Calculus of Variations and Integral Equation

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Lecture No. # 18

Welcome viewers to the NPTEL lecture series on the calculus of variations. This is the 18th lecture of the series. We recall that in the last lecture, we started with sufficient conditions on the integrant of the functional, which we have been considering. And we obtained Jacobi's equation, which ensures that we have central field at the point a of the extremal, and that extremal which we are testing whether it gives us the optimal value of the functional, we would like to see that the point b is within the central field. That means around d the extremals do not intersect. So, that is the condition, which is known as Jacobi's condition.

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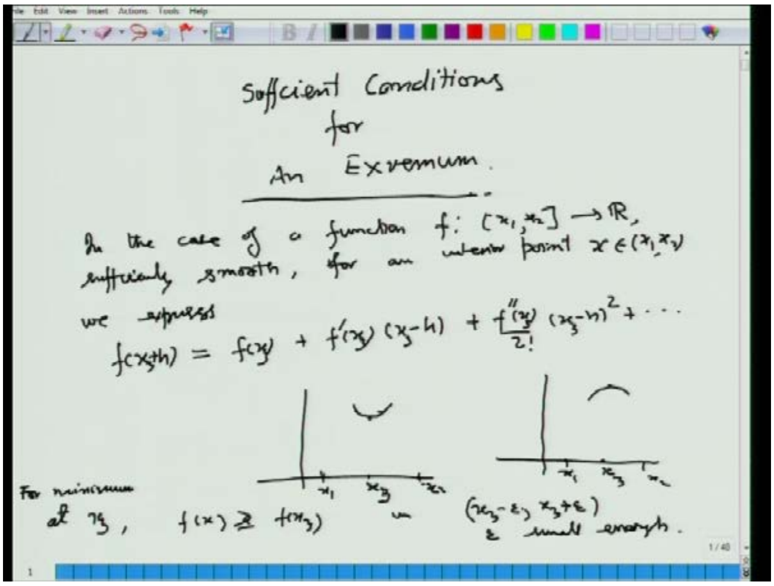

So, we started in the last lecture with the brief introduction to the notion of minima-maxima of a function, where we consider, the case where the function f is defined on the interval x 1 to x 2 into R, sufficiently smooth, so that whatever derivatives we are considering on f are continuous. So, we can expand this f (x) in any at any interior point x in the interval x 1 to x 2. So, we considered this point x 3 here. So, supposing we expand this around x 3 like this. So, all the derivatives will be evaluated at x 3. So, f of x 3 plus h would can be written as f of x 3 plus f prime x 3 into x 3 minus h and f double prime at x 3 upon factor 2 x 3 minus h square and so on.

So, here if we have minimum at this point x 3, which is an interior point of the interval then we can see that around this point, this is the like you have cup kind of a figure here. And similarly, if x at x 3 which is an interior point of the interval x 1, x 2, if at that point, if the local maximum is there, then we have this inverted cup kind of a thing, shape of the curve around this x 3. So, we can see that in the neighboring point here, f in the case of minimum at x 3, f at x 3 plus h, where h is small, may be positive or negative. If it is positive, we are on the right of x 3. If h is negative, we are in of left of the x 3. And so, those values should have been more than the value at f of x 3. So, here we have f of x 3 greater than equal to f of x must be greater than equal to f of x 3, for any x in small neighborhood around this x; that is x 3 minus epsilon to x 3 plus epsilon.

Similarly, here f of x around this x 3 must be the more than, f of x must be a less than the value at f of x 3, which should be the maximum in the neighborhood around this endpoint x 3 that is x 3 minus epsilon to x 3 plus epsilon, in case of maximum at x 3.

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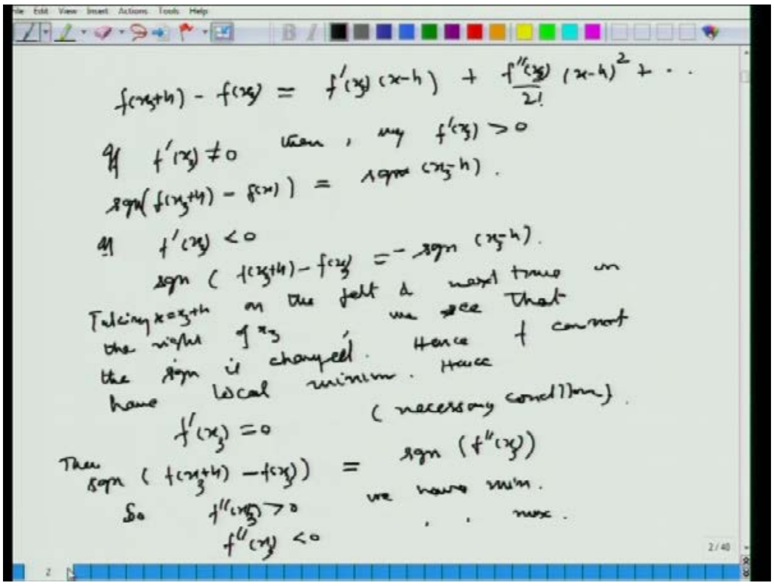

So, if you look at the expansion f of x 3 plus h minus f of x 3, which is a prime at x 3, x 3 minus h plus f double prime at x 3 upon factor 2 x 3 minus h whole square and so on.

Then, we see that this left hand side should have same sign in the neighborhood of x 3. So, therefore, if this prime x 3 is not 0, we see that this term, for a small h, this is the dominating term and the sign of the left hand side will be actually sign of the first term on the right hand side. And therefore, if prime at x 3 is not 0, it changes sign, and therefore, which then the condition of minimality or maximality at x 3 is violated.

So, the necessary condition is that a prime attacks x 3 must be 0, and then we see this sign of this left hand side equal to the sign of the next term, non-zero term, if a double prime at x 3 is not 0, then we see that since x 3 minus h square is coming, which is always non negative here. So, the sign of left hand side will be determined by a sign of f double prime at x 3 for small values of h. And so we see that the testing of the sufficient conditions for the minima or maxima at x 3 are given by that f double derivative a prime f double prime at x 3, if it is positive, then we see that we get minimum at x 3, if f double prime at x 3 is negative, then we get maximum.

