Introduction to Probability & Statistics
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Week - 7
Lecture - 26
Jointly Distributed Continuous Random Variables

Humne ab tak jo dekha tha do asatat yaadricchik char ko ek saath padte waqt joint PMF aur conditional PMF ka concept ab isi ko continuous random variables ke liye dekhenge; jaise hospital me checkup ke samay height aur weight dono measurements ek hi vyakti ke liye liye jaate hain aur unhe ek saath study karna swabhavik hota hai; agar X aur Y continuous random variables ek hi sample space par defined ho to unke liye joint PDF (sanyukt praayikta ghantvya phalan) define kiya jata hai; joint PDF f(x,y) tab valid hota hai jab ye conditions satisfy karta ho: (1) $f(x,y) \geq 0$ for all real x,y; (2) double integral over entire real plane of f(x,y) dx dy = 1,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

jo total probability ko represent karta hai; (3) agar hum koi rectangle define karein $A = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$ to probability ki (X,Y) is rectangle A me aaye hogi integral_a^b integral_c^d f(x,y) dy dx; ye baat rectangle tak seemit nahi fact ye hai ki kisi bhi 2D set A ke liye probability

$$P((X,Y))\epsilon A) = \iint f(x,y)dxdy$$

yani PDF ko us set par integrate karenge; jaise 1D case me PDF ka graph x-axis ke upar hota tha aur probability ko area under curve se measure kiya jata tha, continuous bivariate case me f(x,y) 2 variables ka function hai to graph 3D me draw hota hai x–y plane base hota hai aur f(x,y) z-axis ke along height form karta hai to probability kisi region ke liye us region ke upar bane hue solid surface ka volume ke barabar hoti hai; yani concept wahi pehle area under curve tha, ab volume under surface hai aur isi visualization ko higher dimensions me bhi generalize kiya ja sakta hai, bhale drawing mushkil ho par concept bilkul same rehta hai.To hum aage badhte hain ek example me do counters wale fast–food restaurant me X aur Y random variables define kiye gaye hain: X = proportion of time take-home counter busy hota hai, aur Y = proportion of time sit-in counter busy hota hai; dono proportions 0 aur 1 ke beech values lete hain, yani (X,Y) unit square $D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ ke andar fall karta hai; joint PDF diya hai $f(x,y) = (6/5)(x + y^2)$ jab (x,y) D ke andar ho, aur f(x,y) = 0 otherwise; pehla kaam hota hai verify karna ki f(x,y) valid PDF hai $f \geq 0$ clearly, aur double integral from 0 to 1 over x and 0 to 1 over y of f(x,y) dx dy = 1 isko compute karne par value 1 milti hai, to PDF valid hai; ab isi joint PDF se probability calculate kar sakte hain, jaise $P(X<1/4, Y<1/4)$ ke liye integrate karenge 0 to

1/4, 0 to 1/4 f(x,y) dx dy, aur result milta hai 7/640 ≈ 0.0109; phir marginal PDFs nikaalte hain concept wahi jo discrete case me tha, bas summation ki jagah integration: marginal

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

par kyunki f(x,y) unit square ke bahar zero hai, isliye actually integrate karte hain y = 0 to 1; to $f_x(x) = \int_0^1 (6/5)(x + y^2)\, dy = (6/5)(x + 1/3)$ for $0<x<1$, else 0; isi tarah marginal $f_\gamma(y) = \int_0^1 (6/5)(x + y^2)\, dx = (6/5)(1/2 + y^2)$ for $0<y<1$, else 0; is tarah joint PDF poori probabilistic information contain karta hai aur use karke hum any probability, marginal density, conditional density sab derive kar sakte hain, jaise discrete case me kiya tha — bas yahan sums integral me badal jate hain aur 1D graph ke area ki jagah 2D surface ke volume ko measure karte hain. $x + y^2 \times 6/5$, lekin ab hum integrate kar rahe hain x ke saath, to ye aayega $6/5 \times (x^2 / 2) + y^2x$, aur x ke limits 0 se 1 tak, to milta hai $6/5(1/2 + y^2)$; to marginal PDF of y = $6/5(1/2 + y^2)$ for $0 \leq y \leq 1$, aur otherwise 0; bilkul wahi concept jo discrete random variables me tha, bas summation ki jagah integration aa gaya – aur continuous random variables ke case me integration aam tor par aasaan hota hai, jabki discrete me summation mushkil ho sakta hai; next conditional PDF, notation wahi jo conditional PMF me tha, par ab PDF hai, aur conditional PDF ko define karte hain joint PDF ko marginal PDF se divide karke, lekin tabhi jab denominator (marginal pdf of y) zero na ho; to conditional PDF of x given y = y: $f_x|_\gamma(x|y) = f(x,y)/f_\gamma(y)$, sirf un y ke liye defined jinke liye $f_\gamma(y) > 0$, aur agar x 0–1 ke bahar hai to PDF = 0; example me $f_x|_\gamma(x|y) = (x + y^2)/(1/2 + y^2)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$; agar y=0.5 to $f_x|_\gamma(x|0.5)= (x+1/4)/(3/4)=4x+1/3$ for $0 \leq x \leq 1$, else 0; isi tarah aap $f_\gamma|_x(y|x)$ bhi nikal sakte ho; agla example: joint PDF f(x,y)=24xy jab $0 \leq x \leq 1$, $0 \leq y \leq 1$, aur $x+y \leq 1$ (otherwise zero); iska matlab allowed region ek triangle hai square ke andar: $x \geq 0$, $y \geq 0$, $x+y \leq 1$; is pdf ko verify karne ke liye integral: $\int_0^1 \int_0$^{1−y} 24xy dx dy = 1; phir probability find karne ke liye allowed region ka dhyan rakhna hota hai; exercise:

P(0≤x≤1, 0≤y≤1, x+y≤0.5) = $\int_0$^{0.5} $\int_0$^{0.5−y} 24xy dx dy, jiska answer hota hai 0.0625; is tarah se continuous bivariate distributions ke liye probability, marginal PDFs aur conditional PDFs nikalte hain bilkul discrete jaisi framework, bas sums ki jagah integrals aa jate hain thank you.