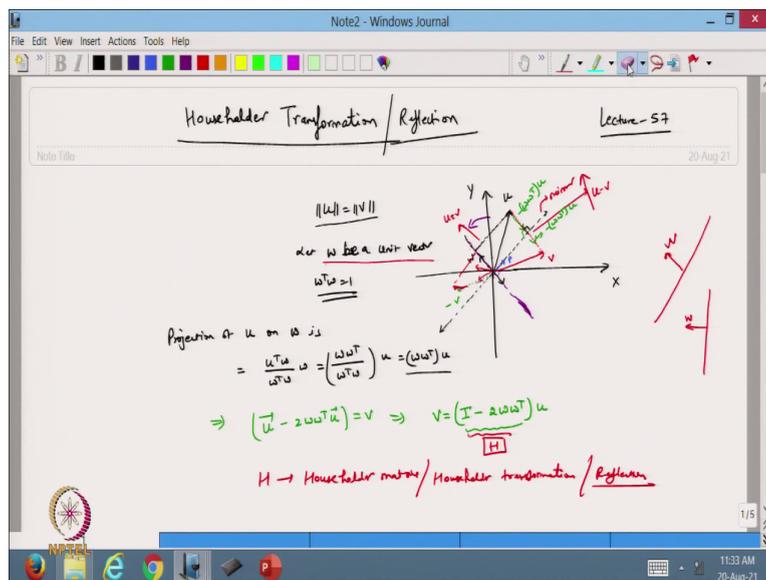


Matrix Computation and its applications
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Lecture - 57
Householder transformation

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Hello viewers, welcome back to the course on Matrix Computation and its Application. So, today, we are going to discuss the last topic of this course, which is Householder Transformation or the reflection.

So, today we are going to discuss, what is a householder transformation or reflection. So, let us suppose a $X - Y$ plane and vector u, v .

Now, I choose a plane $X - Y$ is handled by vector w which is perpendicular to this plane. So, this is my plane or the mirror we also call it. So, if I change the value of direction of w , then the plane will be this one. So, everything depends on this plane.

I call it reflection also or not the reflection I should call it mirror. Why is it called a mirror? Because I want to take the image of u vector, let v is the image of u .

So now, from here I can write that norm of u vector should be equal to the norm of v , it means their magnitude should be the same. And this is possible if I will choose this as a mirror and I will have the reflection of u and that is v . It means that v & u are perpendicular to each other.

$$\|u\| = \|v\|$$

Now, the magnitude of u is same as the magnitude of v and now, I am taking w a unit vector.

Let w be the unit vector such that

$$ww^t = 1$$

Now, let us see the projection. So, projection of u on w is. So, this is suppose my w and I want to take the projection of this one. So, this is my projection of this u on this w and this projection I know that we can define this projection.

Projection of u on w is :

$$\frac{u^t w}{ww^t} w = \frac{w^t u}{ww^t} u = (ww^t)u$$

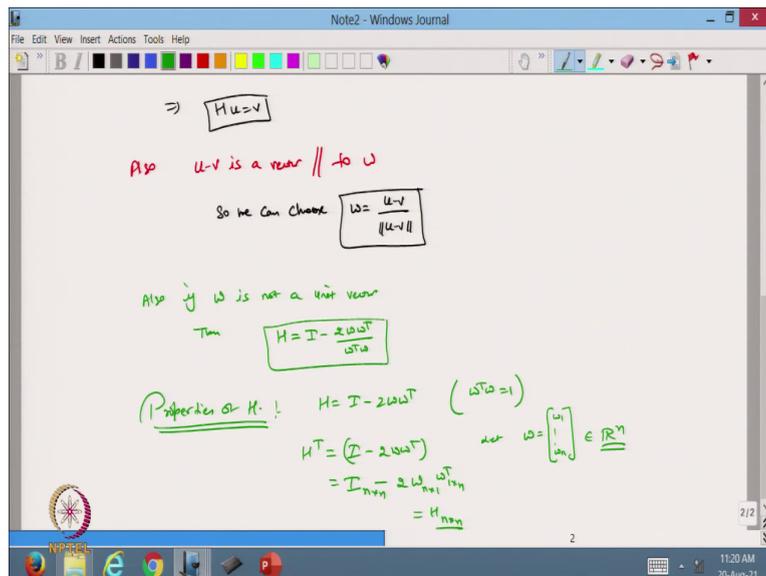
So, this is my projection of u on w . So, that is what we have already seen when we define the projection over a line. Now, you can see that this vector, if I take this vector it will have the length and we are going in this direction. So, I can write that this is equal to $w^t u$. So, this is with the negative sign and I can write as $-w^t u$. This is also $-ww^t u$ in this direction.