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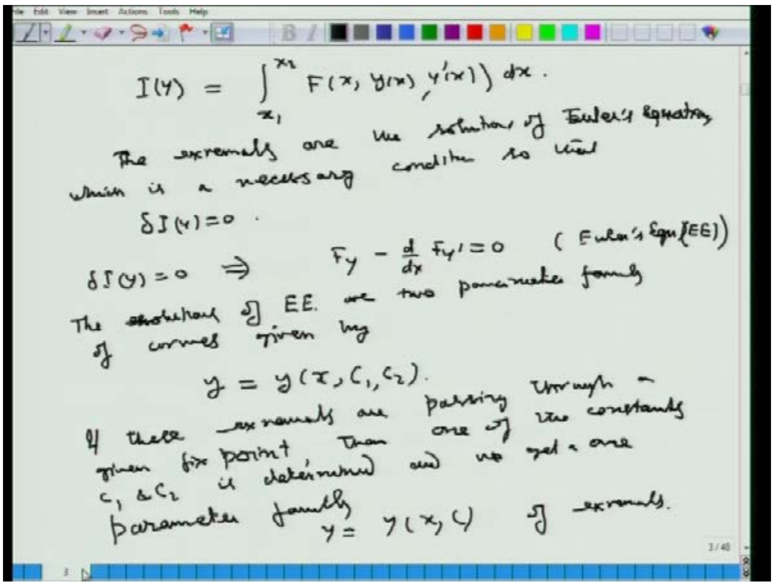

So, the similar concept we would be actually establishing for the functional. So, here a prime at x 3, it being 0 is equivalent to in the case of functional, that the first variation that is delta small delta I is 0, which we had got earlier and this condition implies that F y minus d by d x F y prime equal to 0 that is Euler's equation is the necessary condition here, and so solving that equation, because the second order equation second order ordinal differential equation in y, so it will have solution involving two arbitrary constants. And if we see that the point is fixed, then we can determine one of these two constants, and then, we get to one parameters of family of the extremals.

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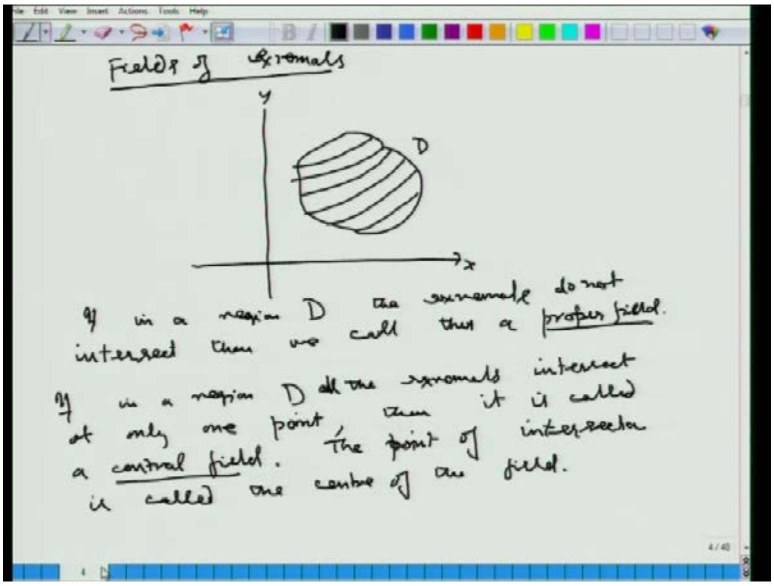

And then we defined the concept of fields, various fields. Here, if the extremals in region D do not intersect at all, such a field is called proper field. And if they intersect at one point and nowhere else, then it is called central field, and the point where all these extremals intersect is called the centre of the field.

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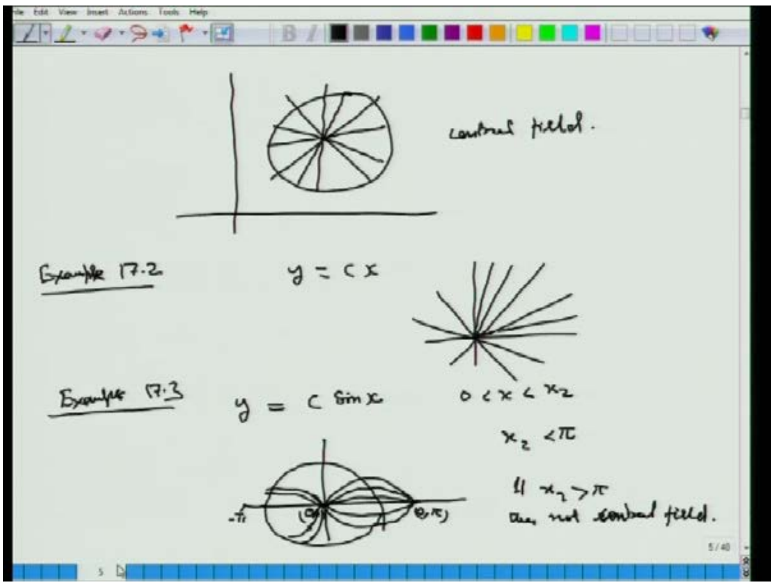

And we had consider these examples like, y equal to C sin x, here at the origin that all the extremals here, all the curves intersect here, and then next they intersect like 0 pi on the right or 0 minus pi on the left. And so, if we have this small neighborhood around 0, then in that it is a central field provided pi or minus pi are not there, 0 pi or 0 minus pi are not there in the domain. So, it forms a central field in the small neighborhood around 0, the origin.

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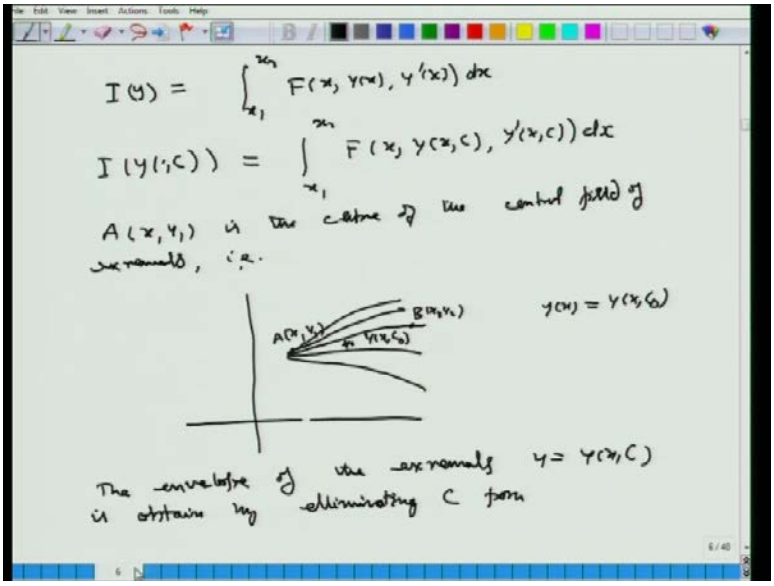

Now, we considered the functional I (y), which is x 1 to x 2 F of x, y, y prime d x, and we consider, it is values only on extremals like we solved this equation here. In this case, in the function case, we solve this f prime at x 3 is equal to 0, so all those x 3, we satisfy this are called critical point. So, here like that they are called extremals, which are satisfying Euler's equation. And so, we have taken that the point A is fixed. So, all these extremals are passing through this point. And we are looking for that extremal here for on which this functional has optimal value, whether minimum or maximum, will we will be the getting certain conditions, sufficient conditions to determined, whether I (y) is having maximum value or minimum value.

