Scientific Computing Using Matlab
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Lecture No. 40
Stirling's Formula and Lagrange's Interpolating Polynomial

Hello viewers, welcome back to the course on Scientific Computing Using Matlab. So, in the last lecture we discussed a problem and we applied the Newton forward and backward formula for that. Today we will go further and we will try to find what will happen if the values of the x lies in between the finite difference table. So, today we will start with the Stirling formula.

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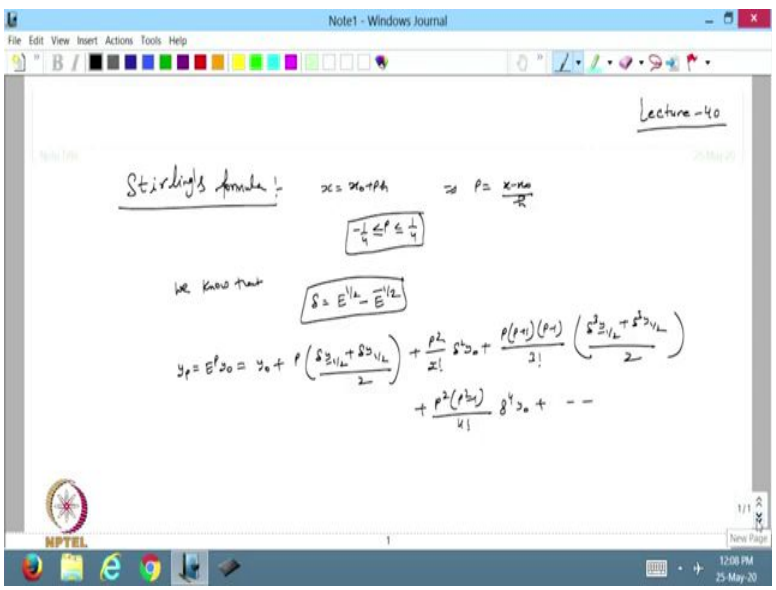

Stirling formula. So, Stirling formula, I am going to give you the direct formula of how this formula will look like. So, I want to find the value of x is equal to $x_0 + ph$. So, in that case I am giving you the value x_0. And from here my $p = \dfrac{x - x_0}{h}$. So, this formula is applicable when the value of p is lying between $-\frac{1}{4}$ to $\frac{1}{4}$. So, this is the value of the p only then we can apply this formula.

And we also know that the E, the shift operator, I want to find the relation between E and δ that my $\delta = E^{1/2} - E^{-1/2}$, so this is we already know. Now, I direct give yout that what is the Stirling formula. So, the Stirling formula is I want to find y_p, so that y_p

$$y_p = E^p y_0 = y_0 + p\left(\frac{\delta y_{-1/2} + \delta y_{1/2}}{2}\right) + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p+1)(p-1)}{3!}\left(\frac{\delta^3 y_{-1/2} + \delta^3 y_{1/2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\delta^4 y_0 + \cdots$$

So, this is the Stirling formula for approximating a value that is lying in between the difference table.

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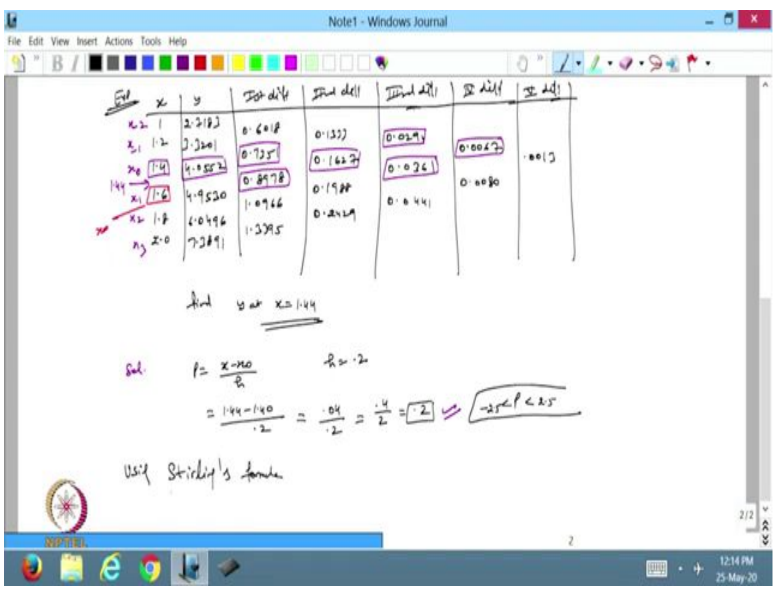

