Introduction to Probability & Statistics
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Week - 1
Lecture - 2
Axioms of Probablity

Ab hum thoda sa set theory ke basics samajhne ja rahe hain, kyunki probability me events aur sample space dono sets hi hote hain, aur har event sample space ka ek subset hota hai. To aage badhne se pehle hum sets ke kuch basic operations dekhte hain jo probability me baar-baar use hote hain. Pehla operation hai complement of a set. Agar koi event A hai, to uska complement, jise hum A^c se dikhate hain, ek naya event hota hai jisme wo sabhi outcomes hote hain jo A me nahi hain, lekin sample space Ω me hain. Visually samjhein to Venn diagram me Ω ek rectangle hota hai aur A ek circle. Jo area circle ke bahar hai, wahi A complement represent karta hai. Isi tarah hum samajh sakte hain ki Ω ka complement kya hoga kyunki Ω me saare outcomes hote hain, Ω^c me koi outcome nahi bachega, to Ω^c equal to ∅ empty set. Aur uska reverse bhi sahi hai empty set ka complement Ω hota hai. Ab doosra relation hai Union of two events, jise hum $A \cup B$ likhte hain. $A \cup B$ me wo sabhi outcomes hote hain jo A me hain, ya B me hain, ya dono me hain. Venn diagram me agar A aur B do overlapping circles hain, to dono circles ka total colored area $A \cup B$ hota hai. Aur teesra important relation hai Intersection of two events, jise hum $A \cap B$ likhte hain, jisme wo outcomes hote hain jo A aur B dono me common hain. Ye teen operations complement, union, aur intersection probability theory ke sabse basic aur powerful tools hain, jinhe hum aage ke chapters me baar-baar use karenge jab hum multiple events ke relationships aur unke probabilities calculate karenge.

Ab hum teesra important relation dekhte hain A intersection B, jo ek event hota hai jisme sirf wo outcomes hote hain jo dono events, A aur B, me common hain. Venn diagram se ye concept aur clear hota hai: agar rectangle Ω sample space hai, yellow circle event A ko represent karta hai, aur blue circle event B ko, to dono ke beech ka overlapped green area $A \cap B$ hota hai wahi region jahan dono events ek saath occur karte hain. Ab ek practical example lete hain

$\Omega = \{1,2,3,4,5,6\}$

$E = \{2,4,6\}$ $F = \{2,3,5\}$

$Ec = \{1,3,5\}$ $Fc = \{1,4,6\}$

$E \cup F = \{2,3,4,5,6\}$

$E \cap F = \{2\}$

To E ∩ F ek simple event hai kyunki usme sirf ek outcome hai. Aap aise aur examples khud practice kar sakte hain A, B ke liye unke complements, unions, aur intersections likhkar. Ab hum probability theory me is baat par focus karte hain ki jab hum koi random experiment karte hain, to uska exact result pehle se nahi pata hota, aur isi uncertainty ko hum ek number ke form me measure karte hain usse kehte hain probability of an event, jise hum likhte hain PA. Ye PA basically batata hai ki event A ke hone ki chance kitni hai yaani ye uncertainty ko quantify karta hai. Iske baad hum ek aur definition dekhte hain Disjoint or Mutually Exclusive Events, matlab aise do ya zyada events jinka koi common element nahi hota, yani A ∩ B equal to empty. aise events jo ek saath kabhi nahi ho sakte. Aur isi ke baad hum probability ke buniyadi niyam, yaani Axioms of Probability, ke taraf badhenge. Ab jaise maine thodi der pehle kaha, axioms yaani buniyadi niya ye wo self-evident truths hain jinpar poori probability theory ka foundation bana hai. Inhe hum basic assumptions bhi keh sakte hain. Probability theory me total teen axioms hote hain, aur ye teeno hi is subject ki neeve foundation hain. Axiom 1 kehta hai ki kisi bhi event A ke liye uski probability hamesha zero ya zero se badi hoti hai mathematically, PA greater than equal to 0. Iska matlab ye hai ki probability kabhi negative nahi ho sakti. Agar koi event kabhi ho hi nahi sakta, to uski probability zero hogi; aur agar event ke hone ka kuch possibility hai, to uski probability hamesha zero se zyada hogi. Ye pehla axiom kaafi simple hai har event ki probability non-negative hoti hai.

Axiom 2 ek tarah se ek normalization rule hai. Ye kehta hai ki sample space Ω omega ki probability 1 hoti hai, yaani PΩ equal to 1. Dhyaan rakhiye, Ω wo set hai jisme experiment ke sabhi possible outcomes hote hain. Jab bhi experiment karenge, result hamesha Ω ke kisi element me hi aayega. Iska matlab Ω ek certain event hai ek aisi ghatna jo hamesha hoti hi hai. Isliye uski probability maximum, yaani 1 hoti hai. In dono axioms se hume ye general property milti hai ki har event A ke liye 0 less than or equal to PA ≤less than or equal to 1.

