Scientific Computing using Matlab
Professor Vivek Aggarwal
Indian Institute of Technology, Delhi
Department of Mathematics
Lecture 45
Cubic Spline

Hello viewers, welcome back to the course on scientific computing using MATLAB. So, today we will continue with the cubic spline as we have started in the previous lecture.

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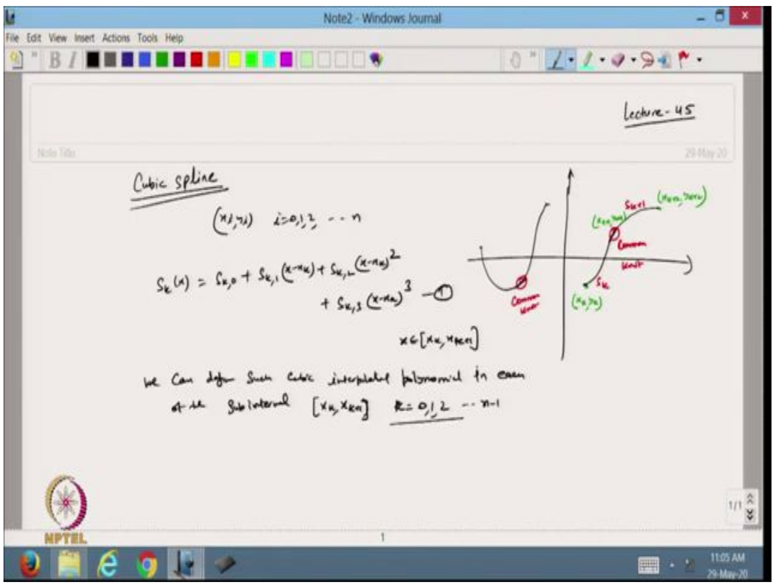

So, today we will discuss the cubic spline. So, as we have discussed that in the previous lecture that we have the data points, few data points are given to us like this one and in each of the sub interval, I want to approximate this with the cubic polynomial such that it satisfies a certain condition at the connecting or the common nodes.

So, like this node I have, so I will define it like this one, then it should go like this. So, the movement at the common node, so that should be smooth so this is a common node. So, let us define this one, so let us say I take maybe, I call it S_k and this is S_{k+1}. It means that this point I am choosing this one is (x_k, y_k) this is (x_{k+1}, y_{k+1}) and this I am taking (x_{k+2}, y_{k+2}).

Now, with this, we have the points $(x_i, y_i), i = 0, 1, 2, \cdots, n.$. Now, define the polynomial $S_k(x)$. So, this is a cubic polynomial I am defining, so I will define

$$S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3, x \in [x_k, x_{k+1}].$$

So, the same I can define in each of the sub-interval so, this is a cubic polynomial. So, I need to find out these 4 coefficients to give me the cubic polynomial in the given sub interval.

So, now this polynomial, so I can define that polynomial. So, we can define such cubic interpolating polynomials in each of the sub-interval $[x_k, x_{k+1}], k = 0, 1, \cdots, n-1.$ So, this way we can define all these n numbers of cubic polynomials.

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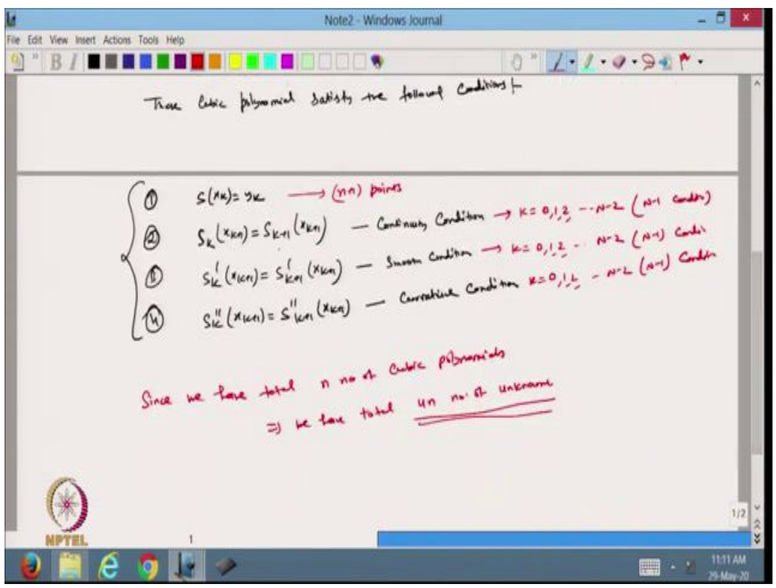

Now, this cubic polynomial satisfies the following conditions.