So, let us do one example and then it will be more clear. So let us take one example. I know the value. So suppose, I start with the value so this value of x is given to me. That is, 1, 1.2, 1.4, 1.6, 1.8 and 2.0, the value of y is given to me, so that is 2.7183, 3.3201, 4.0552, 4.9530, 6.0496 and 7.3891. So, this value is given to me. Now I will make the difference table, so I will write the first difference. So, again this is the 6 values. So, this is 0.6018, 0.7351, 0.8978, 1.0966, 1.3395.

Then I take the second difference. So, the second difference will be 0.1333, 0.1627, 0.1988, 0.2429. Now we will apply the third difference. So, the third difference will be . So, it is 0.0294, 0.0361, 0.0441, so this is the third difference then we will apply the fourth difference. So, the fourth difference will be 0.0067 and 0.0080. And the fifth difference, the last value will be the constant value and this is 0.0013.. So, this is the last value. So, this is my difference table.

Now, the question is that find y at x = 1.44. So, this value I want to approximate using the finite difference operator. Now, from here it is clear that 1.44 lies here. It means if I need to find these values, then I will choose x_0 = 1.40. So, this will be x_0, and $x_{-1} = 1.2, x_{-2} = 1, x_1 = 1.6, x_2 = 1.8, x_3 = 2.0.$ Now, this if it is x_0 this will be y_0 . So, this y_0 I will take the next is I will take the average of these 2 values, then I will second difference, central difference will be this one, then I will choose the average of these 2 values, then I will go here and that is it.

So, using this Stirling formula, I should be able to find the value of the y at x = 1.44. The only condition is that my p lies should lie between -0.25 to 0.25. So, let us see this one. The solution might be $\frac{x - x_0}{h}$. So in this case my h = 0.2. So this is 0.2 it is given to me. So, x = 1.44 So,

$$p = \frac{1.44 - 1.4}{0.2} = 0.2$$

So, the value will be 0.2.

My p should lie -0.25 to 0.25 so, it is satisfying here, so I can apply this one. Somebody can choose this as $x_0 = 1.6$. So, in that case, if I choose this x_0 then the value of p = -0.8. So, everything depends upon which value we are using. So, in this case I will use this value, so, that is, this value is also okay. Now, I will apply this Stirling formula to approximate the value here.

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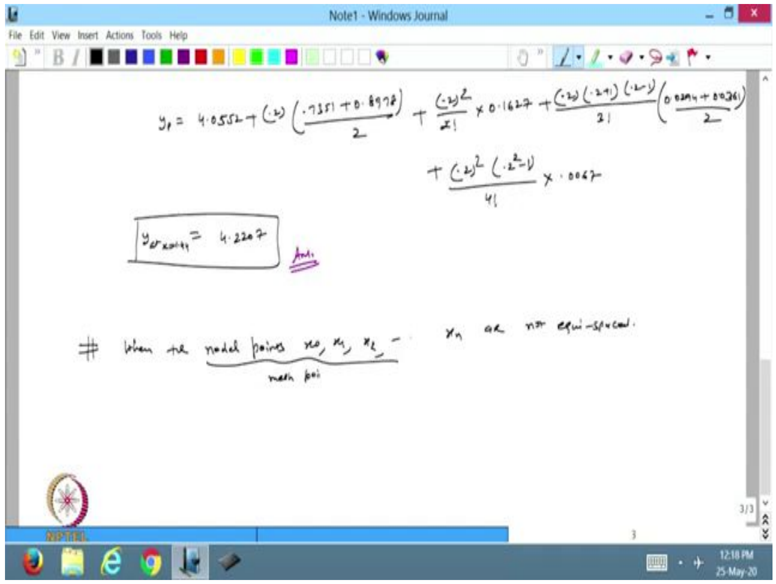

