

**Scientific Computing using Matlab**  
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**Lecture 44**  
**Spline Interpolation**

Hello viewers, welcome back to the course on scientific computing using MATLAB. So, today we are going to discuss how we can do the piecewise interpolation. So, we start with the linear Spline and then we will go for the cubic Spline.

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Lecture - 44  
29 May 20

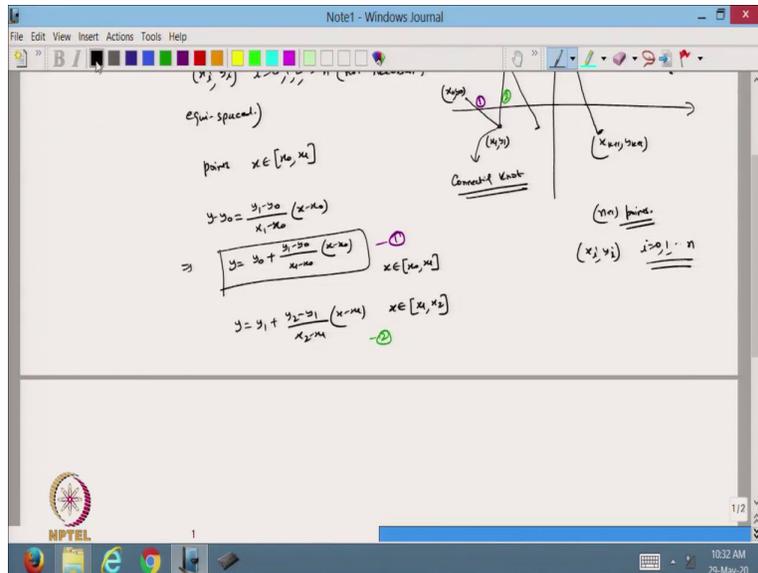
Linear Spline: Let we have data points  
 $(x_i, y_i)$   $i=0, 1, \dots, n$  (not necessarily  
 equi-spaced)  
 point  $x \in [x_i, x_{i+1}]$

$y = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} (x - x_i)$   
 $\Rightarrow y = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i} (x - x_i)$  ①

(n) pairs.  
 $(x_i, y_i)$   $i=0, 1, \dots, n$

Graphical View

The graph shows a coordinate system with a piecewise linear function. The function consists of line segments connecting points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_n, y_n)$ . A point  $x$  is marked on the x-axis between  $x_i$  and  $x_{i+1}$ , and a corresponding point  $y$  is marked on the y-axis, connected by a line segment to the function.



So, let us start with the concept of linear Spline. So, in this case as we know that suppose I have data and some data points are given to me that are not necessarily equi spaced. So, this is my  $(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k), (x_{k+1}, y_{k+1}), \dots, (x_n, y_n)$  So, we have a total  $n+1$  data points and this is given as  $(x_i, y_i), i = 0, 1, \dots, n$ .

So, these data points are given to us. Now, what I do, I try to approximate this one with the help of the piecewise linear function. So, this is my line, then I will take this line, then I will connect these 2 points, I will connect this line and then in the end I will get this line. So, this is the line we are getting from  $x_0$  to  $x_1$  and this you can say as this is the connecting node.

So, we have to take care of what will happen at the connecting node. So, in this case we have that let we have data points  $(x_i, y_i), i = 0, 1, \dots, n$ . not necessarily equi spaced. So, these are not necessarily equi spaced. Now, in this case, I will define the piecewise polynomial that is  $S(x)$  such that. So, now first I will start with the first so, let us try to define what will be the line equation of this line moving from  $x_0$  to  $x_1$ . So, I have the points  $x_0$ , so define the line when  $x \in [x_0, x_1]$ . So, I know how to write the equation of the line, so, like

