Calculus of Variations and Integral Equation

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Lecture No. # 35
Module No. # 01

Welcome viewers to the lecture series on Integral Equation under NPTEL courses. In last lecture, we have considered the Fredholm integral equation of second kind, with non homogenous term and we have discussed in case of separable or degenerate kernel, how the given Fredholm integral equation can be converted into a system of linear equation. And briefly we have discussed that, depending upon the existence of the solution of the system of the linear equation, we can find out solution to the given Fredholm integral equation.

In this lecture, first we considered one more example because, example we discussed in the type, because summation p r x, q r x, d s for the selected problem contains only one term that is p 1 and q 1 term.

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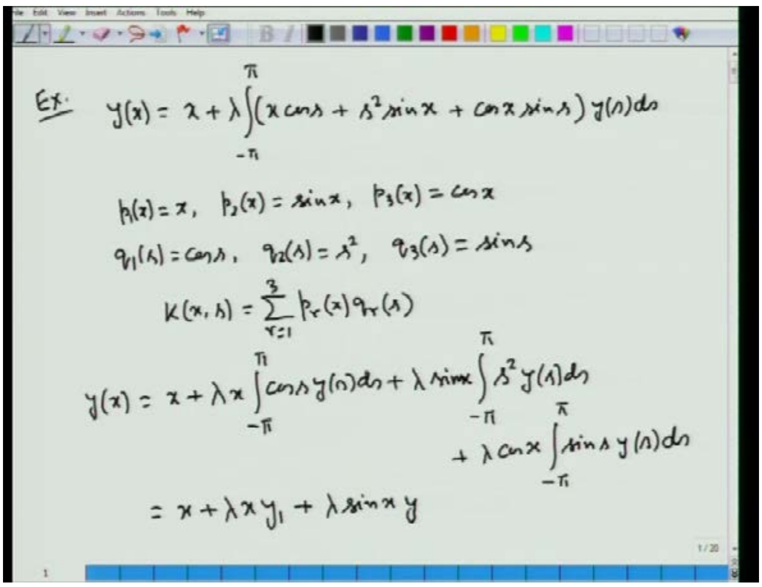

So, in this lecture we can start with this example, that we have to solve this equation, y x is equal to x plus lambda integral minus phi to plus phi x cosine s plus s square sin x plus cosine x sin s y s d s.

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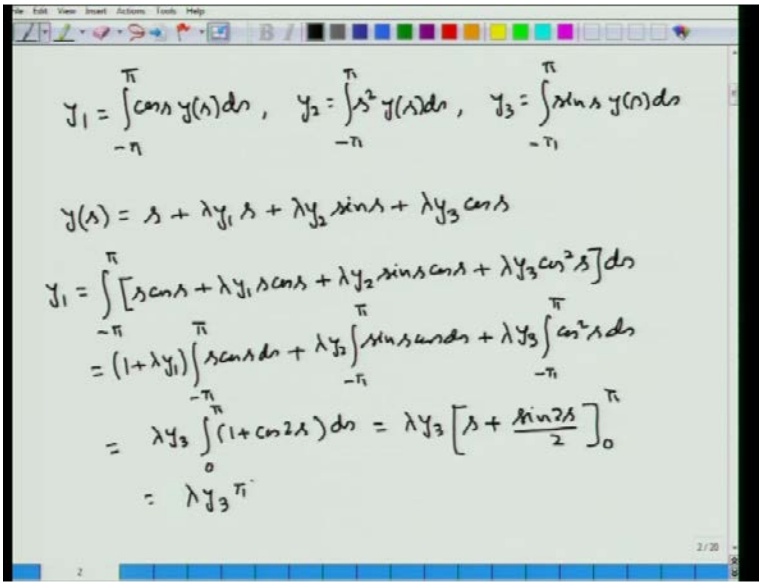

So, with the notation we had discussed, in the last lecture, here p 1 x is equal to x p, 2 x, this is equal to sin x, p 3 x, this is equal to cos x and q 1 s this is equal to cos s, q 2 s this is equal to s square, q 3 s this is equal to sin s and therefore kernel, given kernel k x comma s is nothing but, sigma r running’s from 1 to 3 p r x q r s; so now, we can write the integral equation into this format that is, y x equal to x plus lambda x integral minus phi to plus phi cost s y s d s plus lambda sin x integral minus phi to plus phi s square y s d s plus lambda cosine s integral minus phi to plus phi sin s y s d s and that is equal to x plus lambda x y 1 plus lambda sin x y 2 plus lambda cos x y 3; where this y 1 stands for minus phi to plus phi cos s y s d s y 2 this is equal to minus phi to plus phi s square y s d s and y three this is equal to minus phi to plus phi sin s y s d s.

So, as we have discuss in the last lecture that, now we have to multiple this expression that y x equal to x plus lambda x y 1 x plus lambda sin x, y 2 plus lambda cosine x, y 3 by cosine x, x square and sin x respectably and then we have to integrate from minus phi to plus phi in order to constitute the system of linear equation for y 1 y 2 y 3, so we can do that, alternatively we can think about that y s, this is equal s plus lambda y 1 s plus lambda y 2 sin s plus lambda y 3 cosine s and with these definition for y s we can substitute this y s into the definition for y 1 y 2 y 3, in order to construct the system of linear equations. So, therefore, from the definition for y 1 we can write y 1 equal to integral minus phi to plus phi s cosine s.

Plus lambda y 1 s cosine s plus lambda y 2 sin s cosine s plus lambda y 3 cosine square s d s and from here, we can write this is equal to 1 plus lambda y 1 minus phi to plus phi s cosine s d s plus lambda y 2 integral minus phi to plus phi sin s cosine s d s plus lambda y 3 integral minus phi to plus phi cosine square s d s; now past two integrals initiates as s cosine s is a odd function, sin s cosine s is also an odd function and in both the cases in limit is from minus phi to plus phi, so both of them is equal to 0 and the last integral can be written as lambda y 3 integral 0 to phi 1 plus cos 2 s d s, because minus phi to plus phi can be converted into to 0 to phi and then 2 cos square is nothing but, 1 plus cos 2 s and after integration will be having lambda y 3 s plus sin 2 s divided by 2 limit from 0 to phi and therefore, this will results in this is equal to lambda y 3 phi this will be the result for the first part.