So, we considered here this family and we want to see that this given extremal whether this, which is y (x) equal to y (x, c 0). So, c 0 is a fixed constant parameter value of the parameter c. So that this A the extremal passing through A and B, whether it optimizes or not that is what we would like to see.

So, here first thing is that like that point x 3 was an interior point. Similarly, here this extremal should be an interior extremal that on the left side or right side or above or below of this functional this extremal there should be other extremals as well. So, this extremal should not be the boundary of the region. So that is one condition, which is required.

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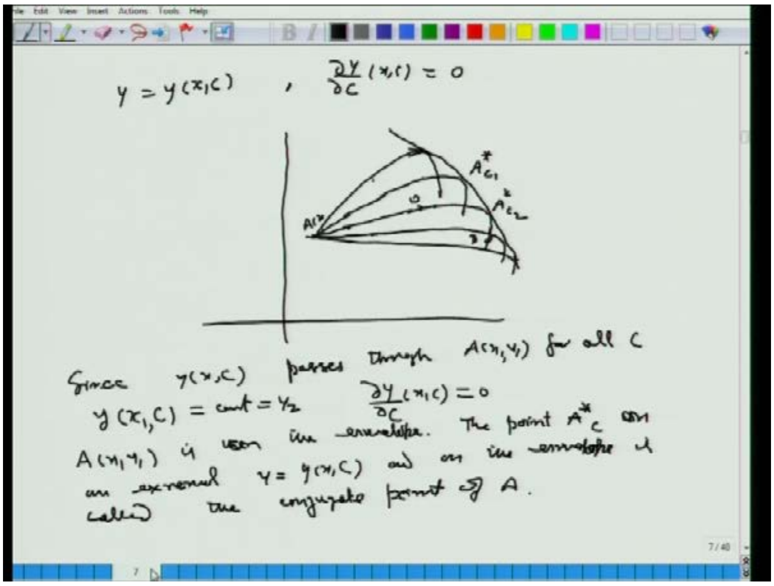

And then, we solve this y equal to y (x, c) and del y over del c (x, c) to get this c; it is called c discriminant curve. So, we find this c discriminant curve. So, that includes this point A (x 1, y 1), because here we have seen that at x 1, this y of x1, because it is on the extremal. So, it satisfies and del y over del c, because all the extremals are passing through that. So, it is a fix number y 2 and therefore, del y over del c (x 1, c) will be 0.

So, A is there on that c discriminant curve and the envelope of this family of y (x) envelope of the family of these extremals is also c discriminant curve, and the picture is like this, here we take various extremals. And the points, which are there on the extremal like one A is there, and then another one, which is there on the extremal as well as on the envelope that we denoted the conjugate point of A. It is denoted as A star c one like c is taking various values. So, we get different extremals. So, correspondingly we get a point on the envelope. And so that is called, so A star c is called the conjugate of A.

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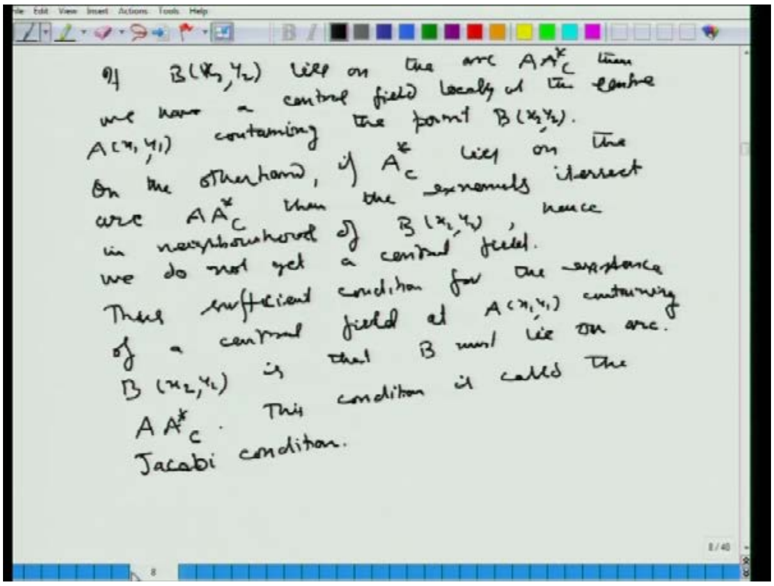

And so the condition is, sufficient condition is that which we had seen that the point B should be this point B should be there on the arc on the part of the extremal passing through A and A star c. So, that this point B should be their prior to this point A star c, and if it is after that A star c, then we see that around A star c various or around that in the neighborhood of B, there are various extremals intersect each other.

And so, the condition that it is a central field is violated that means, the Jacobi condition is not satisfied there. So, we have to see that the Jacobi condition is fulfilled in order to retain the nature of central field.

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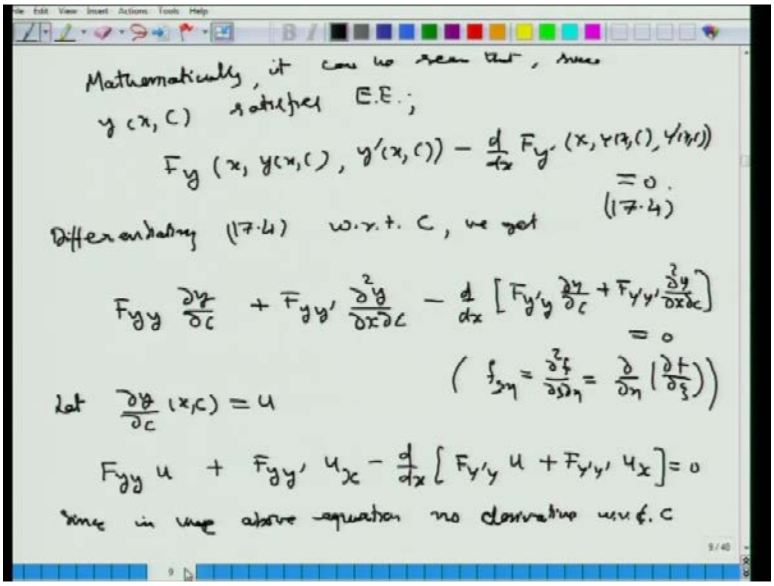

So, that is what is now mathematically derived like this, we differentiate the we differentiate Euler's equation partially with respect to c.