Now, from here let us see. I can write from here that u I take this triangle that is a right angle triangle. Now, I can write it as $u - 2 * ww^t u$. So, I am going in the opposite direction. So, this is if I write this is equal to v .

$$v = (\vec{u} - 2ww^t \vec{u}) = (I - 2ww^t)(\vec{u})$$

So, from here I take that this is my $H=(I - 2ww^t)$ and I call this H as a householder matrix. I also call it householder transformation or I also call it reflection. So, this is a reflection matrix.

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So, I can say that Hu is equal to v . Now, the thing is that, one thing also I can see from here that this vector is v and this. So, I can call this vector as $u-v$ because the vector in this direction is parallel to w .

$$Hu = v$$

Also $u - v$ vector is parallel to w . So we choose

$$w = \frac{u-v}{\|u-v\|}$$

If w is not a unit vector then,

$$H = I - \frac{2ww^t}{w^t w}$$

$$H = I - 2ww^t$$

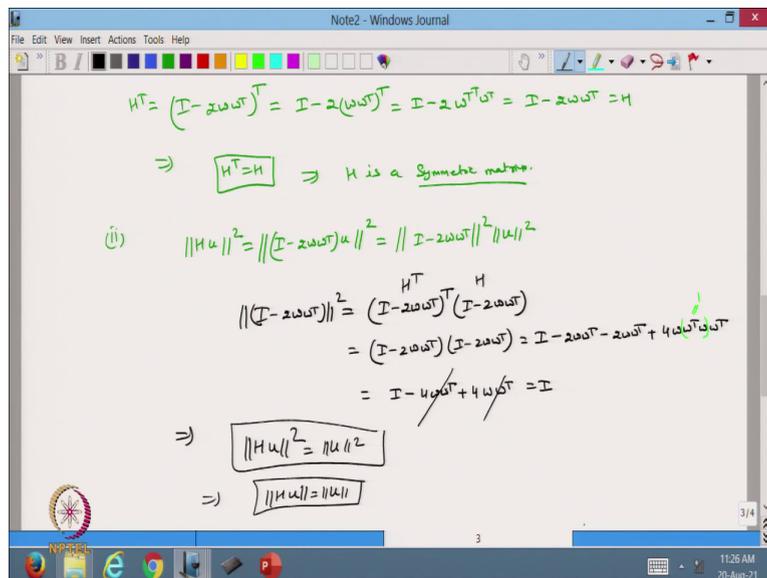
$$H^t = I_{n*n} - 2w_{n*1} w_{1*n}^t = H_{n*n}$$

Now, some properties I want to discuss about this matrix H . Now, if you see from here then this is my v and if you can go back in this direction. So, I can take this $-v$ and if you see from here I can choose this as a mirror and if you see this and maybe I can just I have taken in this direction. So, maybe I can take this v is the.

So, this $-v$ is the image of this u . So, this is my u I am taking the image in this direction now. So, I can say that $-v$ is the image of u when we take this mirror. So, this is the, another mirror that is perpendicular to this one.

Let us discuss some properties of H

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First property: H is a symmetric matrix.

Then I want H^t . So, I can write this as a, $(I - 2ww^t)^t$

$$H^t = (I - 2ww^t)^t = I - 2(ww^t)^t = I - 2w^t w^t = I - 2ww^t = H$$

So, that implies that H is a symmetric matrix.