So, using Stirling's formula, the value of y_p,

$$y_p = 4.0552 + 0.2\left(\frac{0.7351 + 0.8978}{2}\right) + \frac{(0.2)^2}{2!}(0.1627) + \frac{0.2(1.2)(-0.8)}{3!}\left(\frac{0.0294 + 0.0362}{2}\right) + \frac{(0.2)^2(0.2^2 - 1)}{4!}0.0067$$

So, I will stop here and then I will calculate all these values. So, if I do the calculations and my calculation is correct, then its value should be 4.2207. So, this is the value of y at x = 1.44.

So, this is the approximate value and from here I can see that this value 4.2207 is lying here. So, this value seems correct and from here I can see that this function is increasing function also. So, based on this one I can say that this is my answer. So, yeah, now, we have applied the Newton forward, backward and central. So, this is all about that whenever we have the data which is equi-spaced, then we use this method we can apply, we can find the or we can approximate the value of x based on that we have this lies in the finite difference table.

Now, this is about the equi-space. So, let us do what will happen when the nodal points like $x_0, x_1, x_2, \cdots, x_n$ are not equi-spaced. So, in this case I will say that, so nodal points in these points are also called mesh points somewhere it is also sometimes also called the mesh points. So, what will happen when these are not equi-spaced?

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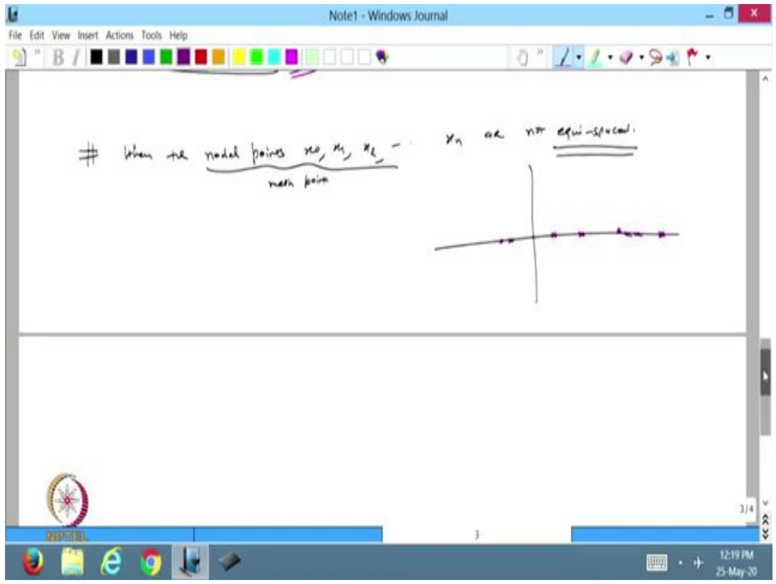

Like I have the data, so my data is given to me, one value is here, another value is here another value is here, then another value is here, this value, this value, this value, this value. So, in this case it is not equi-spaced then I cannot apply Newton forward or backward or central difference as we have discussed. So, in this case we have to develop some other methods to approximate or to find the interpolating polynomial.

