

Introduction to Probability & Statistics
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Week - 7
Lecture - 25
Jointly Distributed Discrete Random Variables

Abhi tak humne ek samay me ek random variable padha discrete ya continuous uske expectation, variance, probabilities kaise nikalni hai, ye sab dekha; lekin aksar ek experiment me ek se zyada measurements hote hain jaise class me randomly student choose karke uske ganit ke marks (X) aur Hindi ke marks (Y) dekhna ye do alag measurements ek hi student se judi hoti hain; isi tarah science ke marks honge to teesra measurement J ho sakta hai; to jab ek se zyada random variables ek saath hon tab unka study kaise karte hain to jaise humne ek discrete random variable ke liye PMF dekha tha, waisa hi ab hum 2 variables ke liye joint PMF dekhenge; agar X aur Y discrete random variables hain jo ek hi sample space par defined hain, to unka joint probability mass function (joint PMF) hota hai $p(x,y)$, aur ye represent karta hai probability of event: $X = x$ AND $Y = y$ simultaneously; ye event actually intersection hota hai: $\{X=x\} \cap \{Y=y\}$, jise hum shorthand me $p(x,y)$ likhte hain; ye probability hoti hai isliye $p(x,y) \geq 0$ hoga har pair (x,y) ke liye, aur jab hum sabhi x aur sabhi y ke upar $p(x,y)$ ka sum karenge to total hoga 1;

$$\sum \sum P(x,y) = 1$$

aur agar koi set A, jo R^2 ka subset ho, uske liye hume probability chahiye ki $(X,Y) \in A$, to hum $p(x,y)$ ko un sabhi (x,y) ke liye sum karenge jo A me aate hain; example dekhte hain: ek petrol pump me 3 pumps hain: 2 petrol ke aur 1 diesel ka; jab vehicles aate hain fuel bharane ke liye, kuch pumps active hote hain to define karte hain X = number of petrol pumps in use, aur Y = number of diesel pumps in use; petrol pump ke 2 pumps hain to X kya values le sakta hai 0,1,2; diesel ka ek pump hai to Y values 0,1; ab inka joint PMF table form me diya hai: Y = 0,1 rows; X = 0,1,2 columns; aur probabilities hain: first column: 0.20, 0.10, 0.20; second column: 0.05, 0.15, 0.30; in sab ko add karne par total 1 aata hai isliye ye valid joint PMF hai; aur is table ka matlab ye hota hai ke har entry $p(x,y)$ represent karti hai probability ki us samay petrol ke X pumps aur diesel ka Y pump simultaneously in-use hon. Agar hum dekhna chahte hain ki probability $X=1$ aur $Y=1$ kya hai to joint PMF ke definition ke tahat ye value hai $p(1,1)$, jo table me di hui hai $x=1$ second row aur $y=1$ second column, to $p(1,1)=0.15$; isi tarah $p(0,0)=0.20$, $p(0,1)=0.05$, $p(1,0)=0.10$ sab values table se directly padhi ja sakti hain; ab agar hume chahiye probability($X \leq 1$), iska matlab $X = 0$ ya 1 hoga, lekin Y par koi restriction nahi hai, yani Y koi bhi value le sakta hai; to $X \leq 1$ ke liye possible pairs honge: (0,0), (0,1), (1,0), (1,1), aur in sab events mutually exclusive hain isliye unki probabilities add karenge: $p(0,0)+p(0,1)+p(1,0)+p(1,1)=0.20+0.05+0.10+0.15=0.50$; isi tarah agar hume $P(X+Y=2)$ chahiye, to X values 0,1,2 ho sakti hain, Y values 0,1; $X+Y=2$ tab hoga jab pair (1,1) ya (2,0) ho,