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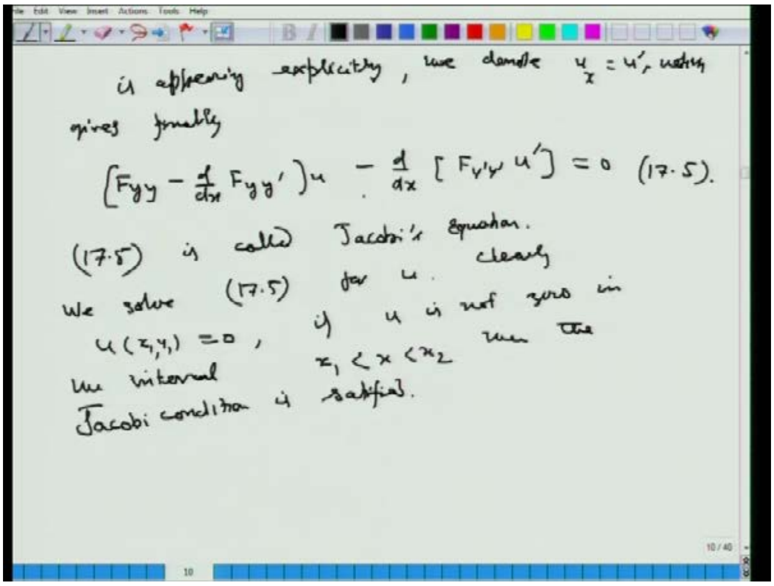

And see that we get here, finally, this equation that F y y minus d by d x F y y prime times u minus d by d x F y prime y prime u prime equal to 0. So, this equation 17.5 which was derived last time is known as Jacobi's equation.

And so, we solve this and get u and we know that it use equal to 0 at A, and then we see that, whether u vanishes anywhere on the interval x 1 to x 2. So, if u is not 0 on this open interval x 1 to x 2, then we see that Jacobi's condition is fulfilled on that interval. Here we will consider various examples, on this concept.

So, let us take this example.

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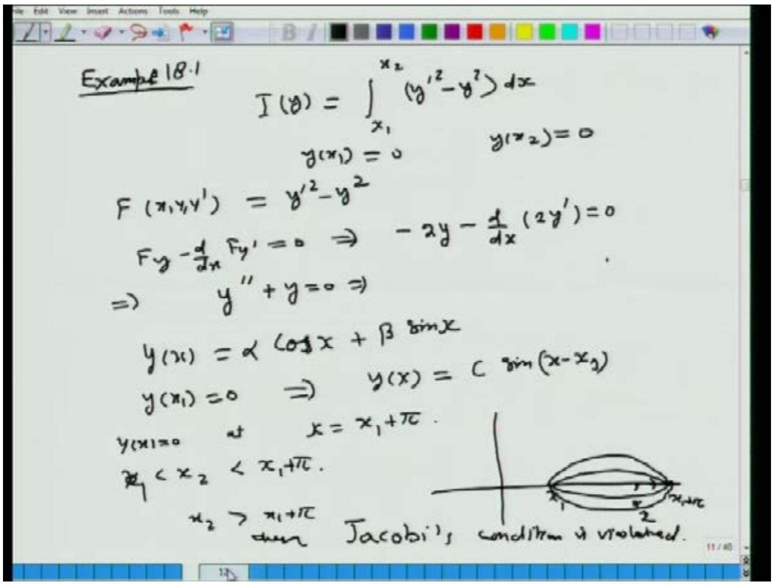

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So, here this functional I (y) is given as integral over x 1 to x 2 y prime square minus y square d x, and y at x 1 is 0, and y at x 2 is also 0.

So, we see that here F is x, y, y prime is y prime square minus y square, and Euler's equation F y minus d by d x of F y prime equal to 0 implies that minus 2 y minus d by d x of 2 y prime equal to 0 or this implies that y double prime plus y equal to 0. And so, we get y (x) equal to A cos x plus B sin x, thus not use A here. So, we get y (x) equal to alpha cos x plus beta sin x. So, here…

Now, this condition that y at x 1 is 0 implies that one of the constants will be determined. So, y (x) will be like some constant C sin of x minus x 1. So, that is what we see that this solution of this will be one parameter family of curves given by this, and here we see that, and then this y (x) will also vanish at… So, next y (x) will be 0 at x equal to x 1 plus pi.

So, this x 2, if 0 less than x 2 is for other x 1 lesser than x 2 is x 1 plus pi, we see that we have a central field, like this we had seen earlier. Here x 1 is there, and then we have this x 1 plus pi here, and these curves are going like this, and if x 2 is before here, then we see that it forms a central field, and if x 2 is greater than x 1 plus pi, then Jacobi's condition Jacobi's condition is violated.

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So, this also can be seen from here, we form…

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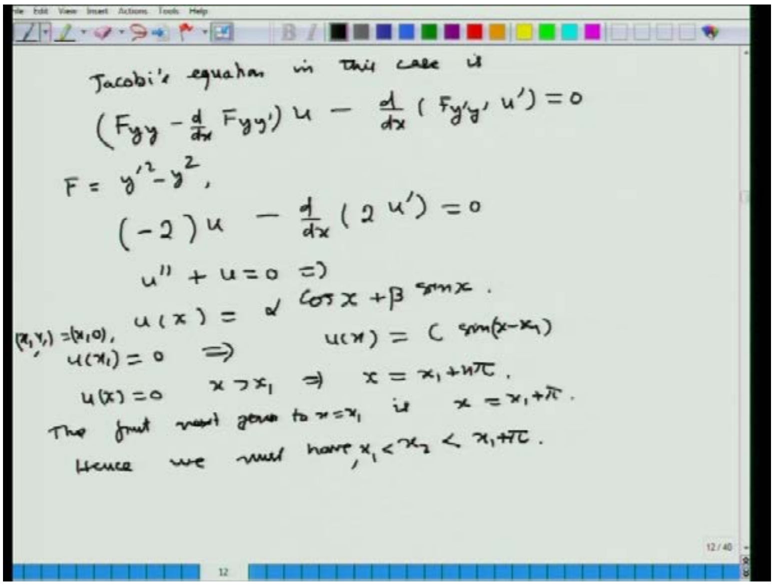

So, let us look at the Jacobi’s equation in this case is…

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So, the Jacobi’s equation is F y y minus d by d x of F y y prime times u minus d by d x of F y prime y prime u prime equal to 0. So, here F, in this case, we have F is y prime square minus y square. So, F y y gives you, minus 2 and this F y prime y will be 0. And so, minus 2 times u and minus d by d x of F y prime y prime it is again 2 here, and u prime. So, this what we get here? It is gives us u double prime plus u equal to 0. And we get, this implies that u (x) is again like alpha sin x plus beta alpha cos x and you can writing anyway, alpha cos x plus beta sin x.

