Introduction to Probability & Statistics
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Week - 6
Lecture - 22
Expectation of A Continuous Random Variable

Maan dijiye ki hamare paas hai ek santat ya yadruccik char X (continuous random variable); uske saath humne dekha do phalans jude hote hain ek jisko hum small f se darshate hain PDF (probability density function) ya praaykta ghanatva phalan, aur doosra CDF jisko hum capital F se darshate hain cumulative distribution function. Maan dijiye ki hamare paas X hai jiska PDF small f hai aur CDF capital F hai; to expected value aur variance kaise nikalte hain ya hum dekhte hain. To expected/mean value of X jise hum E(X) ya mu se darshate hain—uska definition hota hai integral over entire real line of x * f(x) dx. Tipanni: discrete random variables me hum PMF PX se suruaat karte the aur expectation nikalte the sum of x * P(x); yahan continuous case me PMF replace hota hai PDF small f se aur summation replace hota hai integration se—baaki logic same. Chaliye ek example se dekhte hain—ek contractor jo gravel/roadie bechta hai; one-week sale ka distribution continuous hai aur X ki PDF di gayi hai: $f(x)= 3/2\cdot(1-x^2)$ for x between 0 and 1, and 0 otherwise. Pehli observation: PDF di gayi hai to X continuous random variable hai. Doosri observation: f(x)=0 outside interval (0,1), iska matlab X ke values sirf interval 0–1 me hi ate hain. Ab avg kitna gravel ek week me becha jata hai, wo hum E(X) nikalke dekh sakte hain;

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

PDF ki definition dekhke hum range ko 3 intervals me baatenge: –∞ to 0, 0 to 1, and 1 to ∞. Pehle aur teesre interval me f(x)=0 to integrand x·f(x) bhi zero, to dono integrals zero honge; second integral bachta hai:

$$f(x) = \begin{cases} \dfrac{3}{2(1-x^2)} & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

evaluate par milta hai (3/2)(1/2 – 1/4)= (3/2)(1/4)= 3/8= 0.375. To gravel ki expected quantity 0.375 units (ya tons) hogi. Is tarah continuous random variable ki expected value hum integrate karke nikalte hain.Yeh main alag se nahi likh raha, beech wale integral ki kahani aur hai isme fx ek formula se diya gaya hai jo hai $3/2(1 - x^2)$ dx, teesra integral zero hai; to humne pehle setup kar liya ab integration ki baat hai; integrate karte hue 3/2 constant bahar aa jayega aur jo bachta hai wo hai $\int(x - x^3)$ dx limits 0 se 1; main direct likh raha hoon: $3/2 [x^2/2 - x^4/4]$ from 0 to 1; evaluate karne par x=0 par zero milega, x=1 par milega 3/8, ya 0.375; is tarah expected values nikaale ja sakte hain continuous random variables ke liye yeh formula use karke. Jaise pehle definition me likha tha continuous random variable = santat ya satat ya yadruccik char; expected or mean value = apekshit maan ya madhya; hum kehte hain ki yeh random variable X ke madhya ka definition hai jab X ek continuous random variable hai. Ab jo steps discrete random variables ke liye kiye the wahi steps continuous random variables ke liye repeat honge—change itna hoga ki pehle PMF se probabilities/expectations nikaalte the—ab PMF replace ho gaya PDF se, aur sum replace ho gaya integral se. Expected value sirf X ki hi nahi, kisi function of X i.e., E(h(X)) bhi nikal sakte hain; agar X continuous random variable hai with PDF f(x), aur h(x) koi bhi function of X hai to expected value of h(X) hogi ∫ h(x) f(x) dx over real line.

$$E\big(h(X)\big) = \int_{-\infty}^{\infty} h(x)f(x)dx$$

Jaise pehle E(X) me h(x)=x tha, ab h(x) kuch bhi ho sakta hai. Upar wale example ko continue karte hain aur expected value of X^2 nikalte hain, $h(x)=x^2$; $E(X^2)=\int h(x)f(x)\,dx$; phir wahi intervals me baat: $(-\infty,0)$, $(0,1)$, $(1,\infty)$; pehle aur teesre zero; beech wala integrate karenge $x^2\cdot(3/2)(1-x^2)\,dx$; integrand banta hai x^2-x^4 times (3/2); integrate karke $3/2[x^3/3 - x^5/5]$ from 0 to 1; result aata hai value=1/5=0.2. Ab variance of X ko define karte hain: $Var(X)=E[(X-\mu)^2]=E(X^2)-\mu^2$, jahaan $\mu=E(X)$; yeh discrete aur continuous dono ke liye same definition—phir difference yahi discrete me sum+PMF, continuous me integral+PDF. Upar wale gravel example me $Var(X)=0.2 - (0.375)^2$ and approximate value aati hai 0.0594. Standard deviation sigma = $\sqrt{Var(X)}$ hamesha positive liya jata hai; is example me sigma = $\sqrt{0.0594}$. Ab second example dekhte hain do species ek resource ke liye compete kar rahe hain; proportion of resource controlled by species-1 ko X se darshaya gaya hai; X ~ uniform(0,1) hai, to PDF f(x)=1 for $x\in[0,1]$, otherwise 0. Je species adhik resource control karegi—us amount ko define kiya gaya hai $h(x)=\max(x,1-x)$. Agar species-1 zyada control me hai to proportion X hai, warna 1–X; isko function-wise likhte hain: $h(x)=1-x$ for $0 \le x \le 1/2$, and $h(x)=x$ for $1/2 \le x \le 1$. Ab expected value of

$h(X)=\int h(x)\, f(x)\, dx = \int h(x)\, dx$

from 0 to 1; phir interval me divide: $\int_0^{1/2} (1-x)\,dx + \int_{1/2}^1 x\,dx$; integrate: $[x - x^2/2]$ from 0 to 1/2 plus $[x^2/2]$ from 1/2 to 1;

evaluate: pehle part se 1/8, second se 5/8, total=6/8=3/4=0.75; to jo species adhik control karegi, uske control me expected resource hogi 0.75; isi tarah se expectations, variance, functions of continuous random variables ke results nikaale jaate hain.