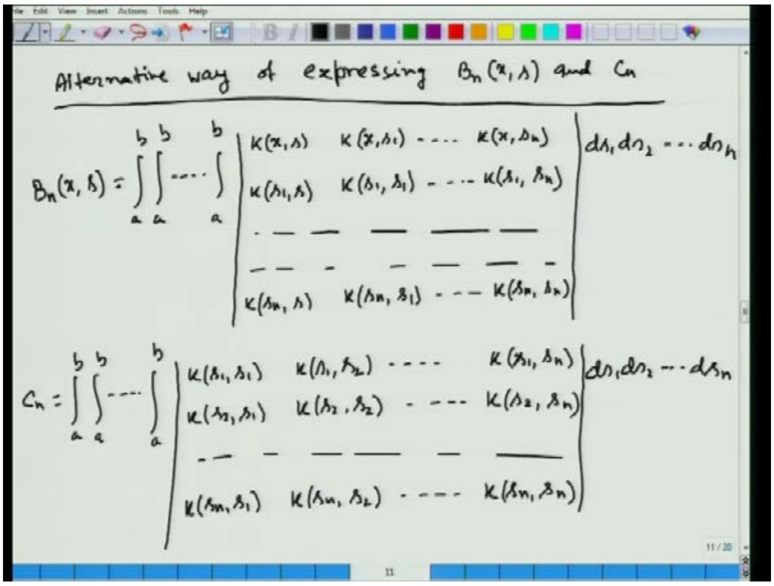

So, the alternative way to expressing B n x and C n.

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B n (x, s) and C n and here this alternative way one sense it is important. That in earlier method, you have observed that calculation of B n (x, s) depends up on the x value of C n. And the expression for B n minus 1 x s and evaluation C n was related to the expression for B n minus 1 (x, s).

Now, in this alternative approach, we do not need the expressions for B n minus 1 in order to calculate C n and the value of C n and the expression for B n minus 1 in order to evaluate B n (x, s). So, first of all we define, this B n (x, s) is equal to integral a to b, integral a to b up to integral a to b. Then determinant K (x, s) K (x, s 1) up to K (x, s n), this is the first row. Second row K (s 1, s) K (s 1, s 1) proceeding in this way K (s 1, s n) finally, last row will be K (s n, s) K (s n, s 1). Last term K (s n, s n), this determinant d s 1 d s 2 up to d s n, this is the expression for B n (x, s).

And C n this is defined by integral a to b, integral a to b, integral a to b, this determinant K (s 1, s 1) K (s 1, s 2) up to K (s 1, s n) this is the first row. Second row K (s 2, s 1) K (s 2, s 2) dot dot K (s 2, s n) proceeding in this way. Last row will be K (s n, s 1) K (s n, s 2) last term is K (s n, s n), then d s 1 d s 2 up to d s n. So, in both these cases although, we have to evaluate these nth ordered integrals, but from previous examples what we have discussed, you can able to understand. Using the property of determinant we will be able to prove that B n will be equal to 0 for values of n say 3 4 and onwards or for example, say 2 3 and onwards.

Now, with this alternative definition for the expression of B n (x, s) and C n, we can solve the last problem, that we have just solved that is the y x equal to f x plus integral 0 to 1 x e to the power s f s d s multiplied by lamda. So, this problem, we are going to solve using these definitions for B n (x, s) and C.

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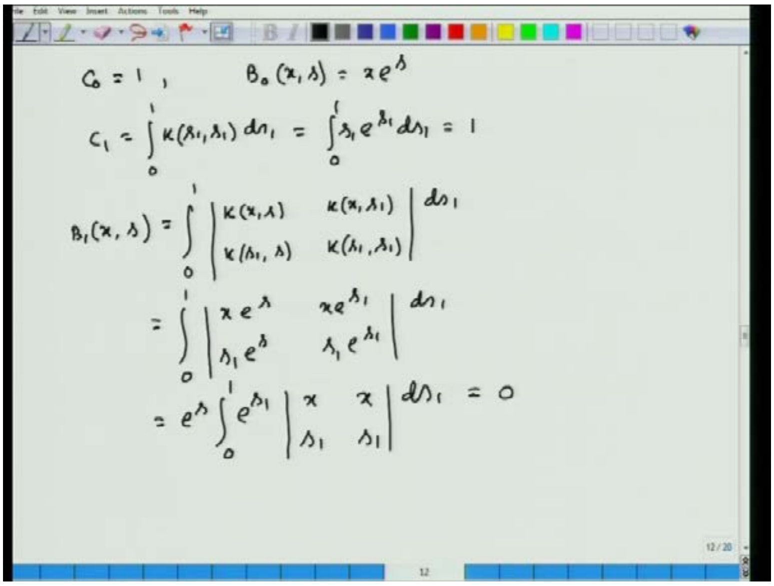