And here, we know that u at x 1 must be 0, because the condition is that here, all the extremals pass through that A, which is (x 1, y 1) and (x 1, y 1) is here. So (x 1, y 1) is actually (x 1, 0) here. And so, we get u (x 1) equal to 0 implies that u (x) as before it should be some constant times sin of x minus x 1.

So, in the same manner here, u vanishes at... So, u then x equal to 0 for x greater than x 1 implies that x must be equal to x 1 plus and pi. And so we have the first 0 will be the first the first next zero to x equal to x 1 is x equal to x 1 plus pi. So, we get… then hence, we must have x 2, which is of course, greater than x 1; it should be less than x 1 plus pi. So, our interval should be such that it should not contain x 1 plus pi, this interval x 1 to x 2 should not contain the point x 1 plus pi, if it contains, if x 2 is larger than this, then obviously, Jacobi's condition is violated.

Now, the next example…

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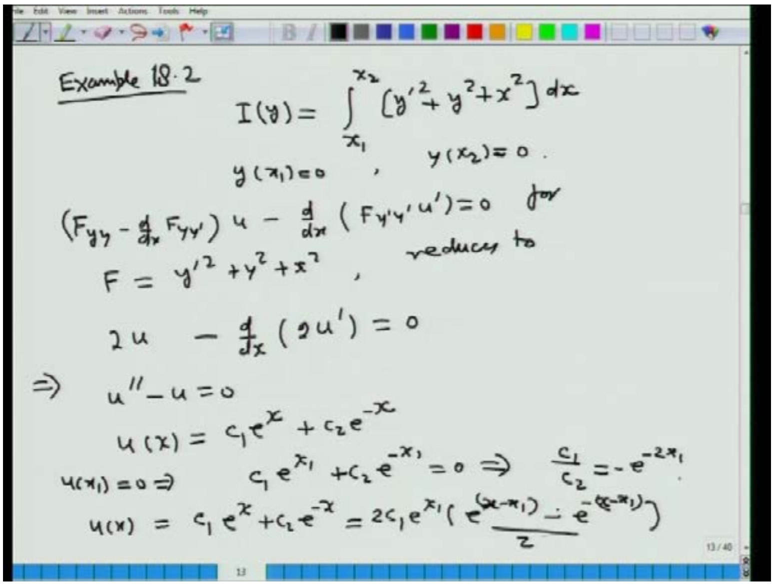

We have this functional I (y) equal to x 1 x 2 integral y prime square plus y square plus x square d x, here y at x 1 equal to 0, again we take y at x 2 equal to 0. So, the points A and B are on the x axis. And here, so the Jacobi's equation so F y y minus d by d x of F y y prime u minus d by d x of F y prime y prime u prime equal to 0, for F equal to y prime is square plus y square plus x square reduces to… This gives us 2 y sorry 2 and this 0. And you get… So, 2 u here minus d by d x, so, I can remove this bracket, 2 u minus d by d x of this again 2 u prime equal to 0.

So, we get here u double prime minus u equal to 0, which has solution u (x) equal to… we can write in hyperbolic form or you can write it like this that u (x) is c 1 e to the power of x plus c 2 e to power minus x. So, u at x 1 is 0 implies that c 1 e to the power x 1 plus c 2 e to the power minus x 1 equal to 0 is implies that c 1 over c 2 equal to minus e to the power minus 2 x 1.

And so, you get u (x) equal to, which is c 1 e to the power x plus c 2 e to the power minus x becomes c 1 e to the power e to the power x minus x 1, and with minus sign here, and minus e to the power minus x minus x 1. And so u multiply by 2 and divide by 2 here, this is plus sign. So, we can take minus of also out and adjust it in this.

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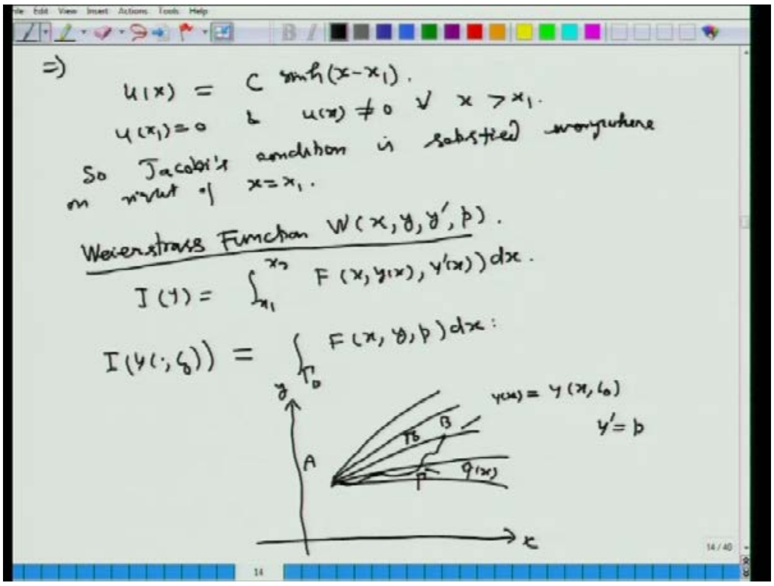

So, we get u (x) equal to some constant C sin hyperbolic x minus x 1. So, that is the solution of this Jacobi‘s equation and it vanishes. So, u (x) of course, u (x 1) equal to 0, and u (x) is not equal to 0 for all x greater than x 1. So, Jacobi’s conditions is verified, is full filled.

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Is satisfied everywhere, on the right of x equal to x 1. So, here it forms a central field everywhere.

In the next, what we would like to see that, once the Jacobi's condition is satisfied, you like to get some other conditions, sufficient condition, which will ensure us the the sufficient conditions. So, that the functional will have minimum or maximum, we will be able to conclude using that sufficient condition.