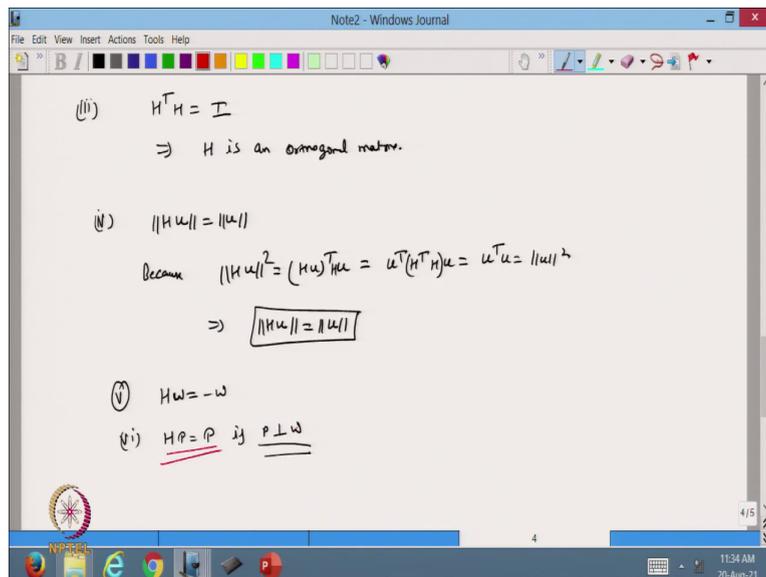
Second Property: $\|Hu\| = \|u\|$

$$\|Hu\|^2 = \|(I - 2ww^t)u\|^2 = \|(I - 2ww^t)\|^2 \|u\|^2$$

$$\begin{aligned}
\|(I - 2ww^t)\|^2 &= (I - 2ww^t)^t(I - 2ww^t) \\
&= I - 2ww^t - 2ww^t + 4w(w^t w)w^t \\
&= I - 4ww^t - 4ww^t \\
&= I \\
\|Hu\|^2 &= \|u\|^2 \\
\|Hu\| &= \|u\|
\end{aligned}$$

So, from here I can write that Hu norm is equal to the norm of u . So, this one we can have.

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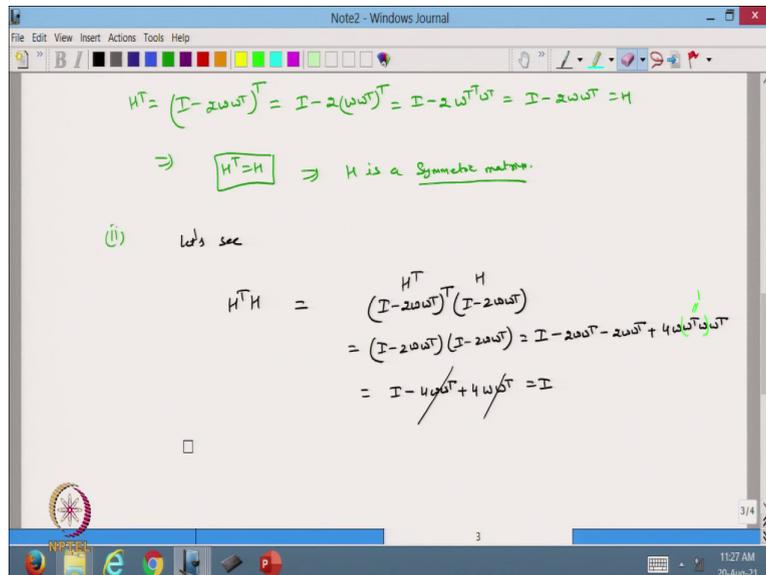


Third property: H is orthogonal

$$\begin{aligned}
H^t H &= (I - 2ww^t)^t(I - 2ww^t) \\
&= I - 2ww^t - 2ww^t + 4w(w^t w)w^t \\
&= I - 4ww^t - 4ww^t \\
&= I
\end{aligned}$$

which implies that H is an orthogonal matrix.

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So, I have taken $H^t H$ and this value we have taken and we found that this is equal to I . So, let us see what we are going to write now.

So, we have just written this value and from here we got this, and now. So, from here we got that H is orthogonal matrix and so, if H is orthogonal matrix, then from here we can say that Hu is equal to u . That is the property of the orthogonal matrix, because I can define from here because $(Hu)^2$ can be written as $(Hu)^t Hu$.