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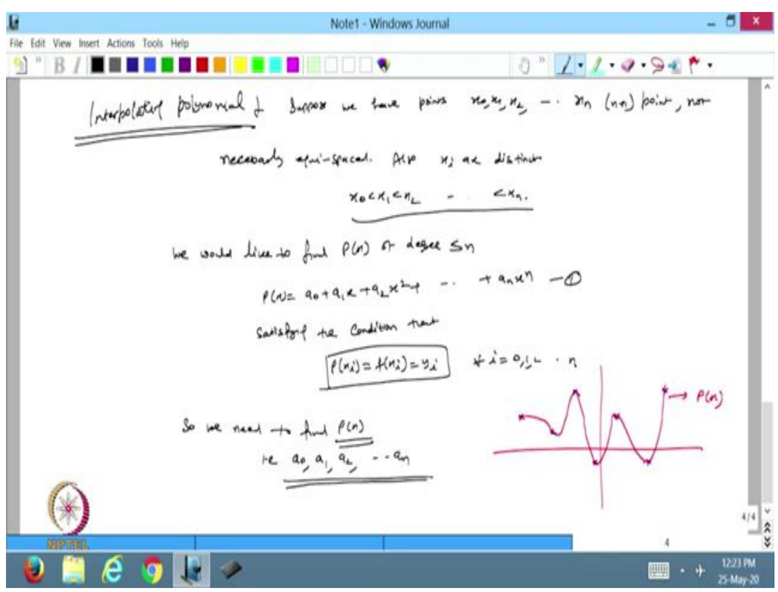

So, let us do this one. Now. So first of all I will try to find how we can find any general interpolating polynomial. So, let us discuss that interpolating polynomial in general. Suppose, suppose we have points, the nodal points $x_0, x_1, x_2, \cdots, x_n$, n+1 points not necessarily equi-spaced. So, this is not necessarily equi-spaced also all x_i, it means I can write this $x_0 < x_1 < x_2 < \cdots < x_n$, so all are distinct

Now, the question is that I know that if this is the n+1 points, then we can approximate a n^{th} degree interpolating polynomial passing through all these points. So, we want to find P(x), the interpolating polynomial of degree less than equal to n. So, my P(x) can be written as $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \cdots (1)$ Because it is satisfying the condition that my $P(x_i) = f(x_i)$, value of the function whatever is given to us is equal to y_i for all $i = 0, 1, 2, \cdots, n.$ So, that is the condition there that is passing through all these points.

So, like this one so, some point is given to me one point is given this, this, that. So I am starting with this polynomial, like this one I am going so this is my P(x) and this value, whatever the points are given to me, this is the points and it is satisfying this condition that it is passing through all these points. Now I need to find what is my polynomial? So, we need to find P(x) and that is the coefficient is $a_0, a_1, \cdots, a_n$.

So, to find the value of P(x) means I should know the value of what is $a_0, a_1, \cdots, a_n$ all these values we have to find out. So, that will be my interpolating polynomial. So, let us try to find this coefficient $a_0, a_1, \cdots, a_n$.

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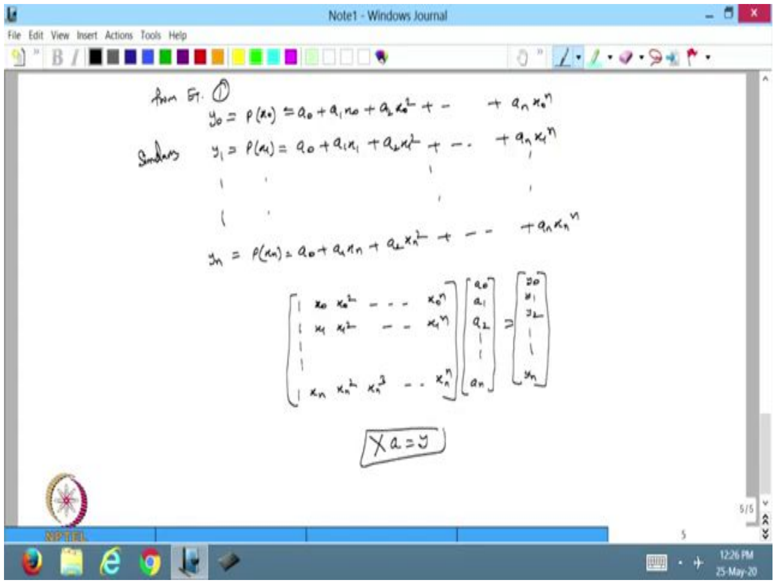

Now, from equation number 1, let us check what is the y at x_0. So, the first initial point, this is my x_0. So at this point I want to find what is my $a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^2$. So this is my basically this is equal to $P(x_0)$. And I know that this $P(x_0) = y_0$, that is known to me. Similarly, I can apply $P(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^2 = y_1$

So, if I go through all this points to me that is given to us. So, this is $P(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^2 = y_n$. Now, to find out the values of $a_0, a_1, \cdots, a_n$, I can convert this equation, the system of equations, into the matrix form.