So, C 0 equal to 1, B 0 (x, s) this is equal to x e to the power s. Now, you can see that C 1 this will involve only one integral, that is integral 0 to 1 d s 1 and here determinant consist of only one term that is, we are having 1 plus 1 determinant that is a scalar quantity K s 1 s 1 d s 1. So, this will be integral 0 to 1 s 1 e to the power s 1 d s 1 this is equal to 1. Then B 1 (x, s) this is equal to integral 0 to 1 second order determinant K (x, s) K (x, s 1), then K (s 1, s) and then K (s 1, s 1) d s 1. And if we substitute the expressions so, this is will be 0 to 1 determinant x e to the power s x e to the power s 1, then s 1 e to the power s 1 and s 1 this will be s 1 e to the power s and this is s 1 e to the power s 1 d s 1.

So, from the first row, first column sorry, first column we can take e to the power s common from the second column we can take out e to the power s 1 column common. So, e to the power s integral 0 to 1 e to the power s 1, then determinants x s 1 x s 1 d s 1 this determinant is 0.

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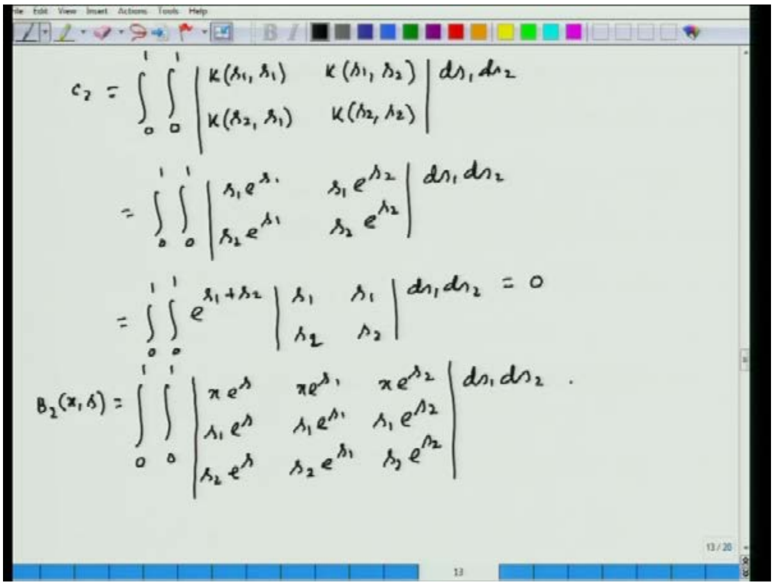

So, clearly B 1 (x, s) is equal to 0. Next, if we calculate C 2, this will be equal to double integral 0 to 1 0 to 1 determinant K (s 1, s 1), then K (s 1, s 2). Second row K (s 2, s 1) and then K (s 2, s 2) d s 1 d s 2, if we substitute the expressions for kernels, it will be s 1 e to the power s 1, then s 1 e to the power s 2, then s 2 e to the power s 1 and s 2 e to the power s 2 d s 1 d s 2. So, if we take out the exponential from the determinant that means taking e to the power of s 1 from the first column. And e to power of s 2 from the second column, then e to power of s 1 plus s 2 multiplied with first column s 1 s 2 second column of the determinant is again s 1 s 2 d s 1 d s 2 so, this is equal to 0.

So, although, here we do not have any link between b 2 and c 2 in order to evaluate them using this alternative method. So, this c 2 is equal to 0 now, in order to understand the rest of the expressions for b 3 and onwards will be equal to 0. We can just have a look at expression for B to x, if we look at expressions for B 2 (x, s), this will be double integral 0 to 1, 0 to 1 a third order determinant. I am not writing the that is K (x, s) I am directly substituting there expressions. First row will consist of x e to the power s then x e to the power s 1 and then x e to the power s 2. Then second row will be s 1 e to the power s, s 1 e to the power s 1 s 1 e to the power s 2.