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

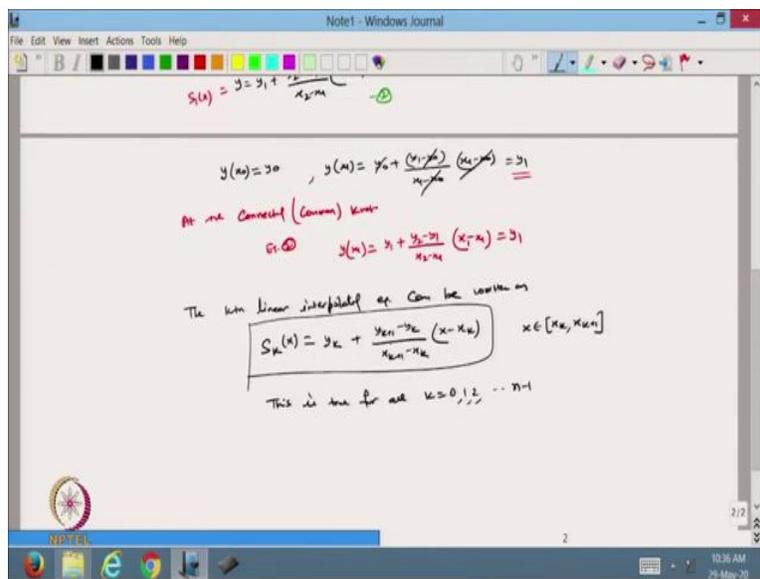
So, this is the way we can define the equation of the line and from here I can write that my y will

be 
$$y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \dots (1)$$

Similarly, the next equation of the line will be so, this is the y when  $x \in [x_0, x_1]$  The same

way I can define 
$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), x \in [x_1, x_2] \dots (2)$$
 So, this way I can define all the lines.

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Now, the thing is that I want to check what will happen at this y at  $x_0$ . So,  $y(x_0)$  I will choose this line like the equation number 1. So, if I put  $x = x_0$ , so, this will cancel and that gives me  $y_0$  so, it is satisfying this one, what would y at  $x_1$ . So, when I put  $x = x_1$  here, I will get  $y(x_1) = y_1$ . So, the equation 1 definitely is passing through this point, it is a line so it will satisfy these conditions.

Now, I want to use what will happen at the connecting node. So, this is my connecting node so at the connecting node or I can say, common node, this is satisfying. What about equation 2? So, in this case, if I put  $y$  at  $x_1$ , what will we get? So, if I come from the left side, I have used this one, and at  $x_1$ , I found that  $y(x_1) = y_1$ .

Now, I want to see what will happen for the next one. So, in this case, I am choosing the equation number 2. So, this would be  $y(x_1) = y_1$ , So, the value of  $y$ , the first linear equation gives the value  $y_1$  and the second one is also given  $y_1$ . So, from here I can write that this equation 1 as I call it as  $S_0(x)$ , and equation 2 I call it as  $S_1(x)$ .

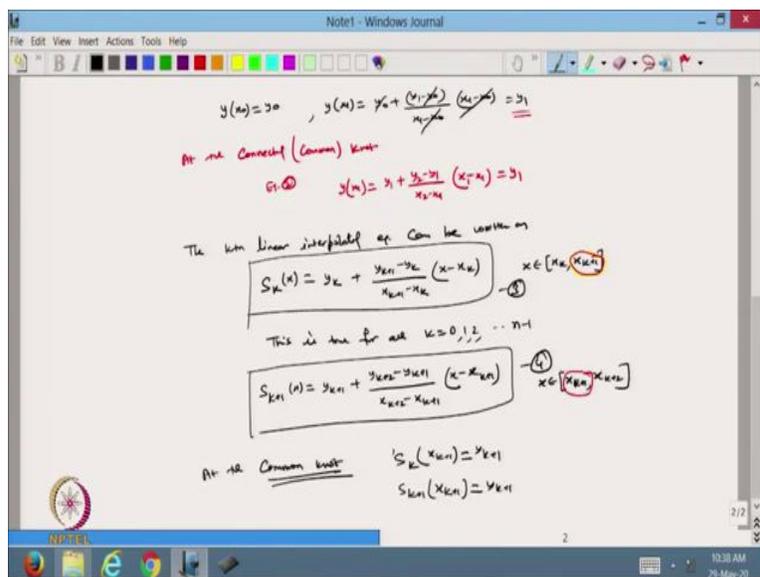
So, in this way I can define, the  $k$ th linear interpolation equation can be written as, so I call it  $S_k(x)$ . So, this is  $S_k(x)$ , so I am talking about the linear so this is equal to

$$S_k(x) = y_k + \frac{y_{k+1} - y_k}{x_{k+1} - x_k}(x - x_k), x \in [x_k, x_{k+1}] \dots (3).$$

So, this is the  $k$ th

linear equation passing through and This is true for all  $k = 0, 1, 2, \dots, n - 1$ . So, therefore, I will get that total  $n$  number of linear equations satisfying this relation.