And then for the second expression y 2, this is equal to integral minus phi to plus phi s square y s d s and this is equal to integral minus phi to plus phi, substituting the expression for the y s, we can find this is s cube plus lambda y 1 s cube plus lambda y 2 s square sin s plus lambda y 3 s square cosine s d s and again here, s cube plus lambda y 1 s cube d s this will be 0, because this is hot function and s square sin s is also a odd function.

So therefore, we are left with the last integral, that is minus phi to plus phi lambda y 3 s square cosine s d s and this will be equal to 2 lambda y 3 integral 0 to phi, s square cosine s d s and after integrating by parts, we find 2 lambda y 3 s square sin s plus 2 s cosine s minus 2 sin s limit 0 to phi and ultimately we will having only one nonzero term, that is from 2 s cosine s when evaluated at s equal to phi.

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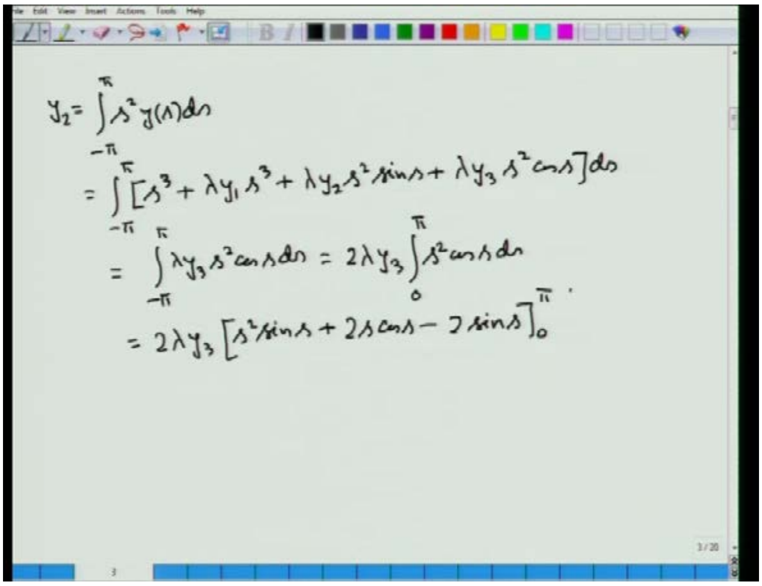

So, substituting this limit, Upper lower limits and upper limits this will results in minus 4 lambda phi y 3 and finally for y 3, this is equal to integral minus phi to plus phi, you can recall the definition, this is minus phi to plus phi sin s y s d s.

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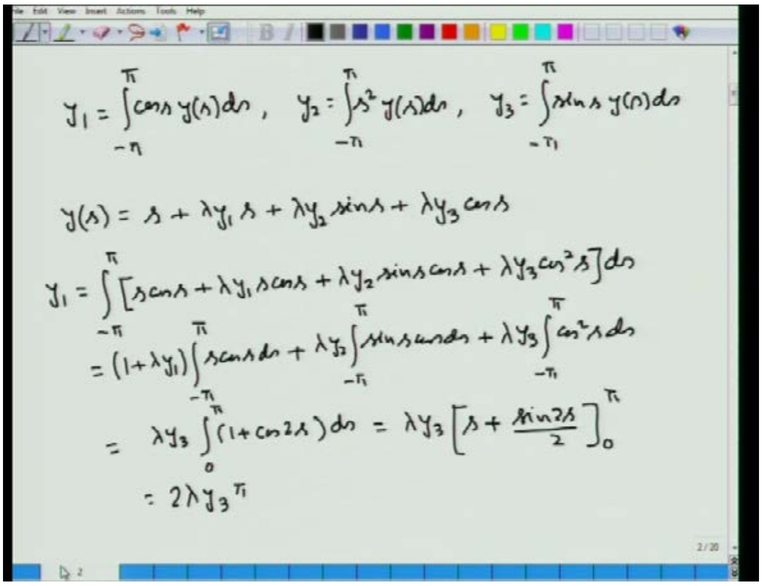

So, this is sin s y s d s and substituting the expression for y s, this reduces to minus phi to plus phi, s sin s plus lambda y 1 s sin s plus lambda y 2 sin square s plus lambda y 3 sin s cosine s d s.

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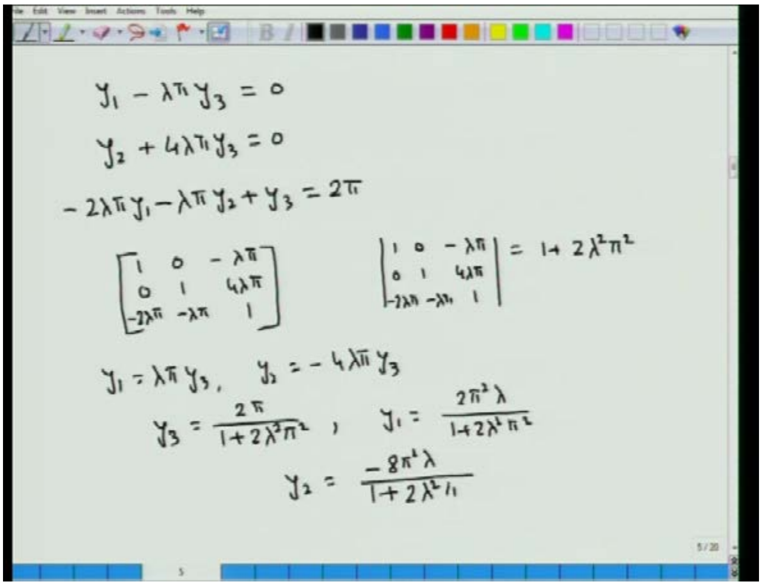

And this will be equal to 2 into 1 plus lambda y 1 integral 0 to phi s sin s d s, because s sin is a even function and from limit is to minus phi to plus phi, then plus 2 lambda y 2 integral 0 to phi sin square s d s and last integral is again equal to 0 because sin s cosine s an odd function and after evaluating this integral, we can find 2 into 1 plus lambda y 1, then limit minus s cosine s plus sin s 0 to phi plus lambda y 2 s minus sin 2 s divided by 2 limit 0 to phi and this will results in 2 phi plus 2 lambda phi y 1 plus lambda phi y 2, this is the result.

