Introduction to Probability & Statistics
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Lecture - 27
Expectations of Functions Of Several Variables

Do continuous random variables X aur Y jinka joint PDF f(x,y) ho, aur marginal PDFs $f_x(x)$ aur $f_\gamma(y)$ ho, independent tab kehlate hain jab har real x,y ke liye $f(x,y) = f_x(x)\cdot f_\gamma(y)$, jo bilkul wahi definition hai jo discrete case me hoti hai, bas waha PMF tha aur yaha PDF; isi tarah jab hum do se zyada random variables $X_1,X_2,\ldots,X_n$ ko saath me study karte hain to unka joint PMF/PDF function n variables par depend karta hai, aur marginal nikalne ke liye baaki variables ko sum ya integrate out kar diya jata hai, while independence ke liye condition hoti hai ki joint PDF ya PMF barabar ho product of marginal PDFs/PMFs ke;

$$P\left(a_1 \le X_1 \le b_1, a_2 \le X_2 \le b_{2,\ldots\ldots}, a_n \le X_n \le b_n\right) =$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \ldots\ldots \int_{a_n}^{b_n} f(x_1, x_{2,\ldots\ldots} x_n)dx_n dx_{n-1} dx_1$$

example ke taur par maan lijiye $X_1,\ldots,X_n$ independent exponential random variables hain parameter λ ke saath, to har X_i ka PDF hai $\lambda e^{(-\lambda x_i)}$ aur joint PDF hota hai $\lambda^n e^{(-\lambda(x_1+\ldots+x_n))}$, $x_i \ge 0$;

$$f\left(x_1, x_{2,\ldots\ldots} x_n\right) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots\ldots f_{X_n}(x_n)$$

$$= \left(\lambda e^{-\lambda x_1}\right)\left(\lambda e^{-\lambda x_2}\right) \ldots\ldots\ldots\ldots \left(\lambda e^{-\lambda x_n}\right)$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

agar ye sab components ek system ke parts hain jo serial me connected ho — jaise 5 tube-lights — to system tab tak chalega jab tak sab components survive kar rahe hain, yani T>t ⇔ X_i>t for all i; isliye $P(T>t) = \Pi P(X_i>t) = (e^{(-\lambda t)})^n = e^{(-n\lambda t)}$, isse pata chalta hai ki system lifetime T bhi exponential distribution follow karta hai parameter nλ ke saath, aur is tareh independence aur joint distributions ke concept ko smooth tarike se do variables se extend karke multiple variables par apply kiya jata hai.ki lifetime of system T small t se jaada hone ka matlab hai ki har component $X_1,X_2,\ldots,X_n$ ka lifetime bhi t se jaada ho, yani events X_1>t, X_2>t,…, X_n>t sab ek saath hon, aur inki probability joint PDF ke integration se milegi, jise hum likhenge

$P(T>t)=P(X_1>t, X_2>t,\ldots,X_n>t)=\int_t^\infty \ldots \int_t^\infty \lambda^n\cdot e^{-\lambda(x_1+\ldots+x_n)}\, dx_1dx_2\ldots dx_n$,

ye separate integrals me toot jaata hai kyunki integrand factor form me hai (sirf independent x_i terms), to each integral ban jata hai $\int_t^\infty \lambda e^{-\lambda x_i}dx_i = e^{-\lambda t}$, aur poore n terms ke liye milta hai $P(T>t)=e^{-n\lambda t}$, isse CDF nikalta hai $P(T\leq t)=1-e^{-n\lambda t}$, aur differentiating gives PDF of T: $f_T(t)=n\lambda\cdot e^{-n\lambda t}$, so T bhi exponential distributed hota hai parameter $n\lambda$ ke saath; ab hum aage badhte hain ki jab do random variables saath me study karte hain, to ek function $H(X,Y)$ ka expectation $E[H(X,Y)]$ kaise nikalte hain — discrete case me $E[H(X,Y)] = \Sigma_x \Sigma_\gamma H(x,y)\cdot P(x,y)$, aur continuous case me

$E[H(X,Y)] = \iint H(x,y)\cdot f(x,y)\, dx\, dy$,

bilkul wahi pattern jo single random variable ke liye hota tha, bas ab double sum ya double integral; example ke liye 5 chits numbered 1–5 se 2 chits bina replacement ke randomly draw karte hain, total possible ordered draws $5\times4=20$, jahan X ho first chit ka number aur Y ho second chit ka number, to joint PMF define hota hai $P(X=x,Y=y)=1/20$ jab $x\neq y$, aur 0 jab $x=y$, iske baad expected values ya function ke expectations isi PMF ko use karke nikale jaate hain — ye saara framework define karta hai ki hum multiple random variables ko ek saath kaise mathematically study karte hain. paanch guna chaar ya bees, to kyunki hum dono chits ko randomly choose kar rahe hain, ye 20 ke 20 pairs equally likely hain, iska matlab humara PXY joint PMF hum likh sakte hain: 1/20 if X aur Y dono $\{1,2,3,4,5\}$ me se hain lekin $X\neq Y$, aur 0 otherwise; yahan X denote karta hai pehli chit par likha number aur Y denote karta hai doosri chit par likha number; ab hum jaana chahte hain ki X aur Y ek doosre se kitne door (different) hote hain on average, yani expected value of $|X–Y|$; iske liye hum function define karte hain $h(x,y)=|x–y|$, aur expectation ki definition use karte hain: $E[|X–Y|] = \Sigma_x \Sigma_\gamma |x-y|\, P(x,y)$, jahan x,y 1–5 ke beech hain aur $x\neq y$; $P(x,y)=1/20$ constant hai isliye bahar aa jayega, bacha $\Sigma|x-y|$; hum table bana kar $|x-y|$ values fill karte hain (1,2,3,4,5 rows/columns with diagonals removed), sab values ka sum nikalta hai 40, therefore $E[|X-Y|]=40/20=2$; ab doosra example continuous case ka: ek packaging company 1kg dry fruits ke boxes banati hai jisme almonds (X), cashews (Y), walnuts ($Z = 1-X-Y$) hote hain; given joint density $f(x,y)=24xy$ for $0\leq x\leq1$, $0\leq y\leq1$, $x+y\leq1$; prices: almonds ₹250/kg, cashews ₹200/kg, walnuts ₹300/kg, to box cost $h(x,y)=300-50x-100y$; expected cost $E[h(X,Y)] = \iint h(x,y)\, f(x,y)\, dy\, dx$, with limits: x from 0 to1, y from 0 to $1-x$; integration ke baad answer aata hai ₹240, jo average cost per box hai; is tarah expected value of $H(X,Y)$ discrete case me double sum se aur continuous case me double integral se nikalta hai, aur ye hi logic general hota hai n variables ke liye, jahan discrete hon to n-dimensional summation, aur continuous hon to n-dimensional integration, function $H(x_1,\ldots,x_n)$ multiplied by joint PMF/PDF ke saath.