Ab aate hain Axiom 3 par, jise hum kehte hain Countable Additivity Axiom yaganniya yogatmak ka niyam. Ye thoda theoretical hai, par bohot important. Ye kehta hai: agar humare paas ek countable yaani ginne yogya infinite sequence of events ho $A_1, A_2, A_3, \ldots$ jisme sabhi events mutually exclusive hain yaani koi do events ek saath nahi ho sakte, unka intersection empty set hai, $A_i \cap A_j$ equal to phi for $i \neq j$, to fir un sabhi events ke union ki probability unki individual probabilities ke sum ke barabar hogi. Mathematically likhte hain:

$$P(A_1 \cup A_1 \cup A_1 \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Iska matlab agar events ek saath nahi ho sakte, to unke hone ke chances add ho jaate hain. Ye hi hai Countable Additivity in teeno axioms non-negativity, normalization, aur countable additivity ke upar hi poori probability theory ka framework bana hai. Ye teeno niyam uski foundation stones hain, aur inke base pe hum aage probability ke alag-alag properties aur theorems jaise addition theorem, complement rule, conditional probability, Bayes' theorem, etc. ko derive karte hain.

Dhyaan rakhiye, ye countable additivity axiom mein ek zaroori baat hai ye jo collection hai, ye mutually exclusive events ka collection hai. Koi bhi do events lein Ai aur Aj, unka intersection agar lein to wo shunya empty hota hai, matlab ye mutually exclusive hain. Aur jab ye mutually exclusive events hain, to inka union lekar unki probability nikalni ho, to hum har event AI ki probability le kar sabhi probabilities ko jod dete hain. Ye teen jo basic niyam hain, unke aadhaar par hum probability ki aur properties dekh sakte hain. Ye jo axioms hain, ye shuruaati niyam hain, inhe prove nahi karte, ye assumptions hain. Lekin ab jo properties likhenge, unhe hum inhi teen niyamon ke aadhaar par prove karenge. Pehli property hai empty set null event ki probability hamesha zero hoti hai. Isse prove karne ke liye hum set theory ke relations aur axiom 3 countable additivity ka use karte hain. Countable additivity axiom kisi bhi mutually exclusive sequence ke liye lagu hota hai, to hum ek sequence lenge jisme A1, A2, A3... sabhi empty set hain. Ye sabhi mutually exclusive hain kyunki empty set ka intersection empty hi hota hai. Inka union bhi empty set hi rahega kyunki sabhi mein koi element nahi hai. Axiom 3 ke hisaab se,

$$\sum_{i=1}^{\infty} P(A_i) = P(\emptyset) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset) = P(\emptyset) + P(\emptyset) + ..$$

$$P(\emptyset) = \sum_{i=1}^{\infty} P(\emptyset)$$

Left side hai Pempty set, right side hai infinite sum of Pempty set. Matlab humein milta hai Pempty set = Pempty set + Pempty set + ... infinite times. Agar ye number x ho $x \neq 0$, to right side ho jaayega infinity × x, aur left side x. Ye tabhi possible hai jab $x = 0$, kyunki probability infinity nahi ho sakti. Isliye Pempty set equal to 0. Is tarah humne pehli property prove kar li ki empty set ki probability hamesha zero hoti hai. Agli property dekhte hain agar do ya kuch limited events mutually exclusive hon to un par countable additivity axiom ka finite form lagu hota hai. A n probability of A n aur sankshipt roop mein hum likhenge probability of the union of n events A1, A2, A3 ... An.

$$P(A_1 \cup A_2 \cup A_3 \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Matlab probability of union AI, jahan i goes from 1 to n, is equal to summation i equal to 1 to n of PAI. Is statement aur axiom 3 ke statement mein farq ye hai ki axiom 3 mein infinite events hote hain, yani anant ghatnayein, jabki yahan hum sirf finitely many events le rahe hain n number of events, jahan n ek finite sankhya hai. Axiom 3 ek stronger statement hai, lekin usse hum property 2 finite additivity prove kar sakte hain; lekin ulta, finite additivity se hum countable additivity prove nahi kar sakte. Ab dekhte hain ye kaise prove karte hain. Humein axiom 3 use karna hai, iske liye humein ek infinite collection chahiye. To hum A1 se AN tak diye gaye events ke baad AN+1, AN+2, ... sabhi ko empty set maan lenge, jaise humne property 1 ke proof mein kiya tha. Humein pata hai A1, A2, ... AN mutually exclusive hain, aur jab hum unke saath empty sets add karte hain to nayi sequence bhi mutually exclusive rahegi, kyunki empty set ke saath intersection lene par hamesha empty set hi milta hai. Isliye ab hamare paas ek countable collection hai mutually exclusive events ka, jisme hum axiom 3 use kar sakte hain. Axiom 3 ke hisaab se, probability of infinite union = infinite sum of probabilities. Ab infinite union dekhein to ye hoga: A1 ∪ A2 ∪ ... ∪ AN ∪ AN+1 ∪ AN+2 ... Lekin kyunki AN+1, AN+2 sab empty set hain, to empty set ke saath union lene par result wahi pehle wala union rahega, yaani A1 ∪ A2 ∪ ... ∪ AN. Isi tarah, right side mein sum lenge to PA1 + PA2 + ... + PAN + PAN+1 + PAN+2 + ... Lekin PAN+1, PAN+2 sab zero hain, kyunki empty set ki probability zero hoti hai property 1 se. To right side bacha PA1 + PA2 + ... + PAN. Dono sides compare karte hain to milta hai: PA1 ∪ A2 ∪ ... ∪ AN equal to PA1 + PA2 + ... + PAN.