So now, if we collect all the equation that we obtain, so from first we have obtain y 1 equal to lambda phi y 3, then y 2 equal to minus 4 lambda phi y 3 and finally, here y 3 equal to 2 phi plus 2 lambda phi y 1 plus lambda phi y 2, so finally we can write down the system of equation as y 1 minus lambda phi y 3 this is equal to 0, then y 2 plus 4 lambda phi y 3 this is equal to 0 and minus 2 lambda phi y 1 minus lambda phi y 2 plus y 3, this is equal to 2 phi.

So, therefore, the co-efficient matrix is consisting of this coefficients, that is 1, 0, minus lambda phi, then 0, 1, 4 lambda phi and minus 2 lambda phi minus lambda phi 1. So, therefore, we have unique solution if this determinant is non-zero; now, if we calculate this determinant, that is determinant 1 0 minus lambda phi, 0, 1, 4 lambda phi minus 2 lambda phi, then minus lambda phi 1, this determinant will be equal to 1 plus 2 lambda square phi square.

So, just for example, if we assume lambda is real; so, therefore, we can say this quantity is always not equal to 0 and therefore, for all values of lambda, this system of equation is uniquely solvable. Now, without going towards the matrix inversion method or Cramer's rule to solve this system of linear equation, directly from this equation you can find y 1, that is equal to lambda phi y 3 and y 2 this is equal to minus 4 lambda phi y 3 and substituting this two expressions in the third equation, we can then solve for y 3 and solving we get, y 3 equal to 2 phi divided by 1 plus 2 lambda square phi square.

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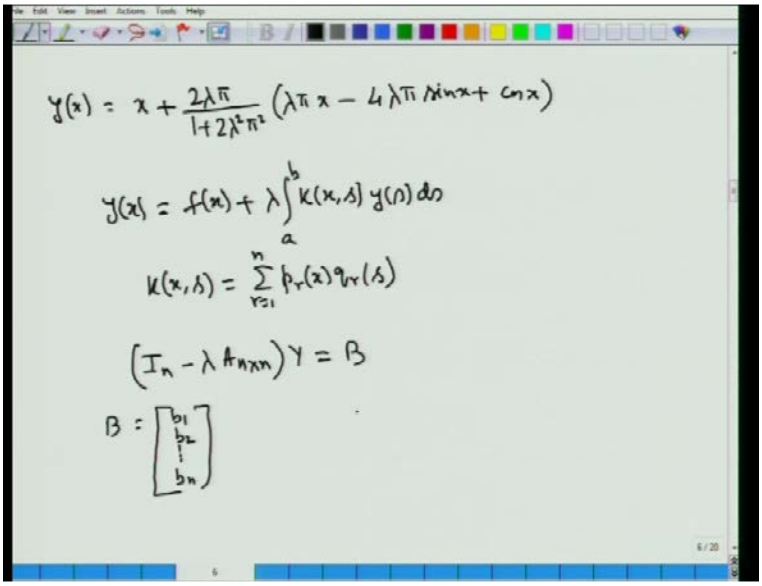

So, with this y 3, immediately we can find y 1, this is equal 2 phi square lambda divided by 1 plus 2 lambda square phi square and y 2 this is equal to minus 8 phi square lambda divided by 1 plus 2 lambda square phi square and therefore, with these expression for y 1 y 2 and y 3, we can find the solution for the given problem that is, y x is equal to x plus 2 lambda phi divided by 1 plus 2 lambda square phi square, this multiplied with lambda phi x minus 4 lambda phi sin x plus cosine x, so this is the solution. That means, we actually, we are substituting these expression for y 1, y 2, y 3 into the first expression, where we have obtain y x is equal to x plus lambda x y 1 plus lambda sin x y 2 plus lambda cosine x y 3, so substituting this values, we can get the general solution.

Now we discuss, different possible situation for the solution of the system linear equation what we have derive in the last lecture, and through which we can understand what is the concept of Eigen values, Eigen functions associated with a linear integral equation that is Fredholm integral equations. So, for that purpose we recall that, the given integral equation y x is equal to f x plus lambda integral a to b k x comma s y s d s, where k x comma s is the degenerate or separable kernel, which is of the form sigma r running from 1 to n p r x q r s can be converted into a system of linear equations which is given by i n minus lambda a n cross n these multiplied with y, that is equal to capital B. So that means, existence of solution of the system of the linear equation, where y is the unknown, column vector with n components y 1, y 2, y 3 up to y n; we can discuss the different possible situations, for the solution of the Fredholm integral equation.

So first of all, you can recall this b was a matrix of the form, some constant B 1, B 2 up to B n and this B 1, B 2 up to B n these are nothing but, the integral a to b f x q r x d x, so therefore, if we assume the first case if we assume that, f x equal to 0, if we assume f x equal to 0, then all the B j are identically equal to 0 and therefore, these B is nothing but and n cross 1 column vector and once n cross 1 column vector is B and therefore, we will be having the system of linear homogenous equation i n minus lambda a n cross n multiplied by y, that is equal to theta n cross 1 and whenever f x equal to 0.