1)$S_k(x_k) = y_k$

2)$S_k(x_{k+1}) = S_{k+1}(x_{k+1})$ (Continuity condition)

3)$S_k'(x_{k+1}) = S_{k+1}'(x_{k+1})$ (smoothness condition)

4) $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$ (Curvature condition).

So, all these 4 conditions are to be satisfied for the given cubic polynomial defined in each sub interval. So, based on this one I can define.

Now, from here, since we have a total n number of cubic polynomials because in a sub interval I define one cubic polynomial So, I have a n sub interval So, total n number of cubic polynomials which implies we have a total 4n number of unknowns. So, we need to find the total 4n number of unknowns. So, this is the number of unknowns I have to find to define the cubic polynomial in each of the sub intervals that satisfy this condition.

Now, from here if you see, then this condition is satisfied for n+1 points because this is true for each x. This is what I am defining so it should be true for $k = 0, 1, \cdots, n-2$. This is also true for $k = 0, 1, \cdots, n-2$ and this is also true for $k = 0, 1, \cdots, n-2$. So, from here I can say that I will get n-1 condition, n-1 condition, n-1 condition, because I have to satisfy this condition at the common nodes.

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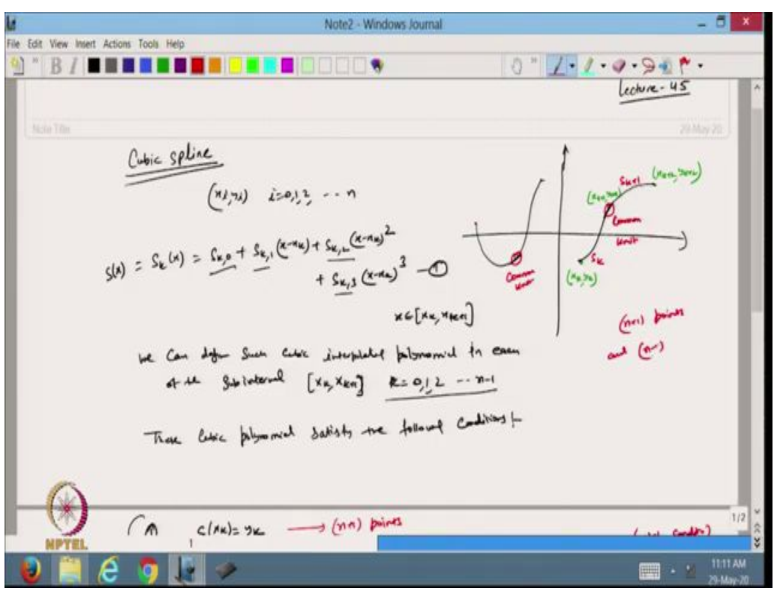

So, in total there are n+1 points and n-1 common nodes so this condition is to be satisfied only at the common node. So, what are the common nodes I am taking, so that is why I am taking n-2. So, n-2+1 is it n-1 so the last will be n-1, So this is a total n-1 condition.

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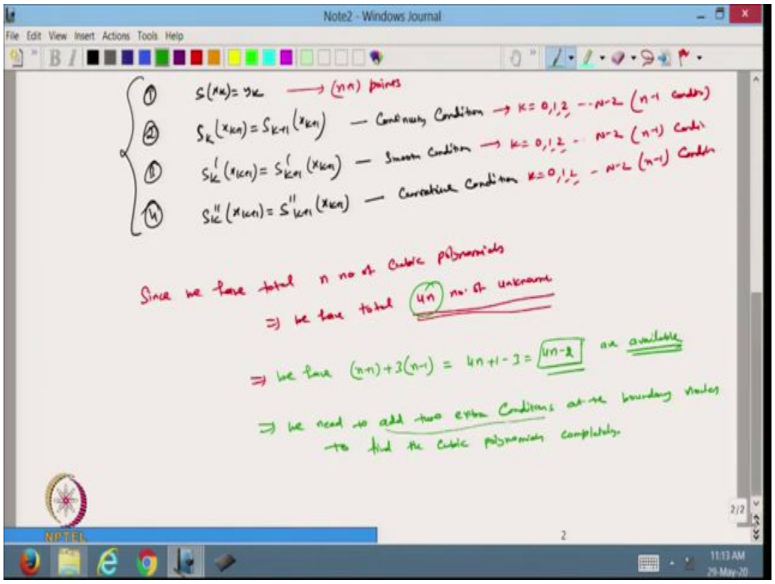