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$$\bigcup_{i=1}^{\infty} A_i = (A_1 \cup A_2 \cup \ldots A_n) \cup (\emptyset + \emptyset + \emptyset)$$

$$= \bigcup_{i=1}^{\infty} A_i$$

$$\sum_{i=1}^{\infty} P(A_i) = P(A_1 \cup A_2 \cup A_3 \ldots)$$

$$= \sum_{i=1}^{n} P(A_i)$$

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

Yahi humein prove karna tha, to proof complete ho gaya. Ab ek example dekhte hain — ek multiple choice question mein candidate ko 4 ya 5 options me se ek sahi option chunna hota hai. Let A be the event ki correct option choose kiya gaya, aur maan lijiye PA = 0.2. Dhyaan rahe, probability hamesha 0 aur 1 ke beech hoti hai. Ab B ho event ki wrong answer choose kiya gaya. Kya hum PB nikal sakte hain? Haan, kyunki A aur B ek dusre ke complements hain A matlab sahi jawab choose kiya, B matlab galat jawab choose kiya. Agar candidate ne ek hi option choose kiya hai, to A aur B complements hain, aur hum likh sakte hain PB equal to 1 – PA equal to 1 – 0.2 = 0.8.

To hamare paas kya hai A union B ka matlab hai ek option choose karna, yani candidate ya to sahi option choose karega ya galat option. Iska matlab A union B poora sample space omega hai, kyunki har situation mein ya to correct answer choose hoga ya wrong answer, aur dono milkar saare possible outcomes cover karte hain. Ab A intersection B dekhein to agar A event ghata hai, matlab sahi option choose hua hai, to wahi option galat nahi ho sakta; ek option ek saath sahi aur galat dono nahi ho sakta. Isliye A intersection B empty set hai, yani A aur B mutually exclusive hain. Property 2, jise hum finite additivity bhi kehte hain, kehta hai ki finitely many mutually exclusive events ke union ki probability unki individual probabilities ka sum hoti hai. To property 2 ke anusaar, probability of omega A union B = probability of A + probability of B. Ab is equation mein teen terms hain — Pomega, PA, aur PB. Humein pata hai Pomega = 1 Axiom 2 ke anusaar, aur PA = 0.2 example se diya gaya. To PB equal to 1 – 0.2 = 0.8. Is tarah humne example se dekha ki in axioms aur properties ka use karke hum unknown probabilities nikal sakte hain.

Ab agar is example ko thoda general roop mein dekhein to ye hamesha sach rahega ki koi bhi event A aur uska complement A′ A complement mutually exclusive hote hain, kyunki A complement ki paribhasha hi ye hai ki usmein koi bhi result A ke saath common nahi hota. Saath hi, A aur A complement ka union hamesha sample space omega hota hai, kyunki dono milkar sabhi possible outcomes ko cover karte hain. Isse ek aur property milti hai: for any event A, PA + PA′ equal to Pomega. Aur kyunki Pomega = 1, to rearrange karne par milta hai PA′ equal to 1 – PA.

Ab agli property dekhein for any event A, probability of A hamesha 1 se chhoti rahegi. Axiom 1 aur property 4 ke milan se milta hai ki probability kisi bhi event ki 0 aur 1 ke beech rahegi. Iska proof: property 3 se hum likh sakte hain PA equal to 1 minus PA′. Axiom 1 ke anusaar, PA′ ek non-negative number hai, yani PA′ greater than or equal to 0. Jab hum 1 me se ek non-negative number ghatate hain, to result hamesha 1 se kam ya barabar aata hai. Isliye PA less than or equal to 1. Aur kyunki probability kabhi negative nahi hoti, PA greater than or equal to 0 bhi hai. Isliye final statement hai 0 less than or equal to PA less than or equal to 1. Yahi property 4 ka proof complete ho gaya.