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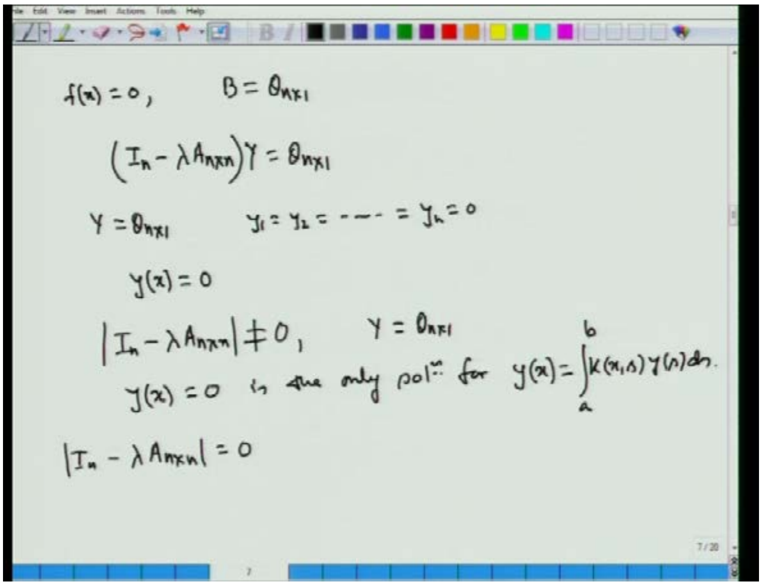

Then, the given Fredholm integral equation of second kind is a homogenous Fredholm integral equation of the second kind and therefore, this system of equation admit solutions y is equal to theta n cross 1, this is the trivial solution and whenever y 1, y 2, up to y n all these quantities are identically equal to 0. So therefore, y x equal to 0 is a solution to the given Fredholm integral equation which is a homogenous equation, because f x equal to 0.

Now, if it happens that I determinant i n minus lambda a n cross n, this is not equal to zero, then unique solution for this homogenous system of linear equation is y equal to theta n cross 1 and therefore, from here you can verify, that y x equal to 0 is the only solution is the only solution for this integral equation y x is equal to integral a to b k x comma s y s d s. So, this is the unique solution and which is identically equal to 0; so, therefore, this homogenous Fredholm integral equation admits only the pervious solution.

Next, we considered case that, determinant i n minus lambda n cross n this is equal to 0, now what is happening here, if this is equal to 0, then at least one of the y j can be assigned some arbitrary nonzero, if this determinant is 0, then we can assign at least one of the y j arbitrary nonzero quantity and then accordingly, remaining y j can be obtain in terms of this y j and this choice of arbitrary values for y j as it is arbitrary. So, therefore, it actually admits infinitely many solutions of the integral equation, so this is very much important point, if this determinant i n minus lambda a n is equal to 0, then we have infinitely many solutions of the integral equations.

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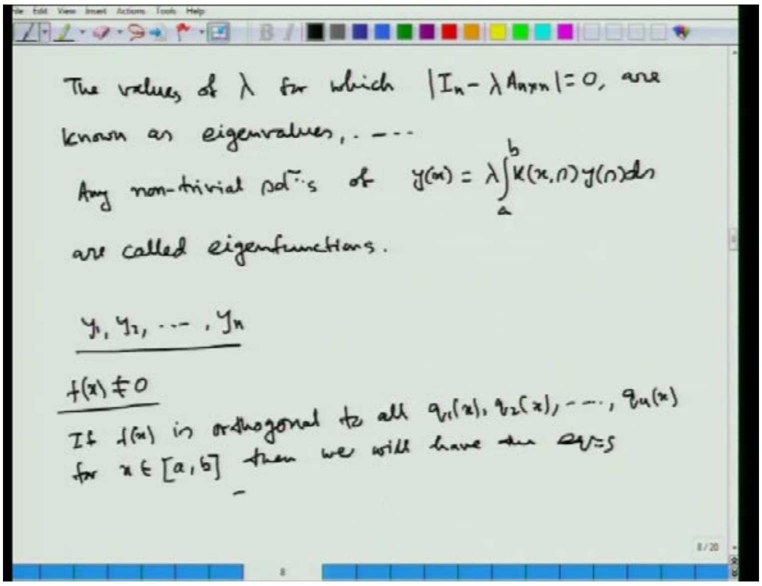

Now, the value of lambda the value of lambda for which this determinant is 0, the value of lambda for which this determinant i n minus lambda a n cross n this is equal to 0, these are known as known as characteristic values or Eigen values, Eigen values of the Fredholm integral equation. And any nontrivial solution any nontrivial solution of the homogenous integral equation associate with the i n lambda, those are actually called the Eigen function; so therefore, any nontrivial solution any non tribunal solution of y x is equal to lambda integral a to b k of x comma s y s d s are called Eigen functions. So, this is the definition or concept associate with the Eigen value and Eigen function of an integral equation.

Now, the point is that if the aim of the constants; that means, among these y 1, y 2 up to y n among these, if m can be assigned to arbitrary values for a given particular value of lambda, then we can find m linearly independent, Eigen functions that can be obtain; so that means, depending upon the how many of this y 1, y 2 up to y n can be assigned as arbitrary constant, then in the solution those p 1 x, p 2 x up to p n x they will be appear and those will be actually Eigen functions corresponding to this arbitrary choice of y 1, y 2 up to y n for which this y j are actually non-zero, of course will explain this idea with help of a particular example. So, up to this we have considered f x equal to 0, next we considered that, f x this is not equal to 0, if f x not equal to 0 still there is possibility, that we can have system of linear homogenous equations corresponding to that, i n minus lambda a n cross n y is equal to b.

And the point is that, if f x is orthogonal is orthogonal to all q 1 x, q 2 x up to q n x, for x belongs to close interval a comma b, then we will have the equation that is, i n minus lambda a n cross n multiplied by y, that is equal to theta n cross 1; so, therefore, as we have discuss earlier, there will be two possibilities, number 1 that determinate of i n minus lambda a n cross n, this is not equal to 0.