So, from here we have (n+1)+3(n-1) So, from here I can find that this is equal to 4n+1-3, and I can write that this is 4n-2. So, now we have total these conditions available. So, this number of conditions are available to us.

So, total we have 4n number of unknowns, but I have the 4n-22 condition available. So, from here I can say that we need to add 2 extra conditions at the boundary values or boundary nodes to find the cubic polynomial completely so that we have to do. So, these 2 extra conditions will see how we can define these 2 extra conditions to find out the cubic spline. So, let us do that how we can define the cubic spline formula.

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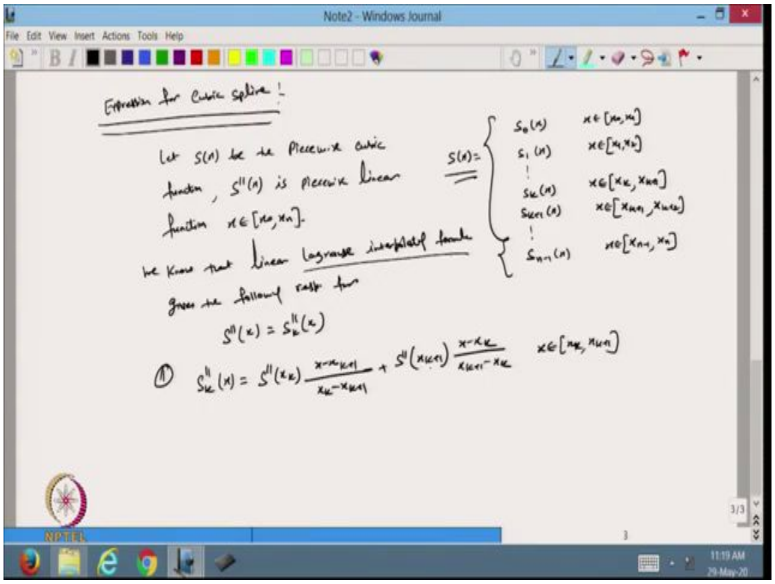

So, let us do that so we define the expression for the cubic spline. Now, let S(x) is the cubic spline for all $x \in [x_0, x_n]$. See, I have defined this S_k that is defined only in the given interval that $[x_k, x_{k+1}]$, but this condition I wrote the S, so it is defining for all complete polynomial cubic polynomials and that is satisfying for each value of k.

So, this is an S_k and so that is I have written S(x), this is equal to

$$S(x) = \begin{cases} S_0(x) & x \in [x_0, x_1] \\ S_1(x) & x \in [x_1, x_2] \\ \vdots & \\ S_k(x) & x \in [x_k, x_{k+1}] \\ S_{k+1}(x) & x \in [x_{k+1}, x_{k+2}] \\ \vdots & \\ S_{n-1}(x) & x \in [x_{n-1}, x_n] \end{cases}$$

So this is my cubic polynomial that is in the piecewise so this is my S(x) and that is a cubic polynomial we have defined in each of the sub-interval so that is why we are defined like this.

So, now let S is the cubic spline is the piecewise cubic function, then if I take S double derivative, then S'' is the piecewise linear function defined on $x \in [x_0, x_n]$, so that is already there. Now, we know that that the linear Lagrange interpolation formula, so we know that Lagrange interpolating formula gives the following representation for, so I am taking the second derivative so at x_k that is equal to this is, i.e. $S''(x) = S''_k(x)$.

So, the first one is,

$$S''_k(x) = S''(x_k) + \frac{x - x_{k+1}}{x_k - x_{k+1}} + S''(x_{k+1})\frac{x - x_k}{x_{k+1} - x_k}, x \in [x_k, x_{k+1}].$$

…(1) So, this we have defined from the concept of Lagrangian interpolating polynomial in the $x \in [x_k, x_{k+1}]$., because in this case if I put $x = x_k$, so this will cancel out, this will be equal to the second derivative of S_k and this will be 0. When I put x_{k+1}, this will be 0 and I will get only this one. So, this is we have defined using the Lagrange interpolating polynomial.