So, we construct a function, which is called Weierstrass Function, Weierstrass Function which is function of the variables x, y, y prime, and p, will explain what is the speed in a moment. So, we have here I (y), which is x 1 to x 2 F of x, y (x), y prime (x) d x. Now here, we are taking it on the extremals only that I (y, c) here. So, if we take this extremals, which is what we are testing that is y (x). So, here we have situation already explain like that that this is a point A (x 1, y 1), and then we have various extremals like this, and then this y (x) here is actually y (x, c 0).

So, here this less… Makes it slightly bigger here.

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So, this is point A and this point B and this is the curve y (x), which is y (x, c 0). So, fixed value of the parameter c, we have this A, B the the extremals passing through these two points, denoted by y (x), and we want to test, whether this extremal actually, extremizes gives you the minimum value of I or the maximum value of I.

So, that is what we would like to see, we would like to get sufficient condition. So that, we can using that sufficient condition, we can ensure that the functional I (y) takes the maximum value or minimum value for this function, which is given by y (x) with this extremals given by y (x), which is y (x, c 0). So, this on this this y prime, we will denote as p to distinguish between any curve, joining these two points.

Let say, this is the another, this is some curve y tilde x, it is not an extremals; it is any curve, permissible curve joining these two points A and B. So, like that we will have… So, we will denote this by gamma 0, and this general curve gamma like this. So, this extremals joining this, these points A and B, so from A to B will denote it as gamma 0. And any other admissible curve joining these two points A and B, it need not be an extremal; we will denote it by gamma. So, we will have this I (y, x 0) on gamma 0, which is F x, y, and p dx.

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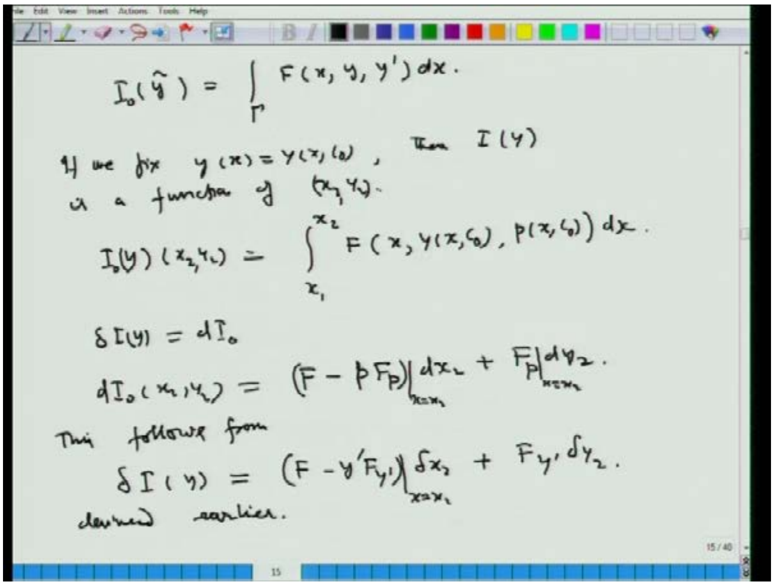

And on any other curve, this I (y) tilde, which is then on gamma F of x, y, y prime d x, now, recall that when we considered this also on y, if we fix this y (x), which is y (x, c 0), then then this functional I (y) is function now of this point (x 2, y 2), because now y is fixed and only the point (x 2, y 2) will changing. So, this is a function like this. So, this I let us say, this is y, and then it is like this (x 2, y 2) which is here you have x 1, which is (x 1, y 1) will fixed and only (x 2, y 2) will changing. So, that is F of x, y (x), c 0, and that is your p on y prime, we are denoting this has p x, which is something as p (x, c 0) d x.

And we have seen that on this the variation delta, since here is a function now. So, this delta I (y) will be denoted as d I. So, here to distinguish it from the other functional, we will write it as I 0. So, this d I 0 and d I 0 we know that, it will be F minus y prime F y prime rather y prime is p here, F p and delta. So, delta x 2 will be written as d x 2.

So, this is the point. So x 1, So this is the function of (x 2, y 2) that is F minus p F p of d x 2 plus F p d y 2. This we had derived earlier, this follows from this follows from our from the earlier one, which we have got the delta I (y) was F minus y prime F y prime delta x 2 plus F, this was evaluated at x 2, and then F y prime delta y 2, derived earlier.

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Here, this since only the point B is moving and therefore, this variation, this functional becomes now function, which we denote by I 0. And so, we have this variation becomes then as usual when it is a function, we denote it by the differential. And so we get differential d I 0 at (x 2, y 2) F minus this of course, is evaluated at this x equal to x 2, this also at x equal to x 2.

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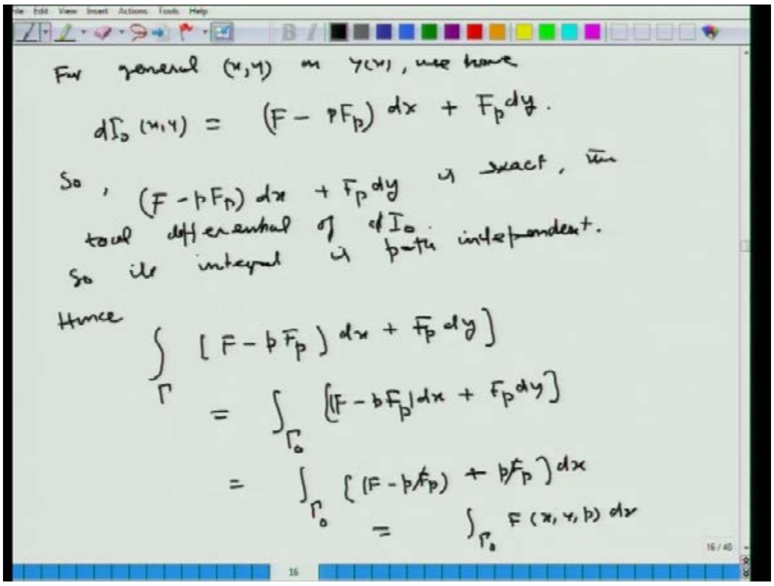

Now here, for general (x, y) so for general (x, y) on y (x), we have this differential d I 0 at (x, y) will be F minus p F p evaluated x at (x, y) general (x, y) d x plus F p d y. So, we see that this is a total differential the right. So, this F minus p F p d x plus F p d y is exact differential, the total differential of d I 0. And so it is integral is path independent.

