Scientific Computing Using Mat-lab
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Lecture No. 31
Gershgorin Circle Theorem for Estimating Eigenvalues of a Matrix

Hello viewers, welcome back to the course on Scientific Computing using Mat-lab. So, in the previous lecture we have discussed the power methods and shift inverse power methods to find out the Eigenvalues of the given matrix. So, today will continue from that. So, in the previous lecture we have discussed what is the meaning of power method.

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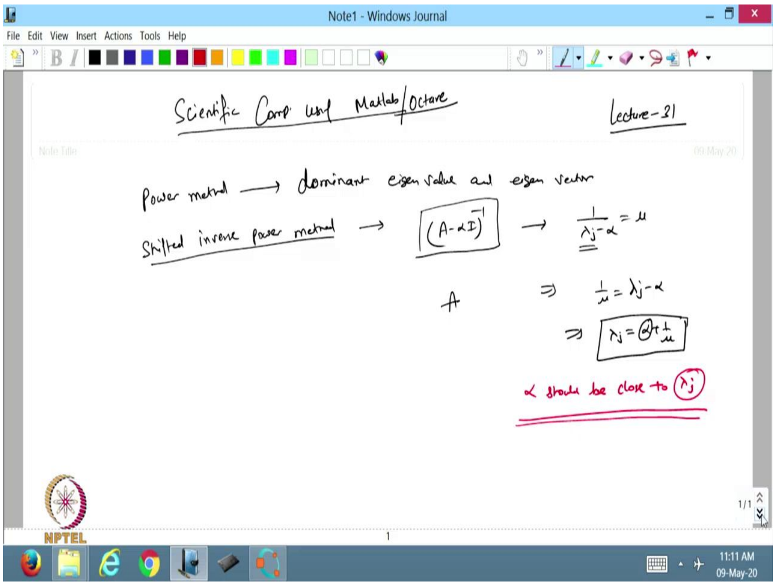

So, we know that the power method gives you the dominant eigenvalue and eigenvector. Then we have discussed the shifted inverse power method. So, that is used for finding the $(A-\alpha I)^{-1}$. So, I have taken the matrix this one so, we are finding the eigenvalue of this matrix and then with the help of this one we are able to find the eigenvalue so, I know that the eigenvalue of this will be $\frac{1}{\lambda_j-\alpha}$ where, alpha we have chosen and from there I can find the value of so, this is my mu.

So, based on this one I can find the value of λ_j. So, from here I can write that

$\frac{1}{\mu} = \lambda_j - \alpha$ and from here I can find $\lambda_j = \alpha + \frac{1}{\mu}$. So, this way we are able to find the value of any eigenvalue of a matrix A. Now, the question comes that how we can choose this alpha because, in the previous lecture we have discussed that the alpha should be close to λ_j whatever the Eigenvalue we are finding but, the question is that we do not know what is the λ_j.

So, if we do not know the λ_j then how we can choose the alpha that should be close to lambda j. So, this is another problem, so that will be taken care of by the Gershgorin so, before that I would just want to do one example.

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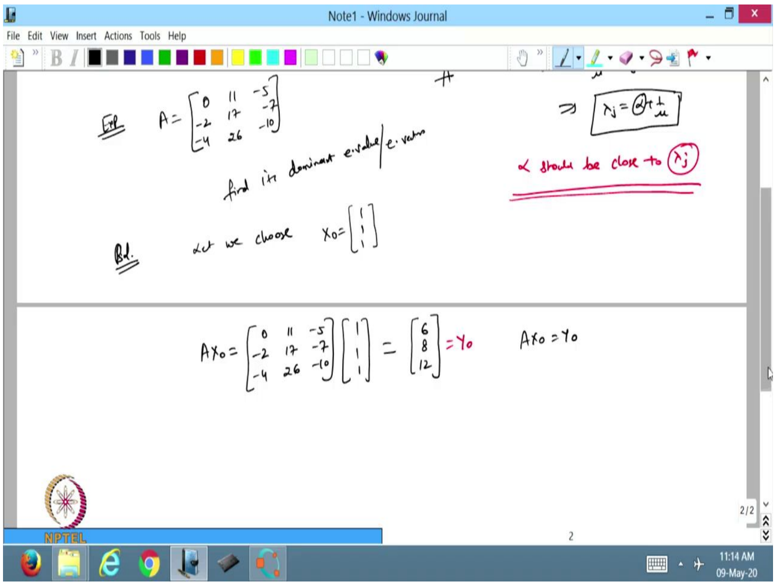

