Introduction to Probability & Statistics

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Week - 2
Lecture - 7
Conditional Probability

Ab hum conditional probability ke baare me padhte hain. Kisi bhi event A ki probability hum tab tak nikalte aaye hain jab hume koi extra jankari na ho. Lekin kai baar aisa hota hai ki hum event A me interested hote hain, par hume itni information mil jaati hai ki koi doosra event B already ho chuka hai. Yeh nayi information humare pehle wale estimate, yani P(A), ko badal deti hai. Isi naye estimate ko hum kehte hain conditional probability of A given B. Matlab event B hone ke baad event A ki probability kya hogi. Pehle ek simple example se samajhte hain: Dilli me 15 February ko barish hone ki probability kuch historical data se nikal sakti hai. Farvari me barish kam hoti hai, par hoti hai. Yani ek baseline probability hogi. Lekin agar usi din hum dekhen ki aasman me kaale ghane baadal hain, to yeh information A ki probability ko badal degi — barish ke chances badh jaayenge. Yahi conditional probability ka concept hai. Ek aur example: ek factory me do assembly lines hain A1 (purani) aur A2 (nayi). A1 se 8 components bane, jisme 2 defective aur 6 not defective hain. A2 se 10 components bane, jisme 1 defective aur 9 not defective hain. Total 18 components me se agar ek random component uthaate hain to probability ki woh A1 se aaya ho 8/18. Lekin agar hume extra information mile ki chosen component defective hai, to ab hum sample space ko sirf defective components tak seemit kar denge total defective 3 hain, jisme se 2 A1 ke hain. To conditional probability ki chosen component A1 ka ho given that woh defective hai ban jaati hai: 2/3. Yeh original probability 8/18 se alag hai. To conditional probability wahi revised probability hai jo extra information milne par update ho jaati hai. Venn diagram se dekhen to event A ek circle hai, event B ek doosra circle. P(A) poore Omega me se A ka hissa hai. Par jab hume pata hai ki outcome B ke circle ke andar hi hai, to ab hamara sample space Omega nahi balki B ho jaata hai. Ab hamara sawaal hota hai: B ke andar se kitna hissa A me bhi aata hai? Yani $A \cap B$ ka area B ke area ka kitna hissa hai. Isi ko mathematically likhte hain: P(A given B) = $P(A \cap B) / P(B)$, provided $P(B) > 0$ ho. Yahi conditional probability ki formal definition hai: $A|B$ ka matlab hota hai B hone ki condition me A ki probability. Yeh wahi concept hai jo real life me baar baar hota hai nayi information milti hai aur hum apni probability ka estimate revise karte hain.