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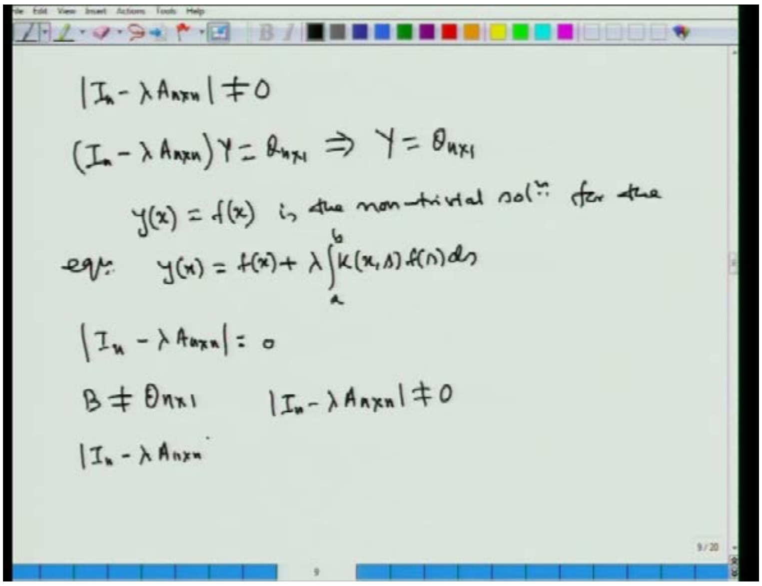

Now, if this is not equal to 0, then the system of linear equation i n minus lambda a n cross n times y, that is equal to theta n cross 1 admits only one solution, that is y equal to theta n cross 1; but in these case, will be having a non trivial solution, that is given by y x is equal to f x, so y x equal to f x is the non trivial solution is the non tribunal solution for the equation y x is equal to f x plus lambda integral a to b k of x comma s f s d s.

And the case when will be having this i n minus lambda a n cross n this determinant equal to 0, then we can find the corresponding Eigen values and therefore, will be having Eigen functions and in that case, solution will be the some of f x and the stiller multiple of Eigen functions of the associated problem. And finally, if we considered that, at least 1 of integral a to b f x q m x is nonzero, so that means, we are assuming B not equal to theta n cross 1 if this happens, then will be having unique solution to the Fredholm integral equation whenever determinant of i n minus lambda a n cross n this is not equal to 0.

So, therefore, for b not equal to theta n cross 1 will be having unique solution to the Fredholm integral equation, if this determinant does not initiates and in case, if this happen that i n minus lambda a n cross n this is equal to 0, then system of equation will be either income system and if the system of the equations are income system, they does not exist any solution are they are redundant; in case of redundant equation will be having infinitely many solutions for the given problem.

So, this is all about discussion regarding nature of solution of the Fredholm integral equation associate with the nature of the determinate i n minus lambda a n cross n and with that the matrix b is a no matrix or it is a non non matrix.

Now, in the last lecture we obtain the resolvant kernel of a integral equation, in these case we are again going to address the same problem and based upon this discussion, we can find out the Eigen values, Eigen functions and we will be able to illustrate all these ideas with help of the same example.

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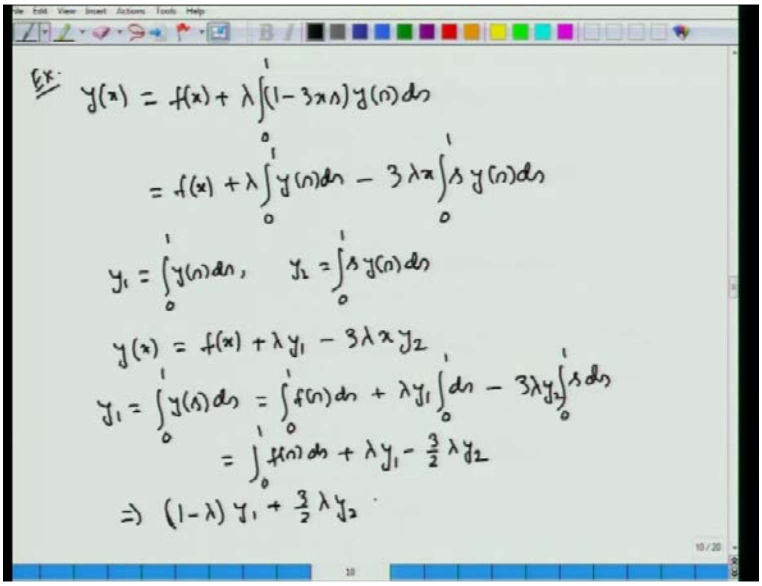

So, the problem is y x y x this is equal to f x plus lambda integral 0 to 1, 1 minus 3 x s y s d s and just for your information you can get this example in the Hildebrand's book on mathematical method or any other book like was was book or carnivals book this very famous example, you can find it in several places on the book of integral equations.

So now, we are going to discuss the solution of the particular problem depending upon various natures of f x that is whether it is 0 or it is non-zero or whenever its orthogonal to the function associated with the given problem, so, first of all we can derived this equation as f x plus lambda integral 0 to 1 y s d s minus 3 lambda x integral 0 to 1 s y s d s and defining y 1 is equal to integral 0 to 1 y s d s and y 2 is equal to integral 0 to 1 s y s d s, we can write y x, this is equal to f x plus lambda y 1 minus 3 lambda x y 2, so this is the expression for y x.

Now, in this expression, that is y a 1 is equal to integral 0 to 1 y s d s substituting this expression for y s, we can find this is equal to integral 0 to 1 f s d s plus lambda y 1 integral 0 to 1 d s minus 3 lambda y 2 integral 0 to 1 s d s, so this will be equal to integral 0 to 1 f s d s plus lambda y 1 minus 3 by 2 lambda y 2 and therefore, from here we have the fast equation of the system of two linear equations, that is 1 minus lambda y 1 plus 3 by 2 lambda y 2 that is equal to integral 0 to 1 f s d s, so this is the first equation.

