Introduction to Probability & Statistics
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Week - 4
Lecture -15
Variance

Hum koi bhi discrete random variable X le jiska PMF ho P(X), to uski expected value ya mean nikaalne ke liye hum X ki sabhi values lete hain, unhe corresponding probabilities se multiply karte hain aur sab results ko jod dete hain;

$$E(X) = \sum_x x\, p(x)$$

jaise agar X = -1, 0, 1 ho with probability 1/3 each, to E[X] = (-1)(1/3) + 0(1/3) + 1(1/3) = 0; lekin agar humein X ke function ka expected value chahiye, jaise $Y = X^2$, to Y ke possible values honge 0 aur 1, jisme Y=0 tab jab X=0 (probability 1/3), aur Y=1 tab jab X=±1 (probability 2/3), to E[Y] = 0 × (1/3) + 1 × (2/3) = 2/3; isse ye samajh aata hai ki expected value of X^2 ko hum direct sum $x^2 \cdot P(x)$ se nikaal sakte hain bina Y ka PMF alag banaye;

$$E(x^2) = \sum_x x^2 P_x(x)$$

Let X be a descrete r.v. with pmf p(x)

Let h(x) be any function then expected value H(x) is

$$E[h(x)] = \sum_x h(x)p(x)$$

practical example mein dukaan waala agar 3 new products 500 rs per piece mein kharidta hai aur 1000 rs mein bech sakta hai (profit 500 per sold), aur unsold pieces company 200 rs mein buyback karti hai (loss 300 per unsold), to agar X = sold pieces ho aur uska PMF given hai 0:0.1, 1:0.2, 2:0.3, 3:0.4, to profit function h(X) = 500X – 300(3 – X), aur expected total profit E[h(X)] = Σ h(x)·P(x) se nikaala jaa sakta hai. Ki kitne piece bachenge agar X piece beche gaye hain to 3 – X bach jayenge, har bache hue piece ke liye unko milega 200 rupaye, to yeh poori unki aamdani hogi un 3 pieces ke liye, lekin shuruaat me unhone 1500 rupaye already kharch kiye hain, to total profit is formula se dekh sakte hain aur isko simplify karenge to milta hai H(X) = 800X – 900, to expected value of H(X) directly nikaal sakte hain: sum of H(X)P(X), jahan X = 0 se 3 tak; H(0) = 800×0 – 900 = -900, P(0) = 0.1; H(1) = 800 – 900 = -100, P(1) = 0.2; H(2) =

1600 – 900 = 700, P(2) = 0.3; H(3) = 2400 – 900 = 1500, P(3) = 0.4; inko add karne par expected profit aata hai 700 rupaye; is tarah humne profit function define kiya given information ke hisaab se aur uske baad directly expected value of H(X) nikala, bina H(X) ke PMF ke; is example me H(X) = 800X – 900 ek linear function hai jisme A = 800 aur B = -900; proposition kehta hai ki linear function ke expectation ke liye E(AX+B) = A·E(X) + B; isko derive kar sakte hain: E(AX+B) = sum over x of (AX+B)P(x), bracket expand karte hain to milta hai AXP(x) + BP(x), is sum ko do alag sums me tod dete hain:

$$E[aX + b] = aE(x) + b$$

$$E[aX + b] = \sum_x (ax + b)p(x)$$

$$= \sum_x axp(x) + bp(x)$$

$$= \sum_x axp(x) + \sum_x bp(x)$$

A·sum XP(x) + B·sum P(x), A aur B constants bahar aa jate hain, sum XP(x) hota hai expected value of X ya E(X), aur sum P(x) = 1 hota hai, to final result milta hai: A·E(X) + B; is tarah yeh proposition proven ho jata hai; pehle wale example me hum expected value of X nikaal kar phir 800·E(X) – 900 directly use karke same answer paa sakte the; aage definition ki taraf badhte hain: expected value se hume middle value milti hai lekin yeh nahi pata chalta ki random variable ke possible values kitni spread hain, iske liye hum dispersion measure variance use karte hain; variance of X with PMF P(X) and mean μx is defined as