So, let us take one example. So, I have a matrix A 0, 11, -5, minus 2, 17, -7, -4, 26 and -10 so, this is my matrix and I want to find its dominant Eigenvalue of Eigenvector. So, in this case I apply the power method so, let us do the power method so, let we choose, first I choose the x0 the initial point so, it can be any value you can choose.

So, let us start with simply 1, 1, 1 so, this is the initial point then I want to find what will be the A into X knot so, if you see from here this is 0, minus 2, 17, -7, -4, 26, -10 and then 1, 1, 1 so, that is the vector and that I multiply this with x0. So, from here if you see I will get so, 1, 11 -5 so, this will be 6 and this I will take so, -2 and -l7, -9 and 17 so, it is 8 and this is 26 -14 so, it is 12.

So, from here I will get this value, so we know that the Ax0 was my y0 so this is my y0. So, this is y0 you can write it as y0.

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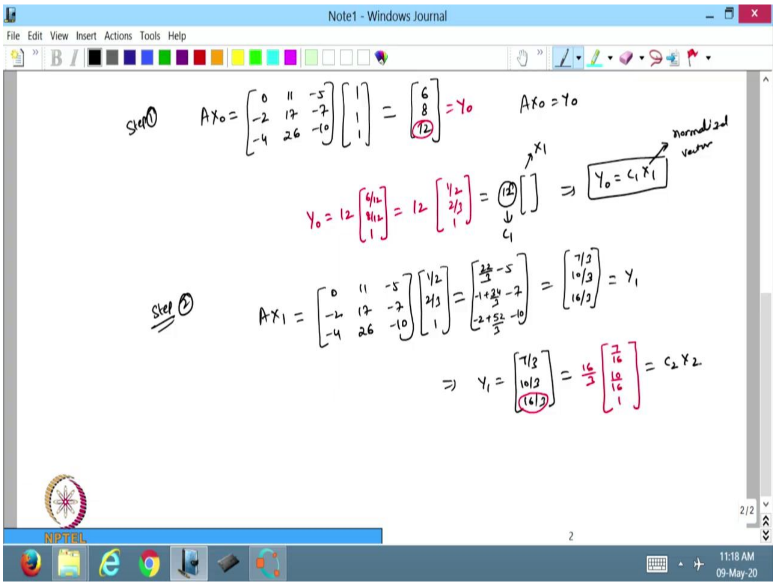

Now, what I do is that I want to make this y0 a normalized vector, so from here what we do I will choose the largest value in this component of this so, that is my largest value 12. So, from here I will write my y0 is equal to I choose the 12 and then take the 12 outside so, from here inside I will get 6 by 12, 8 by 12 and 1 so, from here I will get this value.

So, this is 1 by 2, this is 2 by 3 and then it is 1. So, form her you see that so, this becomes 12 into this vector so, I will call this vector as X1 and we call this vector as C1 so, from here I can write my y0=c1x1 this one where, x1 is a normalized vector so, now from here so, this is my step 1 now, we choose step 2.

So, from step 2 I will do A and now I multiply by x1 this is the normalized vector, so form here I will get my A is again the same matrix and then from here I choose 1 by 2 it is 2 by 3, and 1. S o, if I multiply this one from here I will get 11 by 2 minus it is not 11 by 2 it is 2 by 3 so, it is 22 by 3 - 5 and this I multiply here so, it is minus 1 because, half will cancel out with 2 minus 1 plus 34 by 3 minus 7 and in the end I will get minus 2+26.

