Introduction to Probability & Statistics
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Week - 6
Lecture - 23
Exponential And Gamma Distributions

Abhi hum kuch chand standard continuous distributions ko dekhenge, unme se pehla hai exponential family exponential distribution ya exponential yadruchchhik char. Iski paribhasha: X ko kehte hain exponential distribution parameter lambda ke saath, jahan lambda > 0 hota hai; agar X ka PDF diya ho:

$$f(x;\gamma) = \begin{cases} \lambda e^{-\lambda x} \; if \; x \geq 0 \\ o \quad otherwise \end{cases}$$

f(x)=lambda·e^(–lambda·x), jadi x ≥ 0, aur 0 otherwise. Hum likhte hain X follows exponential distribution with parameter lambda — short form: X ~ exponential(λ). Yani X ek continuous random variable hai jiska PDF hai lambda·e^(–lambda·x) x≥0 ke liye. Isse turant pata chalta hai ki exponential random variable X hammesha 0 ya usse badi values leta hai. Jaise discrete random variables ke liye hum standard distributions dekhte the, waise hi continuous ke liye bhi hum CDF, expected value, variance etc. nikalte hain. Pehle dekhte hain iski cumulative distribution function CDF: F(x) = P(X ≤ x). PDF do intervals me defined hai:

$$if \; x < 0: f(x) = \; p(X \leq x) = \; \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{x} 0dy = 0$$

$$if \; x \geq 0: f(x) = \; p(X \leq x) = \; \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{0} f(y)dy + \int_{0}^{x} f(y)dy$$

$$= \; 0 + \int_{0}^{x} \lambda e^{-\lambda} \; dy = \lambda e^{-\lambda y}$$

Integration karte waqt lambda bahar aa jayega. Integral of e^(–lambda·y) hota hai e^(–lambda·y)/(–lambda). Limits 0 se x; solve karke milta hai F(x)=1–e^(–lambda·x). To result: F(x)=0 if x<0; aur F(x)=1–e^(–lambda·x) if x≥0. Yeh exponential ke liye CDF hai. Ab expected value (mean) nikalte hain:

$$E(X) = \; \int_{-\infty}^{\infty} xf(x)dx = \; \lambda \int_{0}^{\infty} xe^{-\lambda x}dx = \frac{1}{\lambda}$$

Isi tarah $E(X^2)=\int_0^\infty x^2 \lambda e^{-\lambda \cdot x}\, dx$; iske liye do baar integration by parts karna padega; result aata hai: $E(X^2)=2/\lambda^2$; to $Var(X)=E(X^2)-[E(X)]^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$. Aur standard deviation sigma = $\sqrt{Var(X)} = \sqrt{1/\lambda^2}=1/\lambda$ (positive root). Yahan lambda > 0 hota hi hai exponential distribution ke liye. To hum conclude kar sakte hain: exponential RV X ke liye: mean = $1/\lambda$, variance = $1/\lambda^2$, standard deviation = $1/\lambda$. Udāharanon me kaise iska istemāl hota hai, to hum ek example le lete hain. Ek railway bridges ke stress ke upar ek study ki gayi thi, aur dekha gaya hai ki stress distribution on railway bridges of a particular type has an exponential distribution. Jab stress ko kisi particular units me, jaise megapascal (MPa), me māpā jata hai, to ek study ne dekha hai aur suggest kiya hai ki yeh jo stress hai uska distribution exponential hai jiska mean 6 MPa hai – yani exponential distribution with mean (mādhya) 6 MPa. Ab hum question puchte hain: what is the probability that the stress is at most 10 MPa? To agar hum ek symbol X use karein   X = stress on the bridge in MPa – to hume diya gaya hai ki X exponential distribution follow karta hai. Exponential distribution ka parameter hota hai lambda, aur hume diya gaya hai ki expected value of X = 6. Abhi humne dekha ki exponential distribution ke liye E(X) = 1 / lambda hota hai, to iska matlab hamare model me 1 / lambda = 6, therefore lambda = 1/6. To $X \sim$ Exponential(1/6). Ab hume probability chahiye ki stress at most 10 MPa ho, yaani $P(X \leq 10)$. Exponential ke CDF ke hisāb se $F(x) = 1 - e^{-\lambda x}$. Yahan lambda = 1/6 aur x = 10, to $P(X \leq 10) = 1 - e^{-(1/6)\cdot 10} = 1 - e^{-10/6}$. Iska exact value calculator se nikalein to $e^{-10/6} \approx 0.189$, to final approximate answer aata hai $1 - 0.189 \approx 0.811$ (three decimal places tak). Is tarah koi bhi model agar continuous ya discrete random variable ke liye use karein, to jo information di gai hai, usko expectations, variance waale formulas ke saath combine karke example se jude questions ka answer nikal sakte hain. Aage jaane se pehle, exponential distribution ki ek bahut mahatvapurn property dekhte hain jise seedha "memoryless property" bolte hain. Mān lijiye koi component hai – jaise ek tube light ya camera ka koi part – uska lifetime (jeevan kāl) ek random variable X hai jo exponential distributed hai parameter lambda ke saath. Lifetime ka matlab: jab se component ka istemāl shuru hua tab se le kar jis waqt woh fail ya kharab hota hai tab tak ka samay. Alag-alag components ke liye yeh lifetime alag hota hai, isliye X ek random variable hai, aur exponential distribution aksar aise lifetimes ko model karne ke liye use hota hai. Mān lijiye ek fixed time t_0 tak component abhi tak kaam kar raha hai. Yani hume pata hai ki $X \geq t_0$. Ab hum yeh question puchte hain: what is the probability ki yeh component additional time t aur survive kare, yaani t_0 ke baad aur t time tak bhi kaam karta rahe? Isko hum probabilistic notation me likhenge:

$P(X \geq t_0 + t \mid X \geq t_0)$. Conditional probability ke rule se: $P(A \mid B) = P(A \cap B) / P(B)$. Yahan $A = \{X \geq t_0 + t\}$ aur $B = \{X \geq t_0\}$. Lekin dhyaan dein: agar $X \geq t_0 + t$ hai, to zaroor $X \geq t_0$ bhi hoga (kyunki $t_0 + t \geq t_0$). Iska matlab $A \subseteq B$ hai, to $A \cap B = A$. Isliye $P(X \geq t_0 + t \mid X \geq t_0) = P(X \geq t_0 + t) / P(X \geq t_0)$. Ab exponential random variable ke liye $P(X \geq s) = e^{-\lambda s}$ hota hai (kyunki $P(X \leq s) = 1 - e^{-\lambda s}$). To numerator hoga $e^{-\lambda (t_0 + t)}$ aur denominator hoga $e^{-\lambda t_0}$. Dono ko divide karne par milta hai $e^{-\lambda (t_0 + t)} / e^{-\lambda t_0} = e^{-\lambda t}$. Yani $P(X \geq t_0 + t \mid X \geq t_0) = e^{-\lambda t}$, jo EXACTLY waisa hi hai jaise "fresh" component ke liye $P(X \geq t)$ hota. Iska matlab: jo additional lifetime ka distribution hai, woh original lifetime ke distribution jaisa hi hai; purana kitna time beet chuka hai (t_0) isse koi fark nahi padta. Isi ko kehte hain memoryless property: exponential distribution “past” ko bhool jata hai, aur aage ka lifetime distribution same rehta hai chahe kitna bhi samay already nikal chuka ho. Ab exponential ke percentiles dekhte hain. Percentile ka concept: koi p lete hain jahan $0 < p < 1$. 100p-th percentile ko hum eta(p) se denote karte hain. Yeh aisa number hai jiske liye CDF F(eta(p)) = p hota hai.

Exponential ke liye F(x) = 1 – e^(–lambda x). To hume chahiye: F(eta(p)) = 1 – e^(–lambda eta(p)) = p. Isse milta hai e^(–lambda eta(p)) = 1 – p. Dono side natural log lete hain: –lambda eta(p) = ln(1 – p). To eta(p) = –(1/lambda)·ln(1 – p). Isko hum 1/lambda · ln(1/(1 – p)) bhi likh sakte hain. Yani exponential distribution ka 100p-th percentile hai eta(p) = –(1/λ) ln(1 – p). Iske baad ek aur standard distribution aata hai jo isse juda hua hai: gamma distribution. Usko define karne ke liye pehle hum gamma function dekhte hain. Kisi bhi alpha > 0 (alpha positive real number) ke liye gamma function, jise hum capital Gamma(α) likhte hain, define hota hai: $\Gamma(\alpha) = \int_0^\infty x^{(\alpha-1)} e^{(-x)}\, dx$. Yeh math me bahut important function hai. Hume kuch properties chahiye: (1) Agar alpha > 1 ho, to $\Gamma(\alpha) = (\alpha - 1)\cdot\Gamma(\alpha - 1)$. Yeh integration by parts se prove kiya ja sakta hai. (2) Kisi bhi positive integer n ke liye, $\Gamma(n) = (n-1)!$. Yaani integer points par gamma factorial jaise behave karta hai. (3) $\Gamma(1/2) = \sqrt{\pi}$. Yahan π woh number hai jo circle ki circumference / diameter ka ratio hota hai. Kisi general alpha ke liye Γ(α) exact nikalna mushkil hota hai, lekin yeh teen facts hum use karte rehte hain. Ab gamma function ke base par ek PDF define karte hain: $f(x; \alpha) = x^{(\alpha-1)} e^{(-x)} / \Gamma(\alpha)$, agar $x \geq 0$, aur 0 otherwise, jahan $\alpha > 0$. Yeh non-negative function hai, aur agar hum isko 0 se ∞ tak integrate karein: $\int_0^\infty f(x;\alpha)\, dx = 1/\Gamma(\alpha) \int_0^\infty x^{(\alpha-1)}e^{(-x)}dx = 1/\Gamma(\alpha)\cdot\Gamma(\alpha) = 1$. Isliye yeh ek valid PDF hai. Ab general gamma distribution define karte hain:

$$f(x; \alpha, \beta) = [1 / (\beta^\alpha \Gamma(\alpha))]\, x^{(\alpha-1)} e^{(-x/\beta)},\ x \geq 0$$

ke liye, aur 0 otherwise, jahan $\alpha > 0$, $\beta > 0$ parameters hain. Isse hume pata chalta hai ki gamma random variable $X \geq 0$ hota hai. Agar β = 1 ho, to $f(x; \alpha, 1) = x^{(\alpha-1)} e^{(-x)}/\Gamma(\alpha)$ – jo upar wala “standard gamma density” hai. To jab β = 1, X ko hum “standard gamma random variable” kehte hain. Notation me: X ~ Gamma(α, β), aur special case me β = 1 ho to X ~ Gamma(α, 1) standard gamma. Aur agar α = 1 aur β = 1/λ ho, to aap directly check kar sakte hain ki

$$f(x; 1, 1/\lambda) = \lambda e^{(-\lambda x)},\ x \geq 0;$$

yani exponential(λ) ek special case hai gamma distribution ka (α = 1, β = 1/λ). Finally, gamma distribution ke liye mean aur variance: agar X ~ Gamma(α, β) ho, to E(X) = αβ aur $Var(X) = \alpha\beta^2$. Standard deviation sigma = sqrt(Var) = $\sqrt{(\alpha)}\cdot\beta$. Yeh sab aage ke examples me kaafi kaam aayega.