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Now, the same way can define  $S_{k+1}(x)$ , in the next interval, so, I will get

$$S_{k+1}(x) = y_{k+1} + \frac{y_{k+2} - y_{k+1}}{x_{k+2} - x_{k+1}}(x - x_{k+1}), x \in [x_{k+1}, x_{k+2}] \dots (4)$$

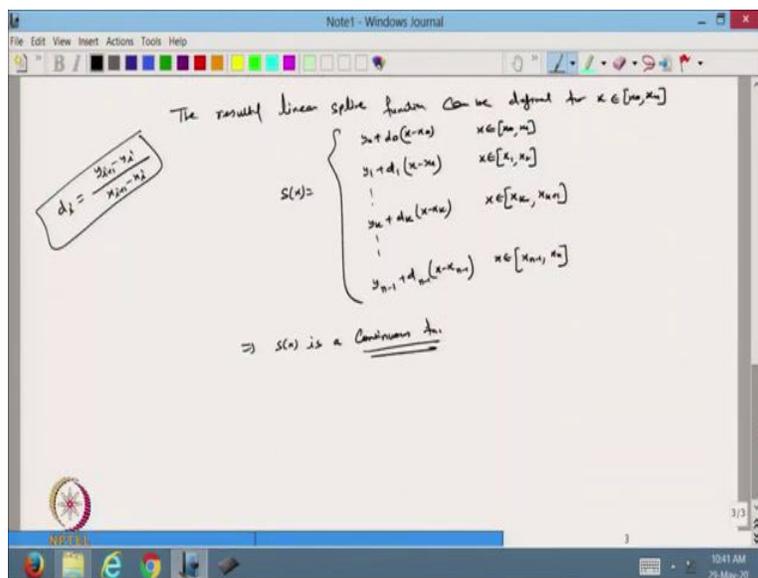
Now,

from here, I can say that at the common node  $x_{k+1}$ , the common node is this one.

So, this is a common node, so I can verify that

$$S_k(x_{k+1}) = y_{k+1}, S_{k+1}(x_{k+1}) = y_{k+1}$$

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So, this is a linear function I have defined, So, now we can write the so, the resulting linear Spline function can be defined for all  $x \in [x_0, x_n]$  such as. So, my  $S(x)$  will be

$$S(x) = \begin{cases} y_0 + d_0(x - x_0), & x \in [x_0, x_1] \\ y_1 + d_1(x - x_1), & x \in [x_1, x_2] \\ \vdots \\ y_k + d_k(x - x_k), & x \in [x_k, x_{k+1}] \\ \vdots \\ y_{n-1} + d_{n-1}(x - x_{n-1}), & x \in [x_{n-1}, x_n] \end{cases}, \quad \text{where}$$

$$d_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, i = 0, 1, \dots, n - 1.$$

So, this is my linear spline and that is we have defined for the whole interval. So, from here I can say that this function  $S(x)$  is a continuous function because at the common node we have seen that if I come from the left side or from the right side that is equal and that is equal to the value of the function given at that point. So, from here I can say that this  $S(x)$  is a continuous function. So, this is my linear splines we have defined so I can take the example also. So, let's take one example as we have taken in the previous lecture also.

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The screenshot shows a Notepad window with the following handwritten content:

Ex Suppose we have data  $f(0)=1$ ,  $f(1)=2$  &  $f(2)=55$   
we can define the linear interpolant Polynomial ①

$S_0(x) = 1 + \frac{2-1}{1-0}(x-0)$

$S_1(x) = 2 + \frac{55-2}{2-1}(x-1) = 2 + \frac{53}{2}(x-1) = 26x - 23$

Verify

$S_0(x=1) = 1 + 2 \times 1 = 1 + 2 = 3$   
 $S_1(x=1) = 26 \times 1 - 23 = 3$

The window also contains a small graph with points (0,1) and (1,2) marked, and a red circle around the point (1,2) labeled "Common Knot".