Fourth Property:

$$Hw = -w$$

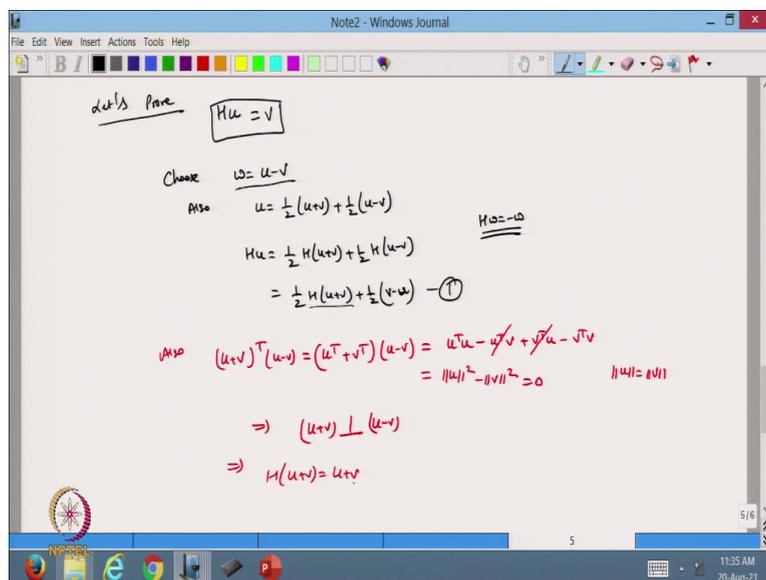
If you see from here if I take the H and applying on w . So, what will happen if I take Hw ? And if you see from here Hw will be just in this direction because this H_1 i am taking here. I am going in this direction by taking the reflection. So, I can write from here then this is equal to $-w$.

Fifth Property:

$$Hp = p \text{ if } p \perp w$$

I take H apply on some vector p and if I take the vector p 's what will happen if I take a vector let us say that vector p is perpendicular to w . So, if I take the vector P that is perpendicular to w it means it will lie in this mirror or in this plane.

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Next thing is the main thing we want to discuss. So, let us see what will happen when I apply on Hu , because I want to show that it should be equal to v . So, I want to show that my Hu is equal to v .

Choose $w = u - v$

Also $u = \frac{1}{2}(u + v) + \frac{1}{2}(u - v)$, implies

$$Hu = \frac{1}{2}H(u + v) + \frac{1}{2}H(u - v), \text{ and } Hw = -w$$

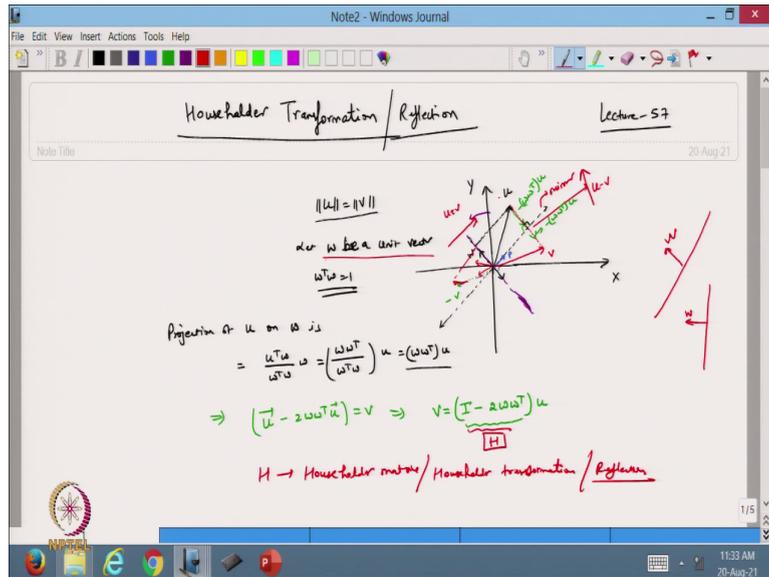
Also, $(u + v)^t(u - v) = (u^t + v^t)(u - v) = u^t u - u^t v + v^t u - v^t v$

$$= \|u\|^2 - \|v\|^2 = 0 \text{ as } \|u\| = \|v\|.$$

This implies $(u + v) \perp (u - v)$

$$H(u + v) = u + v$$

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Now, $u^T v$ or $v^T u$ both are the same. It is just taking the dot product. So, now, from here I can write that this is equal to $u^2 - v^2$ and I know that this is equal to 0, because the magnitude of u is same as the magnitude of v .