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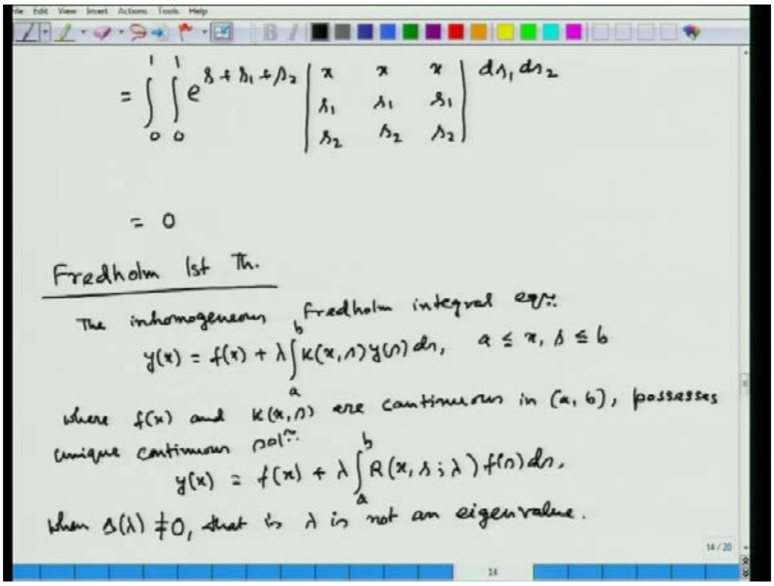

And third row is s 2 e to the power s then s 2 e to the power s 1 and then s 2 e to the power s 2 with respect to d s 1 d s 2. And you can write this is equal to integral 0 to 1 integral 0 to 1 e to the power s plus s 1 plus s 2 multiplied with determinant first row x s 1 s 2, second row column x s 1 s 2, third column x s 1 s 2 all these columns are identical. So, determinant is exactly equal to 0 and therefore, B 2 (x, s) is also equal to 0. So, proceeding in this way and if your able to understand this argument, all other determinants involve with B 3 (x, s) B 4 (x, s) will produce this kind of determinant each of which will be identically equal to 0.

And therefore, all other B 2 (x, s), B 3 (x, s) are identically equal to 0 and also, this kind of determinant will be involved with c j’s also. And therefore, all the c j’s except C 0 and C 1 are identically equal to 0. So, you will be having all this quantities are identically equal to 0. Now, before concluding this lecture I define three theorems of Fredholm and before defining those theorems. I just want to recall that in the last lecture, we have considered one example, where we have discussed about Eigen values and Eigen functions associated with the homogeneous Fredholm integral equation. You can recall that in case of homogeneous Fredholm integral equation, we have Eigen values and Eigen functions.

And with an elastic example, we have discussed in detail that sometimes those equations may not have any nontrivial solutions. And if they have some nontrivial solution, then sometimes they admit infinite number of solutions. So, whenever this lamda satisfies the condition delta lamda equal to 0; that means, if we choose a particular value of lamda, which is a 0 of the function delta lamda.

Or in other words those are the Eigen values of the associative homogeneous problem, then what will be the nature of solution and in which case we will be having infinite number of solutions, those can be summarized by three theorem of Fredholm. So, first of all we consider Fredholm first theorem. It states that, the inhomogeneous Fredholm integral equation y x equal to f x plus lamda integral a to b K (x, s) y s d s, where a less than equal to (x, s) less than equal to b this is the range f x and K (x, s) are continuous in finite interval (a, b) possesses unique solution, (No Audio From 44:43 to 44:49) unique continuous solution (No Audio From 44:53 to 44:59) given by y x equal to f x plus lamda integral a to b R(x, s) lamda f s d s, when delta lamda this is not equal to 0, that is lamda is not an Eigen value.

In case delta lamda not equal to 0; that means lamda is not an Eigen value, then this Fredholm integral equation possesses unique continuous solution. That is given by this one, this R x is lamda. Here I have written in intended to mean the Fredholm resolvent kernel, but of course, by using other method you can also calculate this resolvent kernel which will be a solution of this Fredholm integral equation. Then second theorem states that in this case we are assuming that the particular lamda involve with the equation say lamda 0 which is an Eigen value.

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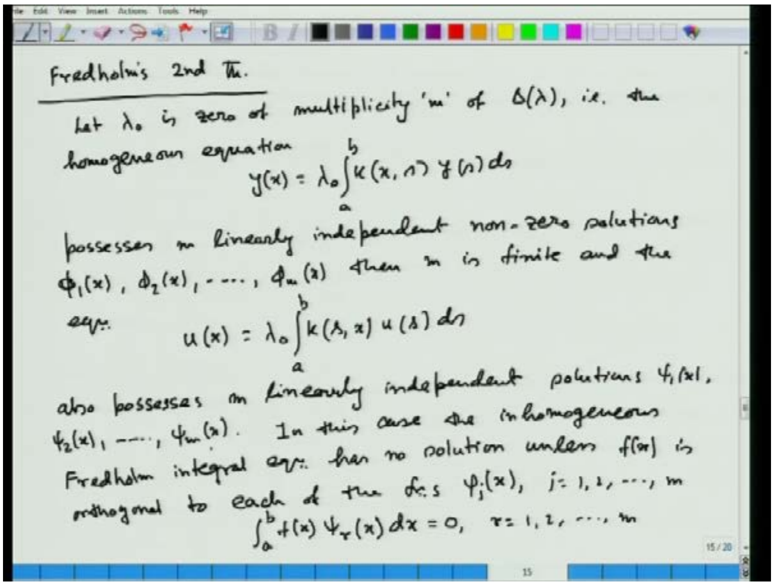