So, suppose I have the data, suppose we have the data  $f(0) = 1, f(1) = 3, f(3) = 55$ . So, in the previous lectures, we asked about the Lagrange interpolation or the Newton divided difference interpolation for this type of data, when we have 3 points of data. So, now in this case, I want to approximate this with the linear spline.

So, this in my 0, this is value 1 and this is 3 so, this is 1 and this is 3. So, at this point, this is the value given to me, suppose, this is 0 1, but this point is  $x = 0$  and  $y = 1$ , another point is 1 and 3 so suppose this is 1 and 3, and the 3 it is 55 so, suppose it is coming very far so, it is 3 and 55. So, in this case I want to approximate this with the linear spline, so I will take this line and then this line.

So, from here, I can define the first line and the second line. So, from here we can define the linear interpolating polynomial as defined in 1, what will it be? So, I am writing this as  $S_0(x)$ . So this will be,

$$S_0(x) = 1 + \frac{3 - 1}{1 - 0}(x - 0) = 1 + 2x = 2x + 1, x \in [0, 1].$$

Now, the same

way I can define what about  $S_1(x)$ , so  $S_1(x)$  is defined in the next interval, so this will be

$$S_1(x) = 3 + \frac{55 - 3}{3 - 1}(x - 1) = 3 + 26(x - 1) = 26x - 23, x \in [1, 3].$$

So,

that is the equation of the line 2, this is my  $S_0(x)$  and this is my  $S_1(x)$ . Now, from here I can verify what is at  $x = 1$  common node, so this is the common node. So, at the common node  $x = 1$  if I put it here so this will be  $S_0(1) = 1 + 2 = 3, S_1(1) = 26 - 23 = 3$ . So, at this value of the connecting node this one the answer is coming this.

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we can define as

$$S_0(x) = 1 + \frac{2-1}{1-0}(x-0)$$

$$S_1(x) = 2 + \frac{5-3}{2-1}(x-1) = 2 + \frac{5-3}{2}(x-1) = 2 + 2(x-1) = 2x - 2 + 2 = 2x$$

Verify

$$S_0(x_0) = 1 + 2 \cdot 0 = 1 + 0 = 1$$

$$S_1(x_1) = 2 + 2 \cdot 1 - 2 = 2$$

$$\Rightarrow S(x) = \begin{cases} 2x + 1 & x \in [0, 1] \\ 26x - 23 & x \in [1, 3] \end{cases}$$

So, from here I can say that my  $S(x)$ , the piecewise function, will be

$$S(x) = \begin{cases} 2x + 1, & x \in [0, 1] \\ 26x - 23, & x \in [1, 3] \end{cases}$$

. So, that is the answer to the question of how we can approximate the given data with the linear splines functions. So, this is the way we can define the linear spline function. Now, so based on this linear spline function, I want to define the quadratic or the cubic spline. So, in our course, we will talk about the cubic spline.

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Cubic spline:

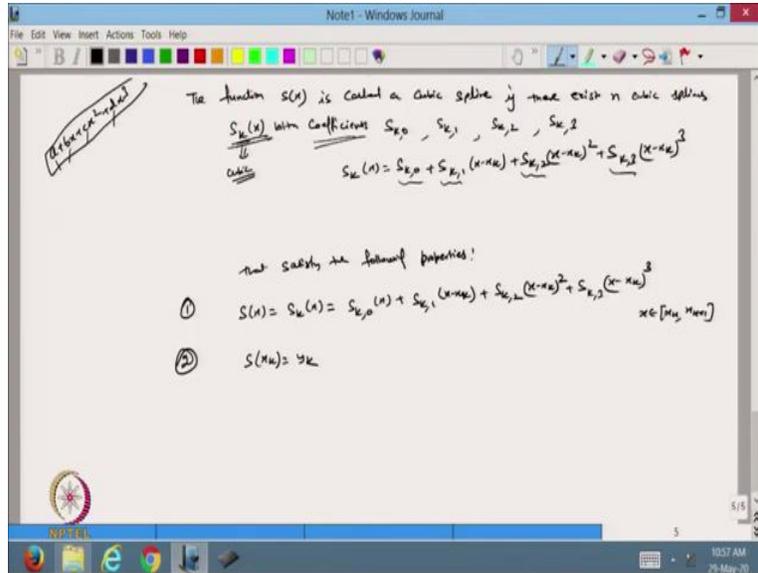
Suppose we have data  $(x_i, y_i)$   $i=0, 1, \dots, n$ .  $(n+1)$  pairs