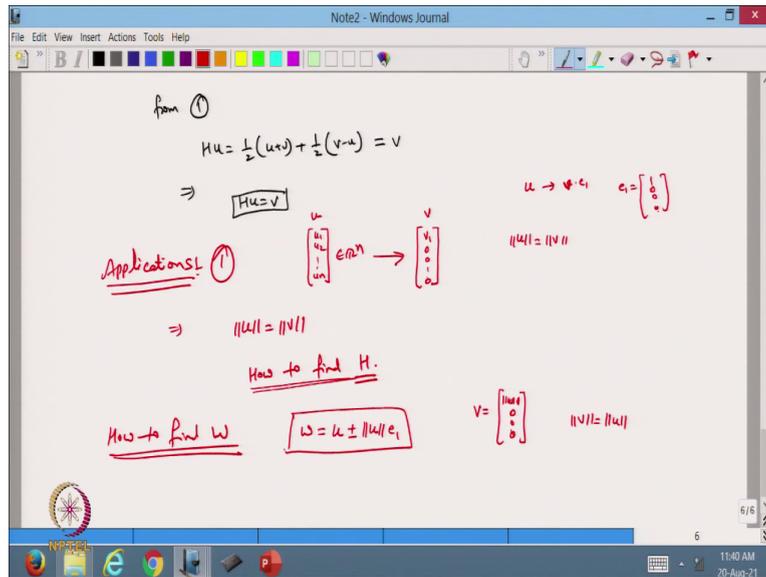
$$\begin{aligned} \text{Also, } (u + v)^t (u - v) &= (u^t + v^t)(u - v) = u^t u - u^t v + v^t u - v^t v \\ &= \|u\|^2 - \|v\|^2 = 0 \text{ as } \|u\| = \|v\|. \end{aligned}$$

This implies $(u + v) \perp (u - v)$

$$H(u + v) = u + v$$

So, from here I can say that this vector $u + v$ is perpendicular to $u - v$ and from the property, this property that any vector, which is perpendicular to w and w is my $u - v$. So, which implies that $H(u + v)$ will be $u + v$.

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So, I can write that from 1 we get Hu becomes $\frac{1}{2}H(u + v)$ is $u + v$. So, I can write this as a $\frac{1}{2}u + \frac{1}{2}v$ and this $v - u$. So, that is $v - u$. So, it becomes now $\frac{1}{2}u$ will cancel out and you will get the v . So, from here we will get Hu is equal to v . So, this is the way we have defined this one.

So, the first application is to suppose that I have a vector. Suppose I am taking u_1, u_2, \dots, u_n belongs to R_n and suppose, I want to convert this vector into another vector. So, that is for my u . So, I want to convert this in another vector v .

Suppose, if v vector has only one and all other elements are 0 it means I am mapping my u into some vector v , which has the same magnitude. So, I can write that v into e_1 where e_1 is just a unit vector $(1 \ 0 \ 0 \ 0)$. And other condition is that the norm of u should be equal to the norm of v . So, this is my v basically.

So, what are the concepts or the things involved and the norm of u should be equal to the norm of v . So, how to find H because to find H_1 need to find w and we have seen that the w is parallel to $u - v$.

So, let us see that how to find H . So, these are few steps that are involved in this case that how to find the value of H . So, this is the procedure.

Now, first thing is that how to find w .