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

So, this is my system of equations in the matrix form because this matrix involves x, so I should write Xa = y we have this. So this is the system of equations.

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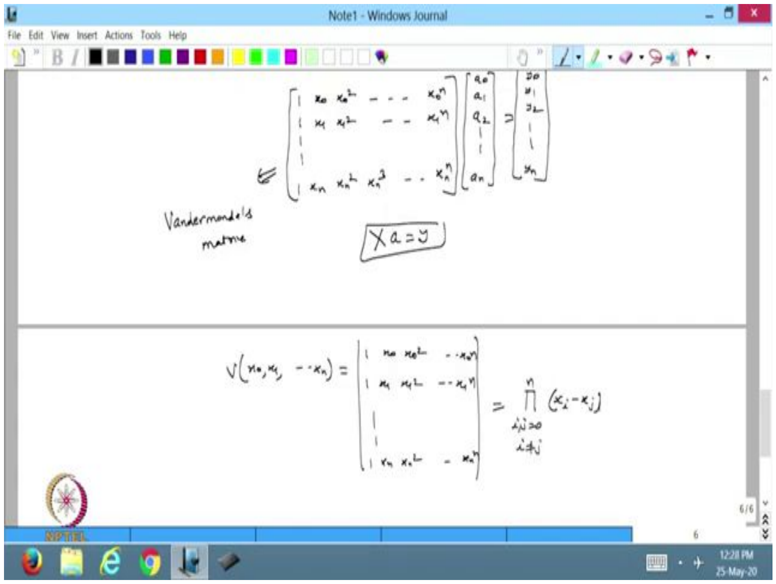

Now this matrix, whatever the matrix we are getting, so this is, this has a special name, and this is called Vandermonde's matrix. Now, if I want to solve the system, then let us see, let us find out what will be the determinant of this matrix. So, let us go for this one. So, from here I can write the

$$V(x_0, x_1, \cdots, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \prod_{i,j=0, i \neq j}^{n} (x_i - x_j).$$

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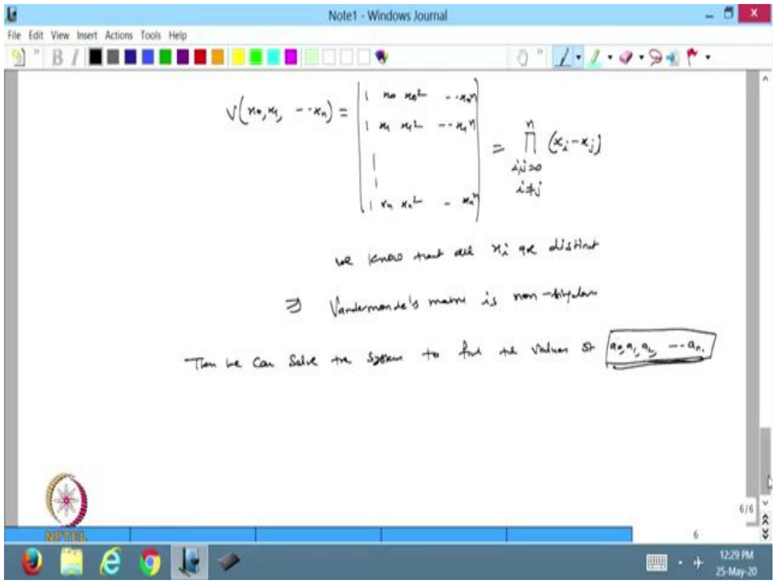

So, that is my determinant in this case. We also know that all x_i. So, which implies that this matrix is non singular, which implies that the matrix Vandermonde's matrix is non singular. But if these values of x_i and x_j are very close to each other, so, in that case this matrix is nearly singular. So, in that case the condition number will be very high, this method is good whenever we find that the value of x_i and x_j are not close to each other, they are distinct and not and the difference is quite good.