So, it is 52 by 3 minus 10 so, if you choose this one now, from here I can find this so, it is 15 so, 22-15 is 7 so, it will be 7 by 3 then, if I do this one again taking the 3 LCM so, it is -1 and -7, -8 and three times so, -34, 24, 34 so, it is 10 by 3 and then it is -12, 36 and 52 so, it is 16 by 3 so, this is my vector I am getting and this is equal to y1.

Now, if I choose from here, my y1 is 7/3, 10/3, 16/3. So, now I find the component which is the largest in magnitude so, largest component is this one because, 16, 10 and 7 all are divided by 3 so, this is the largest component so, I will take this largest component out so, this will be 16/3 and I will get 7/16 inside because, I am dividing by this one so, 3 will cancel out I will get 10/16 and this is 1. So, from here you can see from here that this will become my c2 x2.

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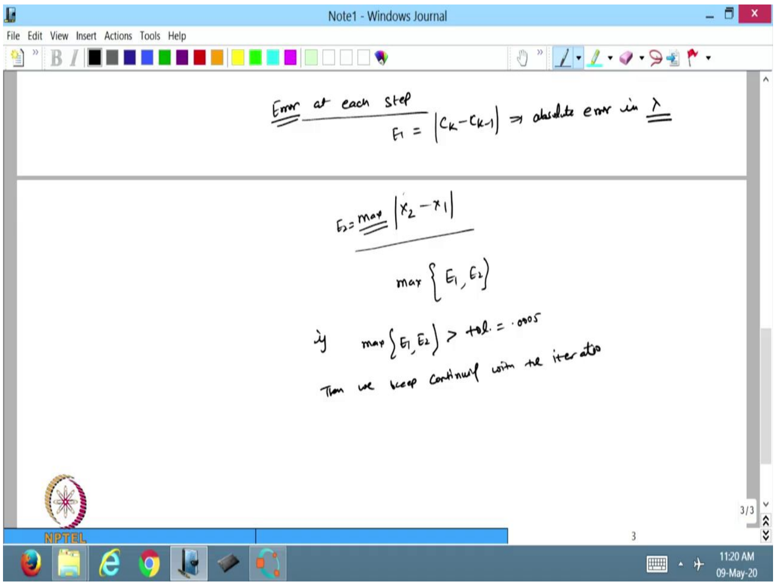

Now, I want to find that in the previous one you can see that my c1 was 12 and now, the c2 is 16/3 so, how will I find the error? So, error at each step because, we have to find the error at each step so, what I will do that I can find error from the two ways, one is that I can find my ck-ck-1 the magnitude so, this is the value I am finding so, like c2-c1 I will find out this one.

So, that is the error in we can say that, the absolute error in lambda or c whatever we are going to find is the Eigenvalue so, this is the one error I can find and another one I can find the Eigen in the error in the Eigenvector. So, in the Eigenvector I will find |x2-x1| then the maxima. So, it is L2 norm I will find out so, from here what I am doing, I am taking the x2 minus the eigenvector will get in the previous step, so that is x1 taking the magnitude and then finding the maximum value. So, then this is called the L2 norm of the given vector.

So, we can take from here, based on this one, how much this error is, so I can take this error as E1 and this error as E2. Now, from here I will find the maximum error. So, maximum of E1 and E2. So, now if the maximum of E1 and E2 is greater than the tolerance so, tolerance maybe I can define .0005 so, if this is greater than tolerance then, we keep continuing with the iterations.