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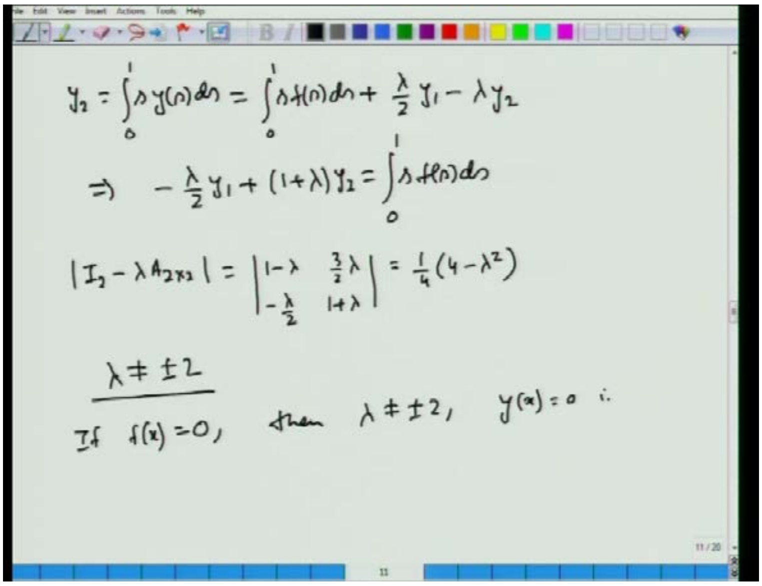

And for the second y 2 is equal to integral 0 to 1 s y s d s and after substituting the expression for y s and evaluating the integral, you can find 0 to 1 s f s d s plus lambda by 2 y 1 minus lambda y 2, and from there we can write, minus lambda by 2 y 1 plus lambda y 2, this is equal to integral 0 to 1 s f s d s and therefore the determinate, that is i 2 minus lambda a 2 cross 2 this is equal to determinate 1 minus lambda 3 by 2 lambda, then minus lambda by 2 and 1 plus lambda, so, this is equal to after simplification will be comes out to be 1 by 4 four minus lambda square.

So in this case we can say that, unique solution exists whenever lambda not equal to plus minus 2, if lambda not equal to plus minus 2, so whether f s equal to 0 or f x not equal to 0, we can find unique solution for the given Fredholm integral equation.

now and the solution can be obtain by substituting the expression for y 1 and y 2, by solving these two equation either in the form involving integral 0 to 1 f s d s and 0 to 1 s f s d s and other wise if f x is given. So, evaluating both this integrals that is 0 to 1 f s d s and 0 to 1 s f s d s, you can find out uniquely y 1 and y 2 and then substituting into required expression, that is y x equal to f x plus lambda y 1 minus 3 lambda x y 2, you can find out the unique solution for the given problem.

Now, we considered the case, that if f x equal to 0 if f x equal to 0 then, for lambda not equal to plus, minus 2 y x equal to 0 is the only solution is the only solution. So, therefore, if f x equal to 0 and lambda not equal to plus minus 2, then given problem admits only trivial solution and now, we can considered the cases that is values of lambda equal to first 2 and then minus 2.

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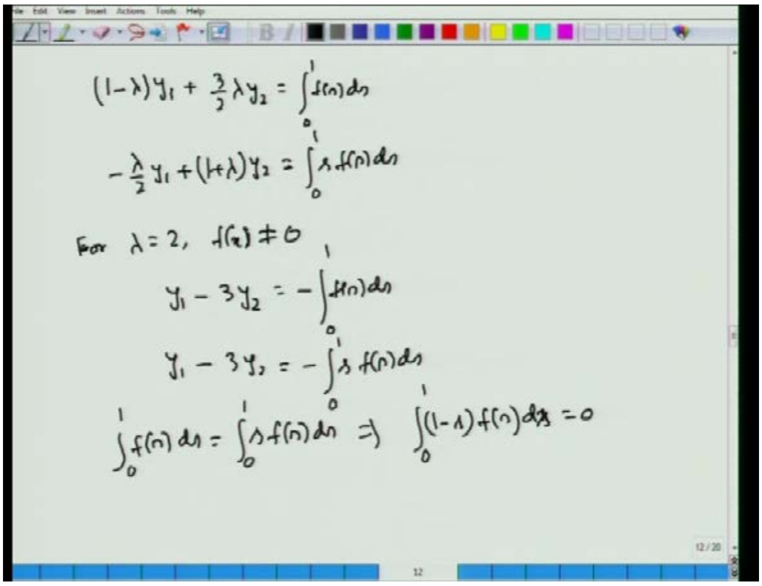

So, we have this system of equation you can recall, that 1 minus lambda y 1 plus 3 by 2 lambda y 2, that is equal to integral 0 to 1 f s d s and minus lambda by 2 y 1 plus 1 lambda y 2 this is equal to integral 0 to 1 is f s d s.

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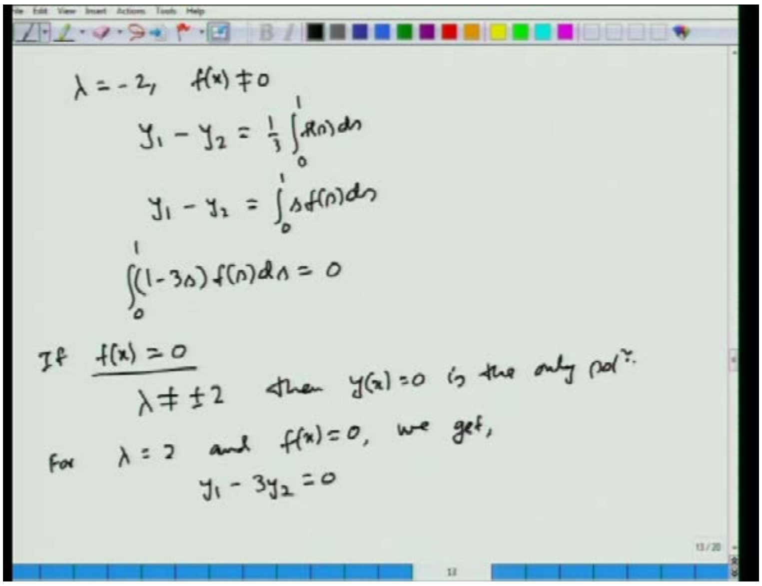

So, here if we assume that, lambda equal to 2 and f s not equal to 0 havoc system of equation reduces to y 1 minus 3 y 2, that is equal to minus integral 0 to 1 f s d s and y 1 minus 3 y 2, that is equal to minus integral 0 to 1 s f s d s, so these are two expressions will be getting when we substituting lambda equal to 0 and with the assumption that f x is not equal to 0.