So, from here I can say that the Vandermonde's, this matrix is a non singular matrix and based on this non singular matrix then we can solve the system to find the values of $a_0, a_1, \cdots, a_n$. So, based on this matrix, whether it is singular or non singular we can find the value of this $a_0, a_1, \cdots, a_n$. So, this we can find out.

Now, the question is that once I get this matrix, this polynomial, interpolating polynomial for any n+1 number of points.

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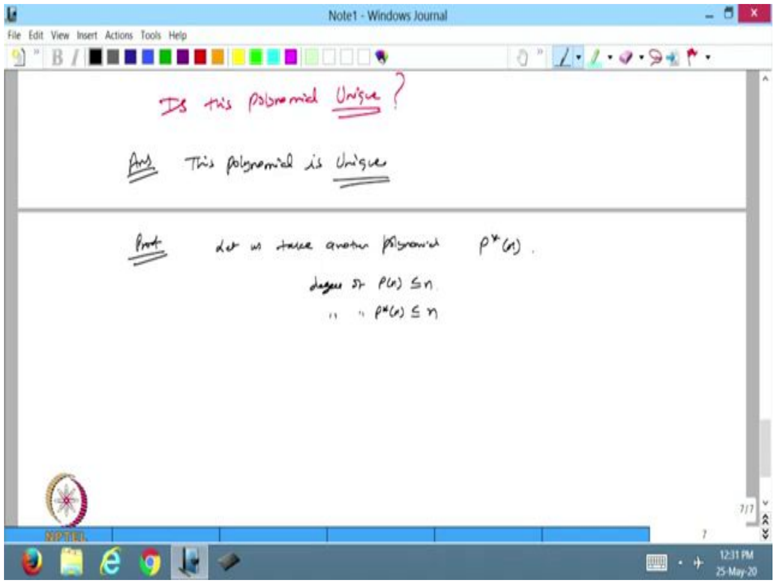

Then the question is, is this polynomial unique. Because here we are not using any of the values, we are just taking the mesh points or the nodal points, whatever is given to us to approximate the coefficient of the matrix or coefficient of the polynomial. So, the question is, is this polynomial unique. So, the answer is yes. So, this polynomial is unique. So, how we can prove this one, so, let us prove. So, let us take another polynomial and we are represented by $P^*(x)$.

So, I know that the degree of P(x) is less than equal to n and also the degree of $P^*(x)$ is also less than equal to n, because if it is passing through n+1 points, so, degree cannot be more than n. So, its degree will be less than or equal to n.

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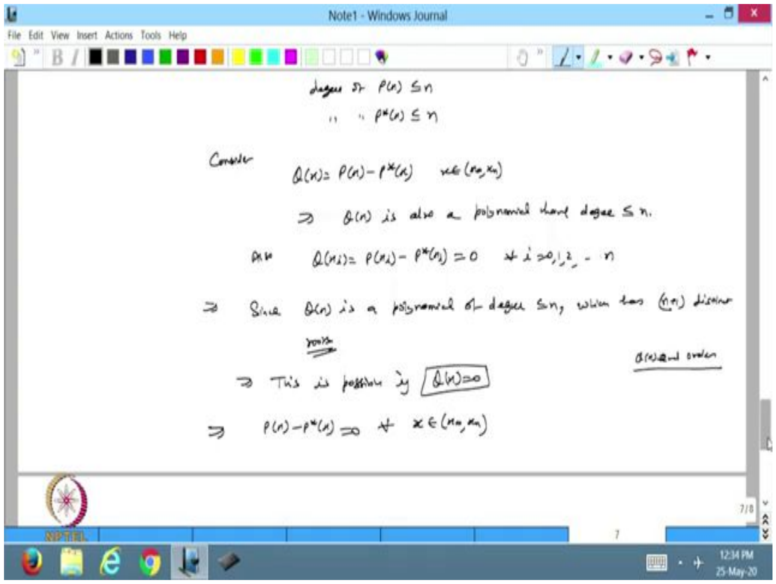