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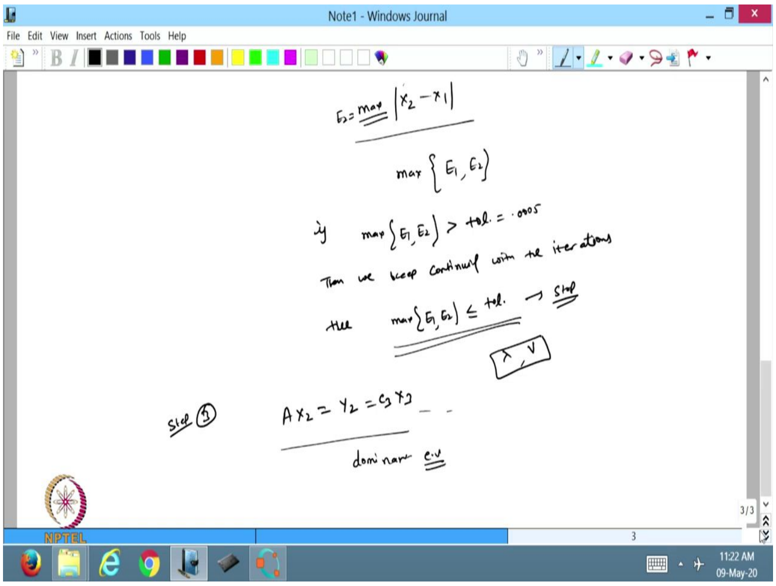

And till till this maximum error is less than tolerance. So, if it is becoming less than tolerance, then will stop less than or equal to so, then in that case it will stop and that is whatever the value will get that will be our lambda the eigenvalue and the corresponding eigenvector.

So, in this case we have able to find these two steps then I can take next step 3 so, that will be A x2 so, from here A x2 is this one then I will substitute I will find whatever the value I will get y2 so, that will be Ax2=y2= c3 x3 and so on so, I will keep doing this one. So, I will find out that in the Matlab code that in how many iterations we are going to find the solution but, this is the algorithm, this is the way we can find the solution so, that will find out in the terms of when we will do the coding in the Matlab. So, this is the way we can find the dominant Eigenvalue.

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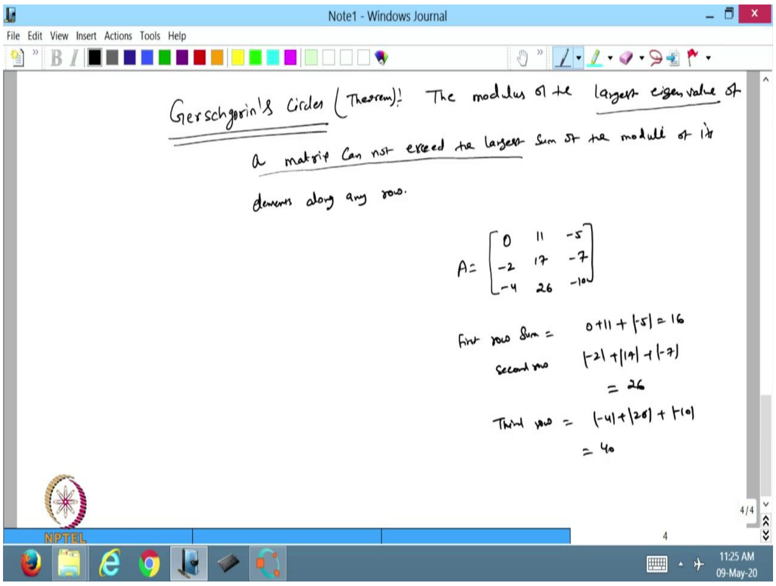

So, this is the example we have done. Now, we are going to start the next topic and that is a very important topic, so this is called Gershgorin's Circles. So, this is basically you can say that it is a theorem. So, what this theorem is saying is that the modulus of the largest Eigenvalue of a matrix cannot exceed the largest sum of the moduli of its elements along any row. So, it says that the modulus of the largest Eigenvalue or the matrix cannot exceed the largest sum of the moduli of its element along any row.