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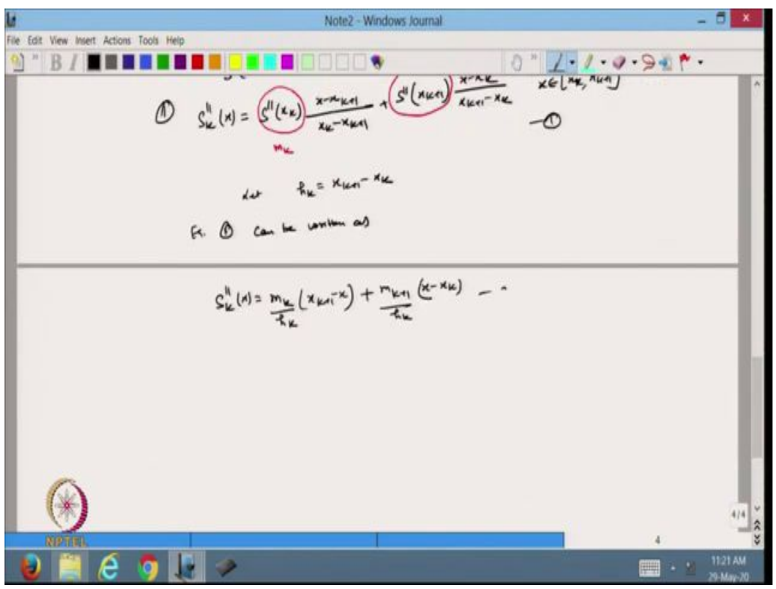

Now, this is a second derivative so we call it m_k this is we call as m_{k+1}. So, this is the corresponding linear interpolating polynomial, the Lagrange interpolating polynomial for the second derivative. Now, let h_k we represent by $h_k = x_{k+1} - x_k$ so from here the equation 1 can be written as $S_k''(x) = \frac{m_k}{h_k}(x_{k+1} - x) + \frac{m_{k+1}}{h_k}(x - x_k) \cdots (2)$ So, we call it equation number 2, this is the way we can define.

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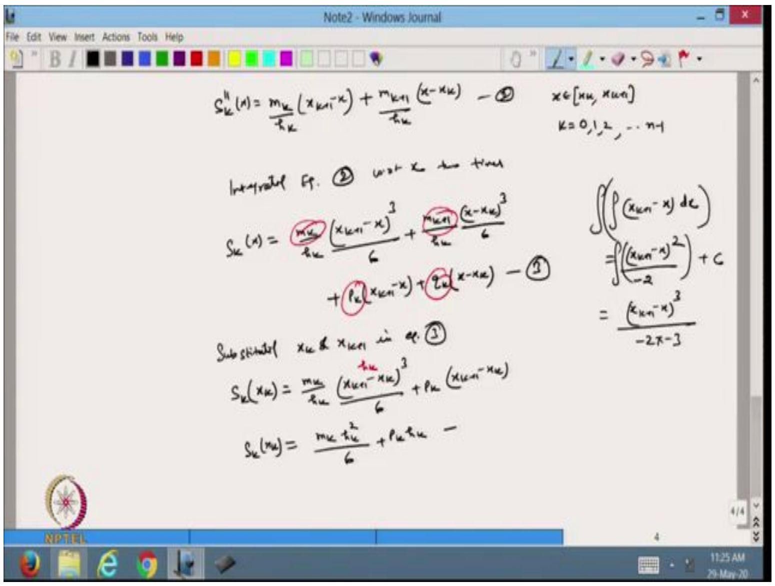

Now, this is true for all $x \in [x_k, x_{k+1}], k = 0, 1, \cdots, n-1.$ Now, integrating equation 2 with respect to x. So, what I do is integrate this 2 times because I want to find what is S_k so 2 times. So, now on the left side, I will get S_k in x. Now, from here I will get $\frac{m_k}{h_k}$, this is the constant value so I am integrating this factor 2 times, so it will be this square.