Now, let us consider, so consider the polynomial $Q(x) = P(x) - P^*(x)$. So, I choose the difference of these 2 polynomials as a Q(x). So, from here I can say that the Q(x) is also a polynomial having degree less than or equal to n, also $Q(x_i) = P(x_i) - P^*(x_i)$. And at x_i, $P(x_i) = P^*(x_i)$, so, this will be 0 for all i. Because this polynomial is passing through this point, all these points whatever the points we have $i = 0, 1, \cdots, n.$

So, at x any of the x_i this value and this value will be the same and then the difference will be 0. So, from here I will get the $Q(x_i) = 0$. Now, from here since Q(x) is a polynomial of degree less than equal to n which has n+1 distinct roots because these all are the roots basically. So, in this case, I have n+1 points that are the roots. So, Q(x) is a polynomial of degree less than or equal to n. So, this is the same as that my Q(x) is supposed to take n = 2 and my Qx is second order.

So, that is second order. So, and I know that the second order polynomial has maximum, not maximum, the second order polynomial has 2 roots. So, but in this case it will be, it is going to have the 3 roots, so that is not possible. Which implies that this is possible if my Q(x) = 0 for all x. And from here, which implies that my $P(x) - P^*(x) = 0 \forall x \in (x_0, x_n)$. Because this is true I am taking $x \in (x_0, x_n)$.

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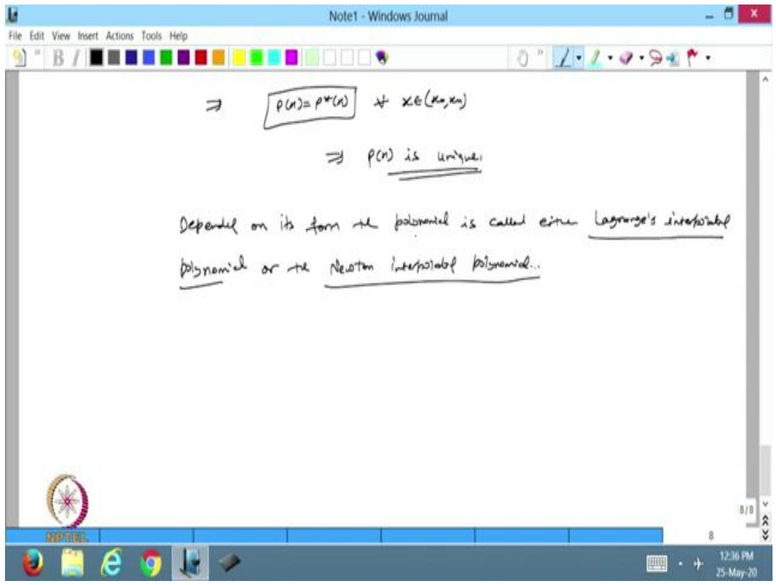

So, from here I can say that my $P(x) = P^*(x) \forall x$. I can say from here that the P(x) is unique, so it is unique. The only question is that it may look like the different one, but ultimately the values are unique. So, from here I can write that depending on its form, the polynomial is called either Lagrange interpolating polynomial or the Newton interpolating polynomial. So, these forms we know that the polynomial will be unique but depending on the form these polynomials have 2 categories.

So, the first category is that either it will be of type Lagrangian interpolating polynomial or it will be a Newton interpolating polynomial. So, we will discuss in the coming lectures what is the meaning of Lagrange interpolating polynomials or Newton divided difference polynomials. Because in this case we are taking the points, mesh points which are not equi-spaced. So, I will stop here today.

So, today we have discussed about the central difference formula, that is the Stirling formula and then we have also discussed that if the mesh points which are given to us or the data points which are given to us are not equi-spaced, then how we can interpolate, approximate the interpolating polynomial, and that polynomial itself is a unique polynomial. So, we will continue from this in the next lectures. So, thanks for watching. Thanks very much.