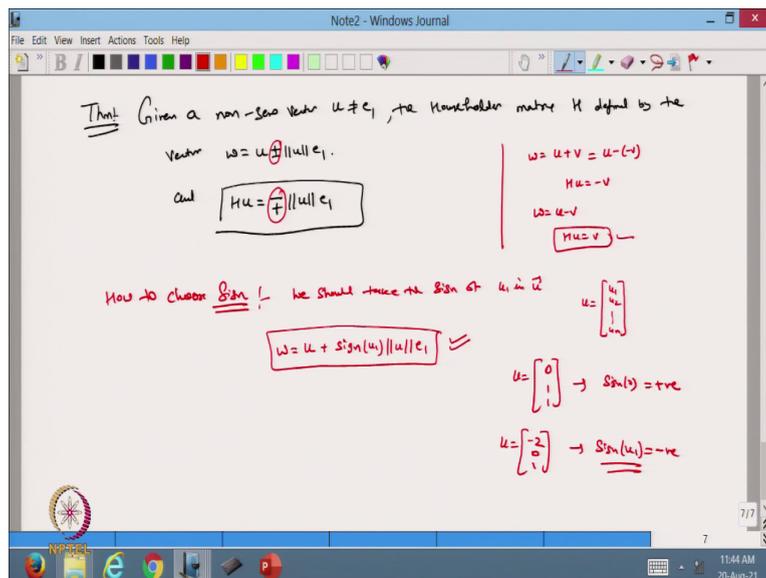
$$w = u \pm \|u\|e_1$$

and

$$Hu = \mp \|u\|e_1$$

So, you can see from here that the norm of v is equal to the norm of u . And here we are writing plus minus because we have seen either it is u minus v or it can be u plus v , if you are taking it in this direction. So, then the w will be in the terms of P . Now, I want to find this value. So, what we are going to write here, let us see this.

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So, I will write one theorem that “Given a non-zero vector $u \neq e_1$, the householder matrix H defined by the $w = u \pm \|u\|e_1$ ”. given a non-zero vector these the household matrix as defined by vector and $Hu = \mp \|u\|e_1$.

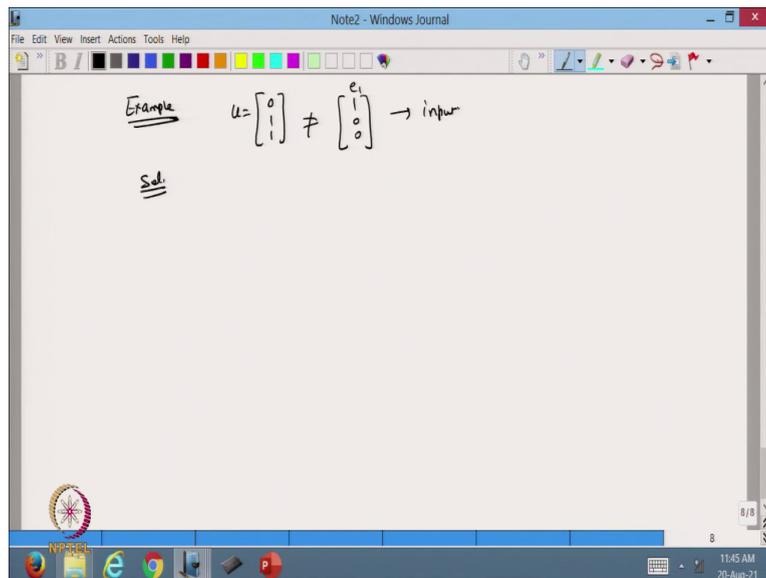
So, here we are writing plus minus if I take w is equal to $u + v$ it means in this case my u is minus v . So, H of u should be in this case minus v . So, I am taking the v here plus sign. So, I

should be taken minus here. If w is u minus v then my H of u should be v because in this case my v is already negative.

So, this is that we should, we should choose we should take the sign of so, whatever the sign of u_1 ; u_1 is basically if I my u is this one, so u_1, u_2 up to u_n . So, I should take the sign of the same as the u_1 is there. So, we should take the sign of u_1 in vector u . It means that my w is always u plus minus sign or not the plus minus I should write now I should write just plus the sign of u_1 whatever the sign is this u_1 . So, that should my w .