So, from here I can write that this will be equal to because integrating $(x_{k+1} - x)$ with respect to dx, that is equal to $\frac{(x_{k+1} - x)^2}{-2}$ so it will be this one. And if I am doing again the integration of this, so this will be again $\frac{(x_{k+1} - x)^3}{(-2)(-3)}$

So, I can write that this will be equal

$$S_k(x) = \frac{m_k}{h_k}\frac{(x_{k+1}-x)^3}{6} + \frac{m_{k+1}}{h_k}\frac{(x-x_k)^3}{h_k} + p_k(x_{k+1}-x) + q_k(x-x_k) \cdots (3)$$

So, now we are able to define this cubic polynomial we know and I need to find what is the value of this m_k, this m_k I want to define what is the value of this p_k and q_k. So, these are all things we need to find out. Now, substituting x_k and x_{k+1} in equation number 3, so this is I want to see S at x_k.

So, you could define $x = x_k$,

$$S_k(x_k) = \frac{m_k}{h_k}\frac{(x_{k+1}-x_k)^3}{6} + p_k(x_{k+1}-x_k)$$ So this will be

$$y_k = S_k(x_k) = \frac{m_k h_k^2}{6} + p_k h_k \cdots (4)$$

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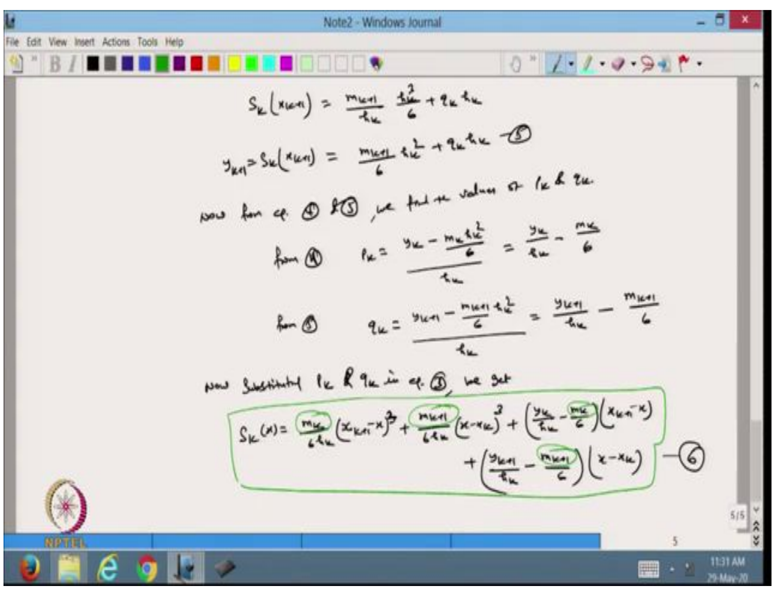

Now, put $x = x_{k+1}$ in equation (3) we will get $S_k(x_{k+1}) = \frac{m_{k+1} h_k^3}{6h_k} + q_k h_k$ So it will be equal to

$$y_{k+1} = S_k(x_{k+1}) = \frac{m_{k+1} h_k^2}{6} + q_k h_k \cdots (5)$$ from equation 4 and 5 so from these 2 equations we find the values of p_k and q_k.

Now, from 4 my p k will be $$p_k = \frac{y_k - \frac{m_k h_k^2}{6}}{h_k} = \frac{y_k}{h_k} - \frac{m_k h_k}{6}$$ and

$$q_k = \frac{y_{k+1} - \frac{m_{k+1} h_k^2}{6}}{h_k} = \frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}$$ Now substituting p_k, q_k in equation (3), we get

$$S_k(x) = \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(x_{k+1} - x) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)(x - x_k) \cdots (6)$$

So, this is the equation I am getting after eliminating the constant value that is p_k and q_k. So, this is the cubic polynomial we are going to get. The only things now we have to define are m_k, m_{k+1}, So, now from here I will try to find out this value of m_k, where we know that this m_k the second derivative at x_k.

So, we will stop here today. And in the next lecture, we will continue with this expression to find out what is the value of this m_k. So, I should stop here today. So, today we have started with the concept of cubic spline and then we have tried to find out how we can define the cubic spline interpolating polynomial for the given data. So, in the next lecture, we will continue with this expression. Thanks for watching. Thanks very much.