So, this is Fredholm's second theorem. It states that let lamda 0 is 0 of multiplicity m of delta lamda. That is the homogeneous equation y x equal to lamda 0 integral a to b K (x, s) y s d s possesses m linearly independent non-zero solutions which are denoted by psi phi 1 x phi 2 x up to phi m x. Then m is finite and the equation u x equal to lamda 0 integral a to b K(s, s) u s d s also possesses m linearly independent solutions which are denoted by psi 1 x psi 2 x up to psi m x.

In this case the inhomogeneous Fredholm integral equation that we have written earlier has no solution unless f x is orthogonal to each of the functions psi j x for j running's from 1 2 up to m. That is integral a to b f x psi r x d x this is equal to 0 for r equal to 1 2 up to m. So, will have non-trivial solution whenever this condition that is integral a to b f x psi r x d x equal to 0; that means, this f x is orthogonal to each of the eigen function psi 1 x psi 2 x up to psi m x and these psi 1 x psi 2 x up to psi m x. These are the eigen functions of the problem defined by u x equal to lamda 0 integral a to b K (s, x) u s d s.

And last Fredholm theorem that is the Fredholm thirds theorem. That gives us whenever this condition is satisfied; that means f x satisfies the condition that is psi r x when multiplied with f x integral a to b 0. That means, it is orthogonal to each of the eigen functions psi 1 psi 2 up to psi m x. Then the nontrivial solution of the equation is given by as a linear combination of this eigen functions. This is the Fredholm's third theorem.

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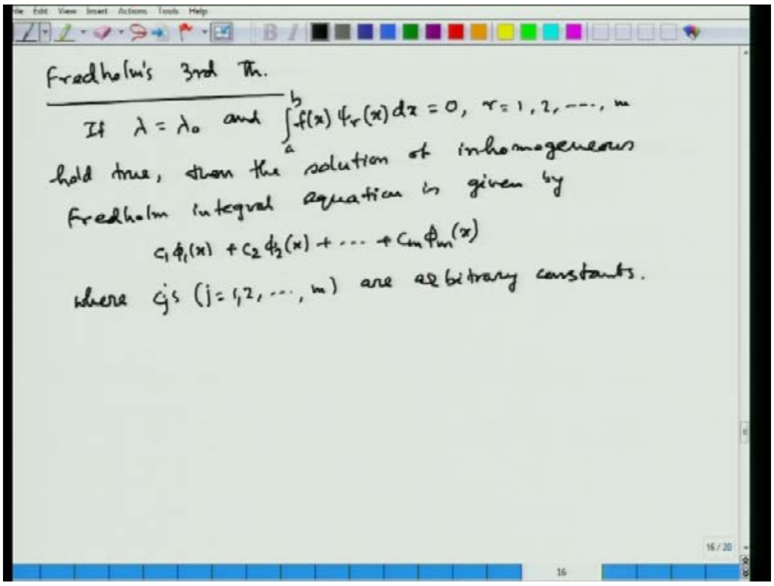

If lamda equal to lamda 0 and integral a to b f x psi r x d x this is equal to 0 where r equal to 1, 2, 3 up to m hold true. Then the solution of inhomogeneous Fredholm integral equation is given by C 1 phi 1 x plus C 2 phi 2 x plus dot dot plus C n phi n x where c j's, j running from 1, 2 m are arbitrary constants. So, these are actually three Fredholm's theorem associated with the Fredholm integral equation the second kind and in case this lamda is not an Eigen value of the associated homogeneous problem. And f x and K (x, s) that is kernel, those are continuous over the interval a, b then we have unique solution.

And in case lamda equal to lamda 0 which is an enfold eigen value or a 0 of delta lamda of multiplicity m. Then we will be having m eigen functions and if f x satisfies this orthogonality condition that is integral a to b f x psi r x d x equal to 0 for r running's from 1 to m. Then we have infinite number of solutions for the inhomogeneous Fredholm integral equation because this C 1 C 2 up to C n these are all arbitrary constants. And the linear combination of this eigen functions that is C 1 phi 1 x plus C 2 phi 2 x plus up to dot dot that is C n phi m x. This will be the solution of this last term will be C m phi m x where c j is ranging from 1 to m. This is the infinite number of solutions for the inhomogeneous Fredholm integral equation.

Today I stop at this point. In next lecture, we will be considering hill ward smith theory for these Fredholm integral equations. Thank you for your attention.