$$V(X) = E[(X - \mu_x)^2]$$

yeh function X minus mean ka square lekar uski expected value hoti hai; variance ko sigma square se darshate hain, aur standard deviation (maanak vichalan) ko sigma se, jo hota hai square root of sigma square; variance hamesha ≥0 hota hai aur standard deviation bhi ≥0 hota hai; agar random variable constant ho to variance = 0, otherwise positive hota hai; examples: random variable X with values 1,2,3 and probabilities 0.2,0.6,0.2 and Y with values 1,2,3,4 and probabilities 0.4,0.3,0.2,0.1; muX = 1×0.2 + 2×0.6 + 3×0.2 = 2, aur muY = 1×0.4 + 2×0.3 + 3×0.2 + 4×0.1 = 2; dono ka mean 2 hone ke baavajood, Y ka spread zyada hai kyunki X sirf 1,2,3 leta hai, jabki Y 1 se 4 tak leta hai, to Y ka variance zyada hoga; analysis karne par ye baat confirm hoti hai.variance calculate Kaise karte hain to variance of X to hai expected value of $(X – \mu x)^2$ aur μx ki value humne nikali hai 2, so X = 1 se 3 tak hum lenge $(X – 2)^2$ P(X); jab X=1 hoga to $(1–2)^2$ =1 hota hai, P(1)=0.2; jab X=2 hoga to $(2–2)^2$ =0 aur middle term 0; jab X=3 hoga to $(3-2)^2=1$, $P(3)=0.2$ to ho gaya 0.4; isi tarah variance of Y bhi nikalein ge $E[(Y - \mu_y)^2]$, aur $\mu_y=2$, to Y =1 se 4 tak $(Y-2)^2$ P(Y); aap exercise ke roop me calculate kar sakte hain aur check karein ge to iska maan 1 aata hai jo variance of X se zyada hai, jo dikhata hai ki Y ka spread zyada hai; ab ek proposition dekhte hain: variance nikaalna har value pe $(X - \text{mean})^2$ karna thoda lamba process hota hai, iske liye shortcut formula use kar sakte hain: $\mathrm{Var}(X) = E(X^2) - (E(X))^2$; isko proof karte hain variance ki definition se: $\mathrm{Var}(X) = E[(X - \mu_x)^2]$; bracket expand karte hain: $X^2 - 2X\mu_x + \mu_x^2$; expectation apply karte hain linearity ke saath: $E(X^2) - 2\mu_x \cdot E(X) + \mu_x^2$; lekin $E(X)= \mu_x$, to middle term ho gaya $2\mu_x^2$, to final result: $E(X^2) - 2\mu_x^2 + \mu_x^2 = E(X^2) - \mu_x^2$;

$$V(X) = E[(X-\mu_x)^2] = E(X^2 - 2X\mu_x + {\mu_x}^2)$$

$$= E(X^2) - 2\mu_x E(X) + {\mu_x}^2$$

$$= E(X^2) - 2{\mu_x}^2 + {\mu_x}^2$$

isse proposition prove ho gaya; ab previous example me random variable Y with values 1,2,3,4 and probabilities 0.4,0.3,0.2,0.1 lete hain; E(Y)=2 jaise pehle nikala gaya, to $\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$; yahan $E(Y^2)= 1^2\cdot 0.4 + 2^2\cdot 0.3 + 3^2\cdot 0.2 + 4^2\cdot 0.1 = 0.4 + 1.2 + 1.8 + 1.6 = 5$; to $\mathrm{Var}(Y)=5-4=1$; yahan subtraction sirf ek baar karna pada, aur calculation kam ho gayi; ab ek aur proposition — variance of $AX + B = A^2\cdot \mathrm{Var}(X)$; yani B ko add karne se variance par koi effect nahi hota, lekin A se multiply karne par variance A^2 times multiply hota hai; proof: $\mathrm{Var}(AX+B)=E((AX+B)^2)-(E(AX+B))^2$; bracket expand karte hain: $A^2X^2+2ABX+B^2$; aur $E(AX+B)=A\cdot E(X)+B$; expectation apply karne pe: $A^2E(X^2)+2AB\cdot E(X)+B^2 - [A\cdot E(X)+B]^2$; isko expand karenge to cross terms cancel ho jate hain aur final result milta hai $A^2[E(X^2) - (E(X))^2] = A^2\cdot \mathrm{Var}(X)$; standard deviation ke liye $SD(AX+B)=\sqrt{\mathrm{Var}(AX+B)}=\sqrt{(A^2\ \mathrm{Var}(X))}$, jo hota hai $|A|\cdot SD(X)$, kyunki hum SD ke liye hamesha positive square root lete hain.