to $P(X+Y=2)=p(1,1)+p(2,0)=0.15+0.20=0.35$; isi tarah $P(X+Y \geq 1)$ dekhna ho to is event ke complement use karna aasaan hai yani $1-P(X+Y=0)$, aur $X+Y=0$ tab hota hai jab $(X,Y)=(0,0)$, to result $1-0.20=0.80$; ab agar $P(X=0)$ chahiye to $X=0$ ke saath Y koi bhi value le sakta hai yani $(0,0)$ aur $(0,1)$, to $p(0,0)+p(0,1)=0.20+0.05=0.25$; isi tarah $P(X=1)=p(1,0)+p(1,1)=0.10+0.15=0.25$; $P(X=2)=p(2,0)+p(2,1)=0.20+0.30=0.50$; ye values hamne table ke row sums se nikaali ye hi marginal PMF of X hoti hai; isi tarah column sums se hamne marginal PMF of Y nikala: $P(Y=0)=0.50$, $P(Y=1)=0.50$; general form me likhen to agar X,Y ke paas joint PMF $p(x,y)$ ho to marginal PMF of X hota hai $p_x(x)=\sum$ over all y of $p(x,y)$, aur marginal PMF of Y hota hai $p_y(y)=\sum$ over all x of $p(x,y)$;

$$p_x(x) = P(X = x) = \sum_y p(x, y)$$

$$p_y(y) = P(Y = y) = \sum_x p(x, y)$$

yani joint PMF ke row sums se X ki marginal distribution milti hai aur column sums se Y ki marginal distribution, aur isi tarah hum joint PMF se individual distributions derive karte hain. Agar hume probability chahiye ki X belongs to A , jahan A real numbers ka subset hai aur X ka PMF available ho to hum ye karte hain: sum over all x in A of $p_x(x)$, lekin marginal PMF $p_x(x)$ ka definition ye hai ki hum joint PMF $p(x,y)$ ko sabhi y ke liye sum karte hain yani $p_x(x) = \sum_y p(x,y)$, isi tarah se probability($X \in A$) ko hum joint PMF ke through bhi likh sakte hain $P(X \in A) = \sum$ over all x in A \sum over all y $p(x,y)$, aur dono approach se ek hi result milega; ab ek example lete hain: X takes values $\{0,1,2\}$, Y takes values $\{1,2,3\}$, aur joint PMF ka table diya hai: $X=0$ row: 0.05, 0.10, 0.20; $X=1$ row: 0.10, 0.10, 0.10; $X=2$ row: 0.20, 0.05, 0.10; inko sum karne par total 1 hota hai isliye ye valid joint PMF hai; marginal PMF of X hum row sums se calculate karte hain $p_x(0)=0.35$, $p_x(1)=0.30$, $p_x(2)=0.35$; marginal PMF of Y column sums se $p_y(1)=0.35$, $p_y(2)=0.25$, $p_y(3)=0.40$; ab conditional distribution dekhte hain: agar hume pata hai $y=2$ to conditional PMF of X given $Y=2$ hoga $p(x|2)=p(x,2)/p_y(2)$; $Y=2$ ke liye $p(0,2)=0.10$, $p(1,2)=0.10$, $p(2,2)=0.05$, aur $p_y(2)=0.25$, to conditionally probabilities hoti hain: $X=0 \rightarrow 0.10/0.25=0.4$, $X=1 \rightarrow 0.10/0.25=0.4$, $X=2 \rightarrow 0.05/0.25=0.2$, ye conditional PMF hamesha add up hokar 1 hota hai; isi tarah $p(y|x)$ bhi nikal sakte hain; ab independence ka concept: do random variables X aur Y independent tab honge jab sabhi pairs ke liye $p(x,y)=p_x(x) \cdot p_y(y)$, agar ek bhi pair aisa mil jaye jisme equality fail hoti ho to X,Y dependent hain; hum example me $p(1,3)=0.10$ dekhte hain, lekin $p_x(1)=0.30$ aur $p_y(3)=0.40$, product=0.12, jo 0.10 ke barabar nahi hai, isliye X aur Y dependent hain; logically bhi independence ka matlab ye hai ki event $\{X=x\}$ aur event $\{Y=y\}$ independent honge yani $P(X=x, Y=y)=P(X=x) \cdot P(Y=y)$; example: fair die do baar giraya jaye— first toss ka outcome X aur second toss ka Y ye naturally independent honge, PMF of $X=1/6$, PMF of $Y=1/6$, to joint PMF $p(x,y)=1/6 \cdot 1/6=1/36$ for all pairs; is tarah independence se joint PMF derive kiya ja sakta hai jab marginals pata ho.