The function  $S(x)$  is called a cubic spline if there exist  $n$  cubic splines  $S_k(x)$  with coefficients  $S_{k,0}(x), S_{k,1}(x), \dots, S_{k,3}(x), - S_{k,n-1}(x)$

Cubic

$$S_{k,0}(x) \text{ is a cubic function } x \in [x_{k-1}, x_k]$$

$$S_{k,1}(x) \text{ --- } x \in [x_k, x_{k+1}]$$



So, in the cubic spline the same way, suppose I have the data. So, suppose this is value  $(x_0, y_0), \dots, (x_k, y_k), (x_{k+1}, y_{k+1}), \dots, (x_n, y_n)$ . So, this value is given to me. Now, what we want is that so in the previous one we have approximated these 2 points with the line and that linear polynomials we have taken with satisfying the continuity condition at the common nodes and then we are able to define the linear spline.

Now, in this case, what do we want to approximate in each sub interval, I want to approximate this with some cubic function. So, this is a cubic function I have defined. Now, what I want to do is define this cube function in such a way that at the common nodes the tangent should be the same.

So, the first condition is that this function should be continuous. The second condition is that it should satisfy that the derivative on the left side should be equal to the derivative on the right hand side, it means that the function has a smooth derivative here. So, that is the smoothness of the function. And the third one is also that, that if I want to see the curvature, the curvature of this should be the same.

So, if I am taking the curvature from the left side on the right side, I should get the same value.

Similarly, it should happen here, it should happen here so if I define from here, like this one, then suppose next point is this, so if this is taking turn like this one, then in this case, if you see the tangent here is this one, but tangent here is this one, so that is not allowed.

So, our function, the cubic function should be smooth enough. So, I can define this as this one and then go like this one. So, the moving of the function should be smooth from one sub interval to another sub interval. So, that is the definition of the cubic spline. So, let us define what the meaning of cubic spline is? So, now we have, so suppose we have the data  $(x_i, y_i), i = 0, 1, \dots, n$ .

So, this is up to n this is given to me, then I call it that the function  $S(x)$  is called a Cubic spline if there exist n cubic splines, so in this case because we have a total n+1 points. So, in this case we have n sub-intervals, so there exists n cubic splines  $S_k(x)$  with coefficients  $S_{k,0}, S_{k,1}, S_{k,2}, S_{k,3}$  such that

$$S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3$$

In the previous one we have taken this one as  $S_k$  and at the time only I was talking about the linear function, but now we are taking the kth. So, this k means, it should be a cubic and  $S_{k,0}(x)$  basically is a cubic function for  $x \in [x_0, x_1]$  so that is there. So, in this case we can define that  $S_{k,0}$  that is defined in the  $x \in [x_0, x_1]$ . Similarly, if I want to define  $S_{k,k}(x)$  is a cubic function for  $x \in [x_k, x_{k+1}]$  So these are the coefficients I will say that these are the coefficients of the cubic spline and then in this way we can define the cubic functions in each sub interval with the condition that so, this is a coefficient that satisfies the following properties.

So, these are the properties to be satisfied, so let I call it  $S(x)$ , the cubic spline. So, this is

$$1) S(x) = S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3, x \in [x_k, x_{k+1}].$$

2)  $S(x_k) = y_k$ . So, at each of the points, this point it should be the interpolating polynomial, it should interpolate the polynomial. So, these are the following three more

properties we have to discuss, so that we will discuss in the next lecture.

So, I will stop here today, and then we will continue with the same cubic spline in the next lecture. So, today we have started with the linear spline interpolating polynomial for the given data and we have tried to solve one example also. And now we will move to the cubic spline. So, that we will continue in the next lecture. So, thanks for watching this. Thanks very much.