So, with f x not equal to 0 will be having this expression, now havoc two equations that is y 1 minus 3 y 2 equal to minus integral 0 to 1 f s d s and y 1 minus 3 y 2 equal to minus integral 0 to 1 f s d s these are in comfortable, unless integral 0 to 1 f s d s, this is equal to integral 0 to 1 s f s d s, so actually, that implies integral 0 to 1 1 minus s f s d s this is equal to 0. So, if these condition does not holds, that is integral 0 to 1 1 minus is f s d s not equal to 0, then the system of equation are in comfortable and therefore, they does not exists any solution.

Similarly, for lambda equal to minus 2 and f x not equal to 0 will be having the system of equation reduce to y 1 minus y 2 that is equal to 1 third integral 0 to 1 f s d s and from the second y 1 minus y 2 is equal to integral 0 to 1 s f s d s and in this case the system of equation are in comfortable unless integral 0 to 1 1 minus 3 s f s d s, this is equal to 0 and these condition holds that is 0 to 1 1 minus 3 s f s d s, then the system of equation actually redundant, because in these case both this equation can be reduce to a single equation.

So now, the case when f x is equal to 0 if f x equal to 0 and lambda not equal to plus minus 2, then y x equal to 0 is the only solution, that we have discuss earlier also and next we can try to calculate the Eigen functions associate with the Eigen values, plus minus 2; whenever this f x equal to 0, now for lambda equal to 2 and f x equal to 0 both this equation can be reduce to y 1 minus 3 y 2 this is equal to 0.

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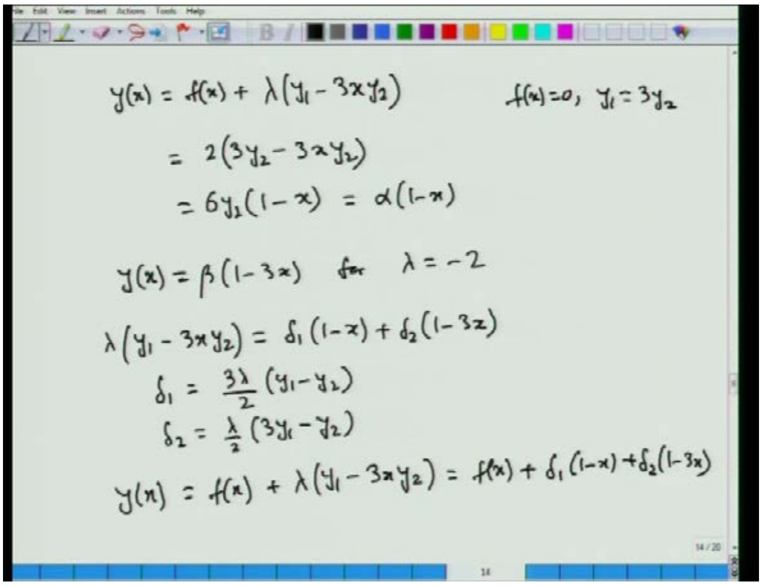

And then from the expression, that is y x is equal to f x plus lambda y 1 minus 3 x y 2 with the substitution, that is f x equal to 0 and y 1 equal to 3 y 2; we can find this is equal to 2 times 3 y 2 minus 3 x y 2, so this is equal to 6 y 2 times 1 minus x, now we can choice y 2 arbitrary quantity and accordingly y 1 will also be the arbitrary quantity and therefore 6 y 2.

Can be written as alpha times 1 minus x, so therefore, this 1 minus x actually Eigen function associate with Eigen value two of the homogenous Fredholm integral equation and proceeding in a similar manner. We can find that, y x is equal to beta times 1 minus 3 x for lambda equal to minus 2, where beta is any arbitrary non-zero quantity and therefore, we can say that 1 minus x and 1 minus 3 x, these are Eigen function corresponding to Eigen values; lambda equal to 2 and lambda equal to minus 2 respectively.

And finally you can check that, this lambda times y 1 minus 3 x y 2, these can be written in terms of linear combination of these two functions, that is 1 minus x and 1 minus 3 x, because this can be done, that this is equal to delta 1 times 1 minus x plus delta 2 times 1 minus 3 x if we choose delta 1 equal to 3 lambda by 2 times y 1 minus y 2 and delta 2, this is equal to lambda by 2 3 y 1 minus y 2. So, from here you can see that, always this lambda times y 1 minus 3 x y 2 can be written as delta minus 1 minus x and delta 2 times 1 minus 3 x.

So this 1 minus x and 1 minus 3 x; they are Eigen functions of the associated problem and therefore the solution, that is y x this is equal to f x plus lambda into y 1 minus 3 x y 2 is nothing, but the some of this given function f x and linear combination of the Eigen functions, that is delta 1 into 1 minus x plus delta 2 into 1 minus 3 x. So that, we have claim in case during the theoretical discussion on this problem that, whenever this lambda is taking the value plus minus 2 and f x is non-zero.