Thus, hence this integral over any curve gamma of this F minus p F p d x plus F p d y is same thing as integral over gamma 0, because gamma and gamma 0 at joining those two points A and B. So, it will be same thing as F minus p F p d x plus F p d y, but on this we see that on gamma 0, it is F minus p F p d x. So, we take on that curve gamma 0, we see that this we get d x taking out, so plus d y by d x here is p, and so, this is p F p dx, and so this gives us… So, this cancels here. And so, we get on gamma 0 is F d x, that is F x y p d x.

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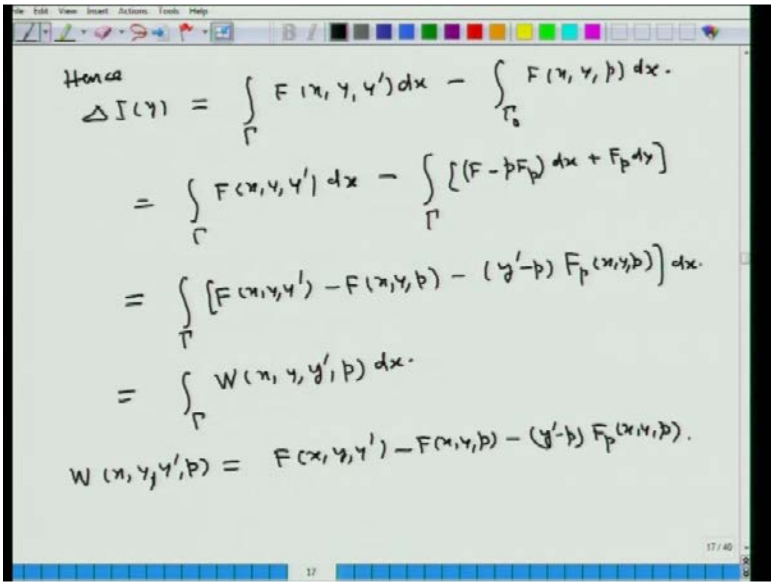

So, therefore, hence this increment delta I (y) will be then over this curve gamma F of x, y, y prime d x minus over gamma 0 F x, y here, because it is p d x. And so, this is gamma F x, y, y prime d x, and this is on the… This gamma 0 can then be written as an gamma of this F minus p F p d x, and then plus F p d y like this.

And so, this will be then here, it can be written as like this, F of x, y, y prime minus this is F x, y, p and minus y prime minus p F p gave, we get this functional d x. So, here on the gamma this is F y x, y that is what we will get here, y prime here. So, this is actually, over this which is W x, y, y prime, p d x, which is Weierstrass Function is given by F x, y, y prime minus F x, y, p minus y prime minus p F p x, y, p.

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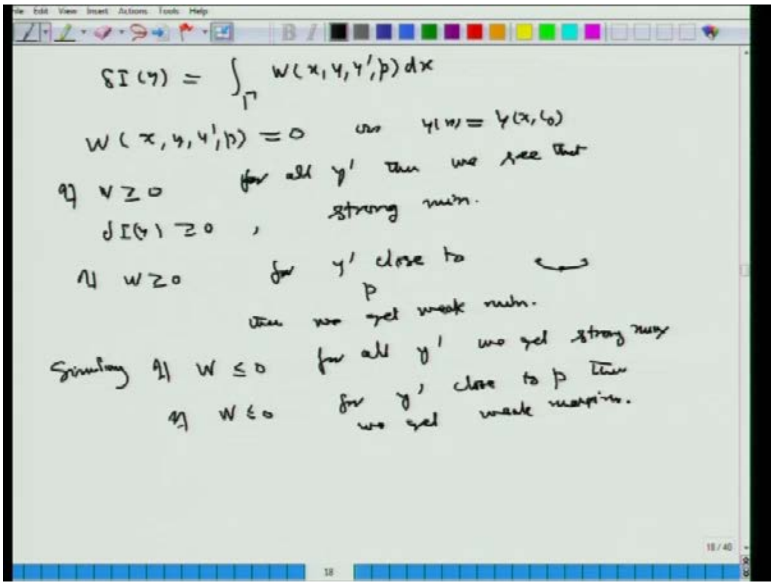

Now here, we see that this, so if delta I is, so here we get delta I (y) has integral over this gamma of W x, y, y prime, p d x. Now, W is 0 on, because their W equal to x, y, y prime, p is 0 on y (x), which is y (x, c 0), because here y prime is p. So, we will see that this y prime equal to p. So, this term will be 0 and here also y prime will be p. So, this term will be these two terms will be equal and they will be canceled.

So, this W equal to 0 on this, and now, if W is greater than equal to 0 in the neighborhood of… So, if W is 0, where W is greater than 0 for all y prime, then we see that delta I (y) will be greater than 0. So, in the neighborhood of this y, we will have strong maximum sorry strong minimum, like the second order variation, we have that is a positive second derivative is greater than equal to 0, we get here minimum.

And so, in this case if delta I (y) is greater than 0, because here than in the neighborhood, you will have it is like this, here you have least value and here in the up it goes, so that the difference here will be positive. So, delta I is greater than equal to 0, we get the minimum and it is independent of y whether y prime, independent of y prime whether y prime is close to p or not, because then as we had defined earlier. So, it is in the zero order of proximity. And so, we say that the minimum achieved is strong.

Now, if W greater than equal to 0 for y prime close to p, then we get weak minimum. Similarly, if W is less than equal to 0 for all y prime, we get strong maximum and if this is if W is less than equal to 0 for y prime close to p, then we get weak maximum. So, that is the case here.

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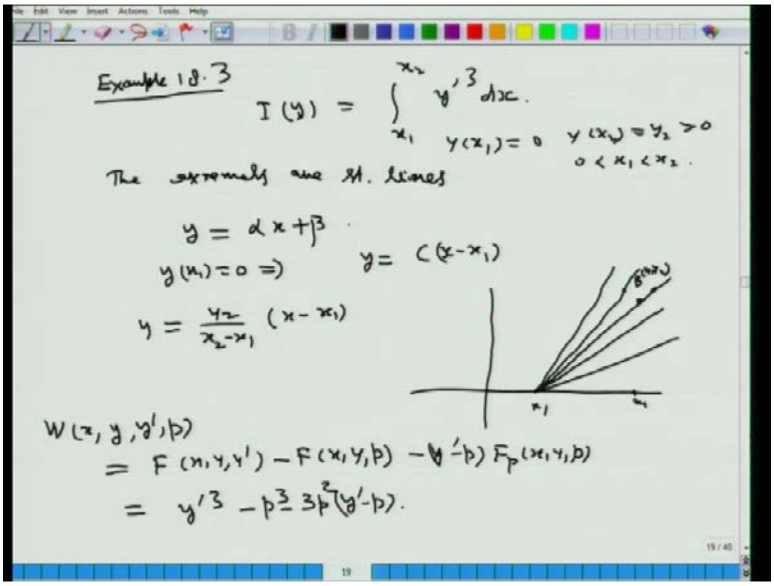

And so let us see the examples now.