So, what is the meaning of this? It says, that if I take a matrix A the same matrix we can take so, it says that 0, 11, -5, -2, 17, -7, -4, 26 and -10. So, in this case what I will do I will find the sum of the first row so, I can write from here that a11 so, from here I can find the first row sum, so that is equal to 0+11+(-5) modulus because, it gives the modulus value.

So, it is 16 the second row, it is -2+17+(-7) so, that is 26 and the third row is equal to -4+26+(-10). So, it is 26+4 30 and it is 40.

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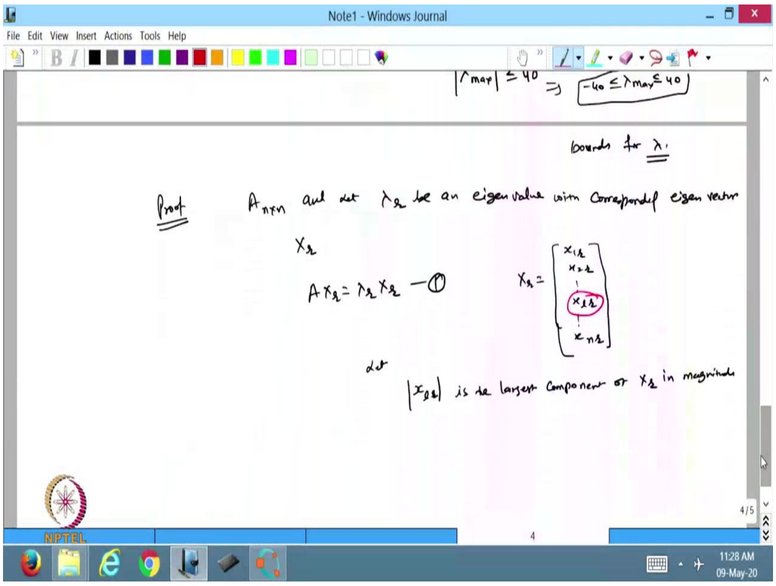

So, from here it says that if we take any eigenvalue so, that is always less than equal to 40 so, modulus of the largest eigenvalue so, from here I can even write lambda maximum so, that will be always equal 40.

So, form here you can even define the range then the lambda max will be -40 up to 40 so, from here we can say that all the eigenvalues of this matrix are lying between -40 so, that is the upper bound for that is the bound for lambda so, this is the way we can find out. So, let us do the proof of the theorem so, let me choose the matrix A n cross n and let λ_r be an eigenvalue with corresponding eigenvector that is I will choose xr so, from here I can write that this is $Ax_r = \lambda_r x_r$ so, that is the definition of the eigenvalue and how we can find the eigenvalue.

Now, this xr you can write is a vector in the column vector, so that is basically x1 r, x2 r, xl r, xn r so that is the component of this vector. Now, let xl r modulus value so, this is the factor that is the largest component of xr in magnitude. So, suppose this is the largest component like, in the previous one we have seen like 16/3 earlier 12 so, this is the largest component, so suppose this is the largest so that is why l has been taken. So, this is the xl r I mean this is the r th vector and l is the largest component of this one.

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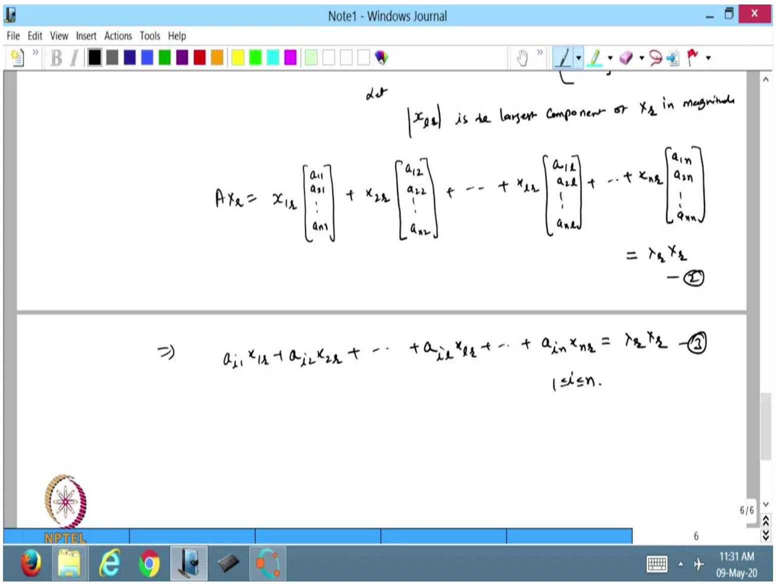