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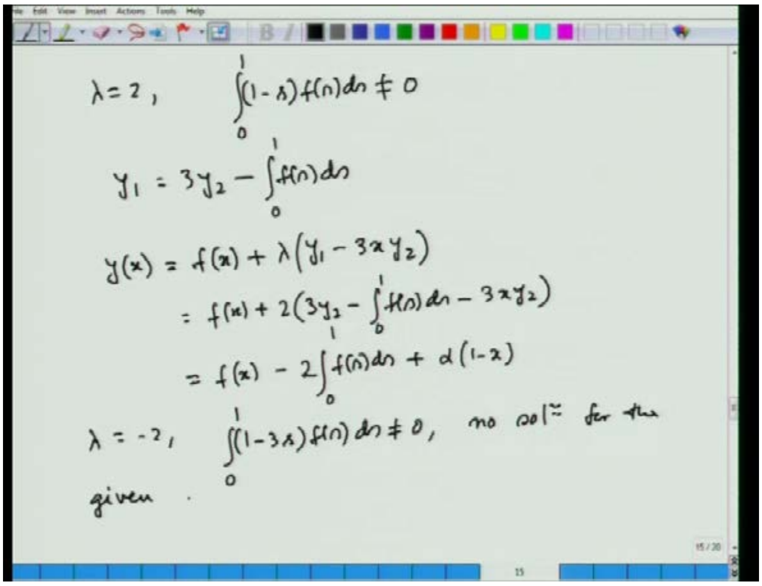

so therefore, the solution can be written as the linear combination of the Eigen functions for some with the given functions f x, for lambda equal to 2 no solution exists if 0 to 1 1 minus s f s d s, this is not equal to 0; if these quantity is not equal to 0, that is the integral 1 minus s f s d s not equal to 0, then they does not exists any solution and in case f s is orthogonal to 1 minus s within the range 0 to 1.

So that means, integral 0 to 1 1 minus s f s d s this is equal to 0 in that case, we can write y 1 is equal to 3 y 2 minus integral 0 to 1 f s d s, this is the result will come up for y 1

And in that case, we can write down the solution of the given problem, that is y x equal to f x plus lambda y 1 minus 3 x y 2 and substituting the expression for this y 1, we can write f x plus 2 into 3 y 2 minus integral 0 to 1 f s d s minus 3 x y 2 and ultimately, this is coming out the f x minus 2 integral 0 to 1 f s d s plus alpha times 1 minus x, because 6 y 2 minus 6 x y 2 can be clubbed into 6 y 2 times 1 minus x and 6 y 2, this y 2 is an arbitrary quantity in this case, for lambda equal to 2 then 6 y 2 can be written as arbitrary quantity alpha and hence, the solution is given by y x is equal to f x minus 2 integral 0 to 1 f s d s plus alpha times 1 minus x.

And in that case in this case you can easily verify that any non-zero values of alpha you can get an infinite set of solution for the given problem. For example, substituting alpha equal to 1 2 3 4 and so on, in all the cases you will be having y x equal to f x minus 2 integral 0 to 1 f s d s plus this either 1 into 1 minus x or 2 into 1 minus x these are the solution of the given (()) equation, so in this case, we will be having infinite number of solutions and for lambda equal to minus 2.

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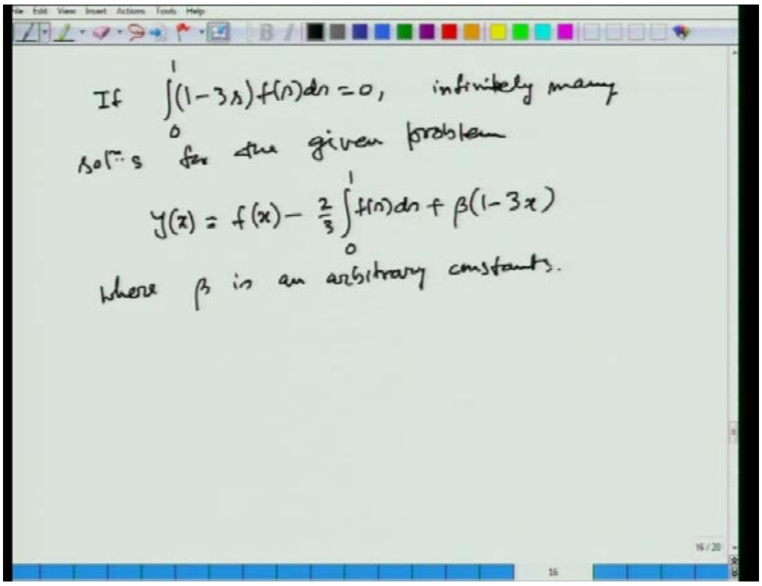

And if these condition is satisfied that is 0 to 1 1 minus three s f s d s this is not equal to 0 then no solution for the given equation, then they does not exists any solution for the given equation and for lambda equal to minus 2, we can again find the solution for the given problem in a similar fashion; that means, if we find that integral 0 to 1 1 minus 3 s this f s d s this is equal to 0, then we can find infinitely infinitely many solutions for the given problem.

And in this case the solution can be obtain as y x is equal to f x minus 2 third integral 0 to 1 f s d s plus beta times 1 minus 3 x, where beta is an arbitrary constants, so this clearly shows under different possibilities, that is whenever f x is identically equal to 0, then will be having system of linear homogenous equation and depending upon values of lambda, we can have either unique solution or no solution and in case f x equal to 0 and that determinate, that is i n minus lambda a n cross n not equal to 0, we will be having the trivial solution that is y x equal to 0 is the only solution.

And in case that, determinate is non-zero, then we will having infinitely many solutions and another point, that we have keep in mind that only f x equal to 0 will not be corresponding to the system of linear homogenous equations that is i n minus lambda a n cross n y equal to theta n cross 1, in case the non homogenous part of the given Fredholm integral equation f x satisfies the condition that it is orthogonal to each of the functions q 1 x q 2 x up to q n x, within the interval a comma b.

Then still will be having system of linear homogenous equation and depending upon the nature of the determinate, whether this is equal to 0 or this is nonzero will be having the case of unique solution or no solution and with these examples we have illustrated, that in some situations when Eigen values and Eigen functions exists for the given problem, then solution to the given Fredholm integral equation will be the some of f x plus linear combination of the Eigen functions and all these items can will be address in terms of Fredholm theory. Where this Fredholm theory relates with the solution of a Fredholm integral equation with the resolve kernel from the for the Fredholm integral equation and there we can discuss the same thing in terms of the three Fredholm theory. So, in this lecture I can stop at here next two lectures will be on three theorem of Fredholm and halberd smith theory, those are related with the Fredholm integral equation, thank you for your attention.