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This would be now… So, now, let us consider this I (y) equal to x 1 x 2 y prime cube d x, we see that here; it is a function of y prime only. So, the extremals are and the conditions, let say y at x 1 is 0, and y at x 2 is y 2, we take this to be positive and 0 less than x 1 less than x 2.

So, the extremals here on straight lines y equal to something like alpha x plus beta, and y at x 1 0 implies that y equal to C x minus x 1, and here it is y 2 so, you get y equal to y 2 over x 2 minus x 1 into the x minus x 1. So, this situation here is like this, these are this is the point x 1 and x 2 here. So, this straight lines form the central field here, and let say here x 2, B is here that is (x 2, y 2) and y 2 is taken to be positive here. So, it is here somewhere.

Now, here the Weierstrass Function W x, y, y prime p is F x, y, y prime minus F x, y, p minus y prime minus p F p x, y, p. So, this comes out to be, this is y prime cube minus, now y prime is to be replaced by p. So, we get p cube, and then y prime minus p into F p means, 3 p square into 3 p square, so let say minus 3 p square.

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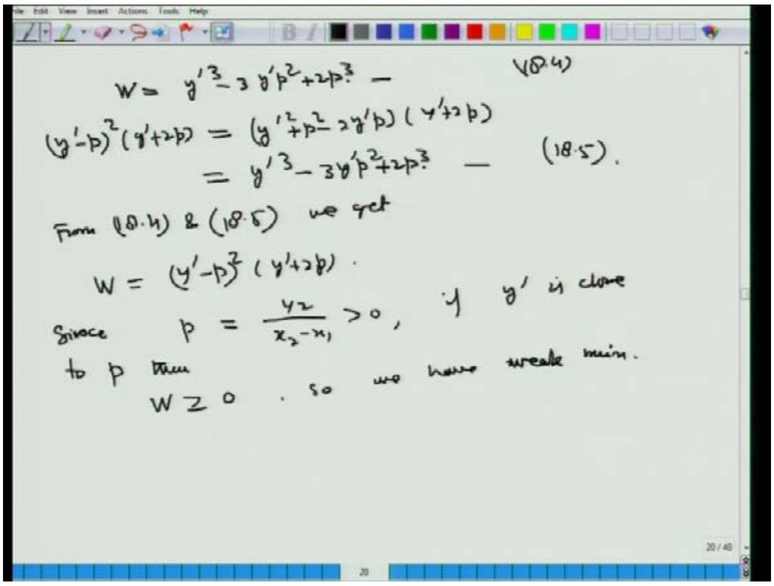

So, here we see that, we get W equal to… So this can be seen that, it is y prime cube minus 3 y prime p square plus 2 p cube, let us put at as 1. And then if we look at this, y prime minus p square into y prime plus 2 p, this comes out to be y prime square plus p square minus 2 y prime p into y prime plus 2 p. And this is then finally, y prime cube minus 3 y prime p square plus 2 p. So, this is let me put 0, it as 18.4 and this as 18.5.

So, from 18.4 and 18.5, we get that W equal to y prime minus p whole square y prime plus 2 p. So, here since, p is since p which is y 2 over x 2 minus x 1 is positive. So, if y prime is close to p, then W is greater than equal to 0. And so, we have weak minimum here.

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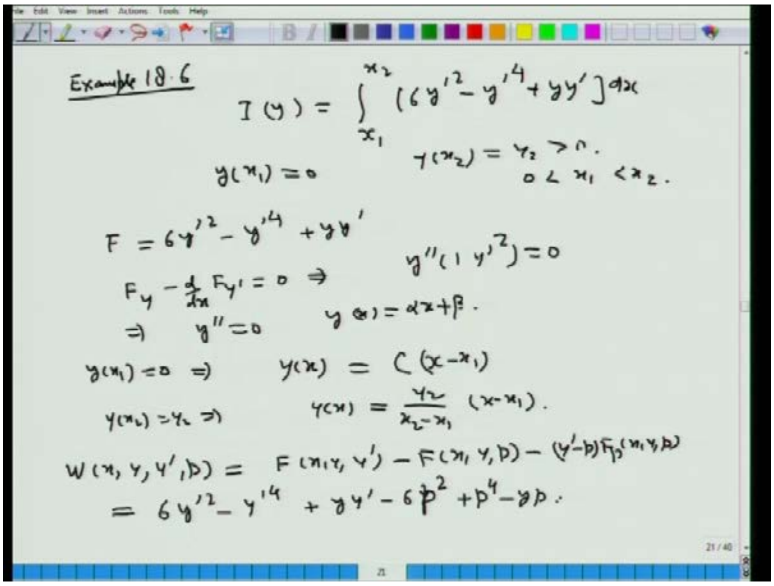

Now the next example, we considered.

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This I (y) equal to integral x 1 to x 2 6 y prime square minus y prime to the power 4 plus y y prime d x. Here, again we have y of x 1 equal to 0 and y at x 2 equal to y 2, we take positive and 0 less than x 1 less than x 2. Here, we have to find first the extremals.

So, here F is 6 y prime square minus y prime 4 plus y y prime. And so, F y minus d by d x of F y prime equal to 0 implies that y double prime into 1 minus y prime square equal to 0. So, this is not 0, this implies that y double prime equal to 0. So, y (x) equal to alpha x plus beta, family of straight lines. So, y at x 1 equal to 0 implies that now we get y (x) equal to C x minus x 1 and passing through this.

So, y at x 2 equal to y 2 implies that y (x) equal to y 2 over x 2 minus x 1 into x minus x 1, which is already depicted in the previous example. And here we see that now, this Weierstrass Function here is y (x) y W x, y, y prime p, which is F x, y, y prime minus F x, y, p minus y prime minus p F p x, y, p, which gives us… Now, in this case 6 y prime square minus y prime 4 plus y y prime minus 6 p square plus p to the power 4 minus y p.

Now, we will see that we will actually get a quadratic equation in that and seeing the roots of that equation, we will be able to conclude in which region, we have a strong minimum, strong maximum, weak maximum or weak minimum that we will be discussing in the next class. Thank you very much for your attention please.