Now, from here now we can multiply this xr by this matrix so, A is my matrix so, I can write my A xr can be written as so, first component x1 r and then the first column of the matrix so, that is a11, a21, up to an3 and plus x2 r a12, a22 … an2 then xl r so, the component will be a so, it is 1 r , 2 r so, it is 1 l , 2 l, up to an l because, this l is the largest component we have taken and in the end I will get xn r so, that is a1n, a2n,...,ann so, this is how we can multiply the matrix $Ax_r = \lambda_r x_r$ so, this is what we can write. I will call it equation number 2.

So, from here now if I write this one so, I can write this in the terms of ai,

$a_{i1}x_{1r} + a_{i2}x_{2r} + \ldots + a_{il}x_{lr} + \ldots + a_{in}x_{nr} = \lambda_r x_r \quad 1 \leq i \leq n$. so, this is we have taken corresponding to lambda r. So, I will take this part on so, we will consider it only up to this so, this is true for all this one so, that is the equation number 3.

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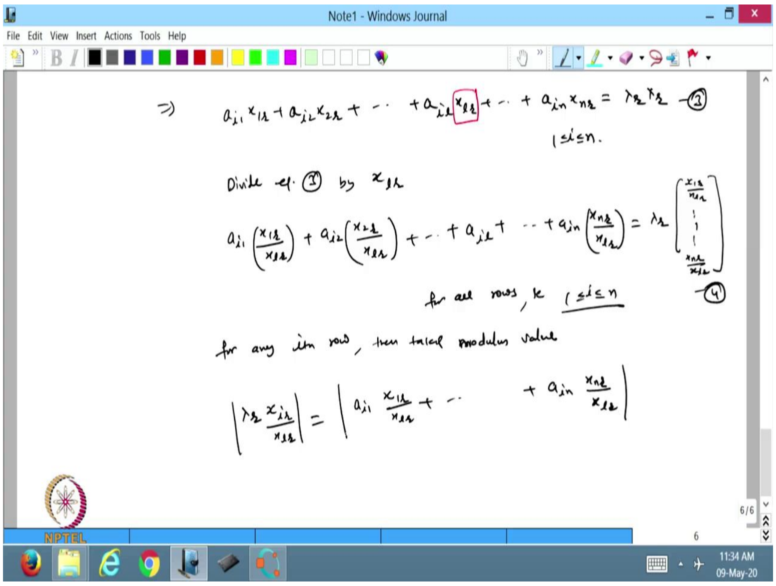

Now, what I do is that this is the maximum element I know in terms of magnitude so, divide we divide equation 3 by xlr so, I will get

$$a_{i1}\left(\frac{x_{1r}}{x_{lr}}\right) + a_{i2}\left(\frac{x_{2r}}{x_{lr}}\right) + \ldots + a_{il} + \ldots + a_{in}\left(\frac{x_{nr}}{x_{lr}}\right) = \lambda_r \left[\frac{x_{1r}}{x_{nr}}, \ldots, \frac{x_{nr}}{x_{lr}}\right]^T$$

so, this is what I am getting: that is the equation number 4.