So, yeh conditional probability of A given B ka concept hai yaani jab event B ho chuka ho tab event A ke hone ki nayi probability kya hogi. Example ke liye maan lijiye ek camera shop me 60% buyers extra memory card (event A), 40% buyers extra battery (event B), aur 30% buyers dono memory card aur battery ($A \cap B$) kharidte hain. Agar hume nikalna ho ki given customer ne battery li hai, memory card lene ka chance kya hai, to hum $P(A|B) = P(A \cap B)/P(B) = 0.3/0.4 = 0.75$ use karte hain, jo dikhata hai ki battery lene ki information se memory card lene ka chance 0.6 se badh kar 0.75 ho gaya. Isi tarah, agar given hai ki customer ne memory card liya hai, to battery lene ki probability $P(B|A) = 0.3/0.6 = 0.5$ hogi. Ab agar given hai ki customer ne battery li hai, to memory card na lene ki conditional probability $P(A^c|B)$ nikalne ke liye pehle $A^c \cap B$ nikalte hain, jo hamesha $B - (A \cap B)$ ke barabar hota hai, to $P(A^c \cap B) = 0.4 - 0.3 = 0.1$. Fir $P(A^c|B) = 0.1/0.4 = 0.25$ milta hai. Dhyaan dene layak baat ye hai ki conditional probabilities $P(A|B) + P(A^c|B) = 0.75 + 0.25 = 1$ hota hai, jo hamesha sahi hota hai, bilkul waise hi jaise normal probability me event aur uske complement ka sum 1 hota hai. Yeh property kisi bhi do events A aur B ke liye hamesha sahi rahegi, sirf example ke numbers ke liye nahi. Agar hum dono taraf ke terms ko ek non-zero number P(B) se divide karein, to hume milta hai ki conditional probability of A complement given B aur 1 minus conditional probability of A given B ka relation hamesha valid hota hai. Isi wajah se example me $P(A|B) = 0.75$ aur $P(A^c|B) = 0.25$ mila, jinka sum 1 hai, aur yeh har situation me sach hota hai. Ab agar hum conditional probability of B complement given A nikalna chahen, to directly property use kar sakte hain: yeh hoga 1 minus conditional probability of B given A, aur $P(B|A)$ humne part (B) me 0.5 nikala tha. Aage badhte hue ek simple observation yeh hai ki conditional probability ke definition ko rearrange karne par ek useful identity milti hai: $P(A \cap B) = P(A) \times P(B|A)$, aur isi ka ulta likhne par $P(A \cap B) = P(B) \times P(A|B)$ bhi milta hai. Ab hum ek naya type ka example dekhte hain jahan events time ke hisaab se ek ke baad ek hote hain. Maan lijiye ek urn me 5 black aur 6 red balls hain, total 11 balls. Ek ball randomly draw karne par black aane ki probability 5/11 aur red aane ki probability 6/11 hogi, jo simple equally-likely outcomes ka case hai. Lekin example 2 me hum do balls ek ke baad ek nikalte hain aur pehli ball draw karne ke baad usse wapas urn me nahi rakhte. Yani pehle draw ke baad urn me sirf 10 balls bachti hain. Hum B1 aur R1 define karte hain (pehla draw black/red), aur B2 aur R2 define karte hain (dusra draw black/red). Ab hume second ball ke red hone ki probability P(R2) chahiye, jo seedhe nahi nikal sakti kyunki wo pehli ball ke color par depend karti hai. R2 event do tarike se ho sakta hai: (1) pehli ball black (B1) aur dusri red (R2), ya (2) pehli ball red (R1) aur dusri red (R2). Dono mutually exclusive hain, isliye $P(R2) = P(B1 \cap R2) + P(R1 \cap R2)$. Ab conditional probability formula use karte hain: $P(B1 \cap R2) = P(B1) \times P(R2 \mid B1)$. Pehli ball ke liye $P(B1) = 5/11$ hai. Agar pehli ball black aa gayi, to urn me 4 black aur 6 red balls bachti hain, total 10, to $P(R2|B1) = 6/10$, isliye $P(B1 \cap R2) = (5/11)(6/10) = 30/110$. Usi tarah $P(R1 \cap R2) = P(R1) \times P(R2|R1) = (6/11)(5/10) = 30/110$. Isliye $P(R2) = 60/110$. Aage ek aur example: ek mobile shop me 3 brands beche jaate hain jisme 50% customers brand 1, 30% brand 2, aur 20% brand 3 kharidte hain. Sab phones ek saal ki warranty ke saath aate hain. Warranty period me repair ki zaroorat padne ki probability brand 1 ke liye 25%, brand 2 ke liye 20%, brand 3 ke liye 10% hai. Events A1, A2, A3 define karte hain (kis brand ka phone kharida), aur B define karte hain (phone ko warranty me repair chahiye). $P(A1)=0.5$, $P(A2)=0.3$, $P(A3)=0.2$. Conditional probabilities: $P(B|A1)=0.25$, $P(B|A2)=0.2$, $P(B|A3)=0.1$. Ab question poochta hai: randomly chosen customer ne brand 1 bhi kharida aur repair ki zaroorat bhi padti hai yani $P(A1 \cap B)$. Formula use karte hain: $P(A1 \cap B) = P(A1) \times P(B|A1) = (0.5)(0.25) = 0.125$.