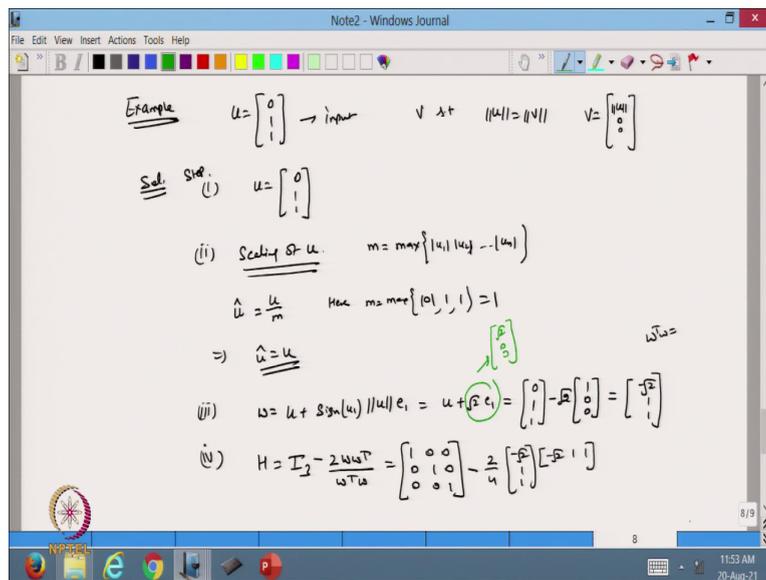
Suppose, my u is $0 \ 1 \ 1$. So, in that case the sign of 0 is positive. So, I should take the positive value. Let my u is $\text{minus } 2 \ 0 \ 1$ then in this case I will take the sign of u_1 is negative. So, I have to take the negative sign here. So, this is the way we can choose the w . Now, let us solve one example that how we can find out this value. So, let us take one example

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Suppose, I take u as $(0 \ 1 \ 1)$ and I want to convert this into $(1 \ 0 \ 0)$, means I want to reflect this one on e_1 such that the vector norm u and the vector v should be the same. So, this is what we are going to do.

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This is my u_1 and vector v such that, the norm of u is equal to norm of v and my v is basically the u 0 0. So, the step 1, my u is $(0 \ 1 \ 1)$. Step 2: I will take the scaling of u because just for the numerical stability we will do the scaling of u . We find the m . So, that is the maximum norm of u_1, u_2, \dots, u_n .

$$\text{Step 1: } u = (0 \ 1 \ 1)^t$$

$$\text{Step 2: Scaling of } u, \quad m = \max\{|u_1|, |u_2|, \dots, |u_n|\}$$

$$\hat{u} = \frac{u}{m} \quad \text{here} \quad m = \{|0|, 1, 1\} = 1$$

$$\text{implies:} \quad \hat{u} = u$$

$$\text{Step 3: } w = u + \text{sign}(u_1)\|u\|e_1 = u + \sqrt{2}e_1 = (-\sqrt{2} \ 1 \ 1)$$

$$\text{Step 4: } H = I_3 - \frac{2ww^t}{w^t w} = [1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1] - \frac{2}{4}[-\sqrt{2} \ 1 \ 1] [-\sqrt{2} \ 1 \ 1]$$

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Handwritten mathematical derivation in a Windows Journal window:

$$\Rightarrow H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & 1 \\ \sqrt{2} & 1 & 1 \end{bmatrix} \quad \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$H = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & -1/2 & 1/2 \end{bmatrix} \rightarrow \text{Householder matrix}$$

$$Hu = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = -v$$

\Rightarrow $Hu = -v$ and $\|u\| = \|v\|$

So now, from here I am able to find my H is

$$H = [1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1] - \frac{1}{2} [2 \ -\sqrt{2} \ -\sqrt{2} \ \sqrt{2} \ 1 \ 1 \ \sqrt{2} \ 1 \ 1]$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2}/2 & 1/2 & -1/2 & -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2}/2 & -1/2 & 1/2 & -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

So, this is my H and you can see from here that H is a symmetric matrix and also the orthogonal matrix. So, H is my householder matrix.

So, from here you can say that my Hu is $-v$. If you see from here, because in this case if

So, in today's lecture we have discussed the transformation that is the householder transformation. And in that transformation we shown that the H is orthogonal, H is symmetric and satisfying some other properties

And, if the vector is a having higher dimension like $5 * 1$ or maybe $10 * 1$ then finding the householder matrix is quite cumbersome. So, we have to take the help of some programming language to find out the householder matrices and its application. So, in the next lecture we will take the help of MATLAB or Octave. So, I hope that you have enjoyed this lecture.

Thanks very much.