Now, if I choose and this is true for all rows so, that is i, is moving from 1 to n. Now, for any ith row so, I just take any ith row for any ith row first I choose ith row and then take modulus value so, I will get $\left|\lambda_r\left(\frac{x_{ir}}{x_{lr}}\right)\right| = \left|a_{i1}\left(\frac{x_{1r}}{x_{lr}}\right) + \ldots + a_{in}\left(\frac{x_{nr}}{x_{lr}}\right)\right|$ so, this value we have taken.

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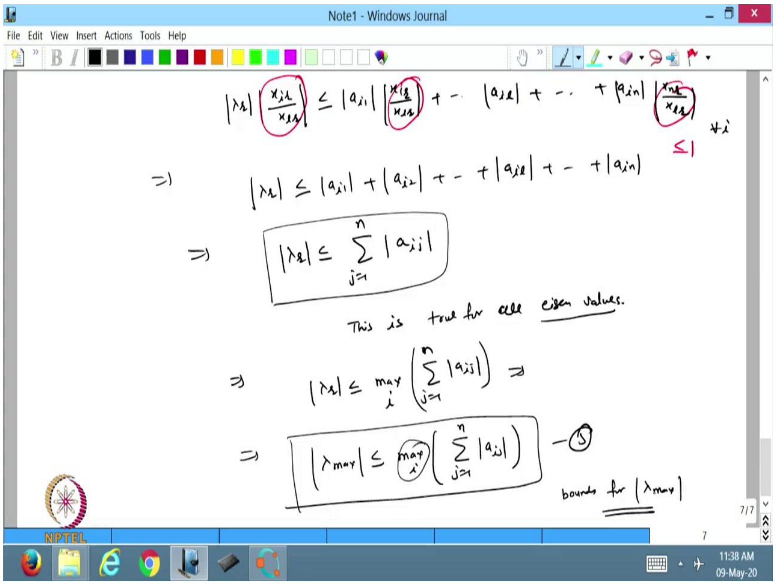

Now, from here we know that this can be written as

$$|\lambda_r|\left|\frac{x_{ir}}{x_{lr}}\right| \le |a_{i1}|\left|\frac{x_{1r}}{x_{lr}}\right| + \dots |a_{il}| + \dots + |a_{in}|\left|\frac{x_{nr}}{x_{lr}}\right|$$

. So, this one I have taken now, I know that all this component if you see then all this component, this component, this component, this component, all or less than equal to 1.

Because, we have chosen this xl are the maximum element so, this all values will be less than 1 so, from here I can write that my

$$|\lambda_r| \le |a_{i1}| + |a_{i2}| + \dots + |a_{il}| + \dots + |a_{in}|$$

. So, from here I can add that this

$$|\lambda_r| \le \sum_{j=1}^{n} |a_{ij}|$$

.

So, this is changing so, this is I can take aj and j is moving from 1 to n. So, this is what we are taking from here now, this is true for all Eigenvalues, so this is true for all Eigenvalues. So, from here I can write that, so I can write from here that lambda r modulus can be written as less than maximum of this 1. So, this is also true and from here I can write that lambda max will be always less than this because, this is true for all so, that is true for all Eigenvalues I will choose the maximum Eigen values in the modulus. So, that will be also less than equal to this value.

So, that is I can write as a fifth so, from the fifth we can say that whatever we want to show that the modulus of the largest Eigenvalue of the matrix cannot exceed the largest sum of the corresponding row sum modulus of the row sum. So, in this case I am taking the modulus of each element in the row and then I am taking the summation and then we are taking the maximum of all this summation.

So, from here I can say that the Eigenvalue maximum Eigen will be always less than this one. So, this is upper bound for the lambda x, upper as well as lower. So, from here you can write that this is bound for lambda max.

So, let us stop here, today we have started with some examples to find the dominant Eigenvalue using powers method and then we have to find value of alpha in the case of shifted inverse method we are taking the help of Greshgorian theorem. So, that we have discussed today so, in the next lecture we will continue from the Gershgorian. So, I hope you have enjoyed this one thanks for watching.

Thanks very much.