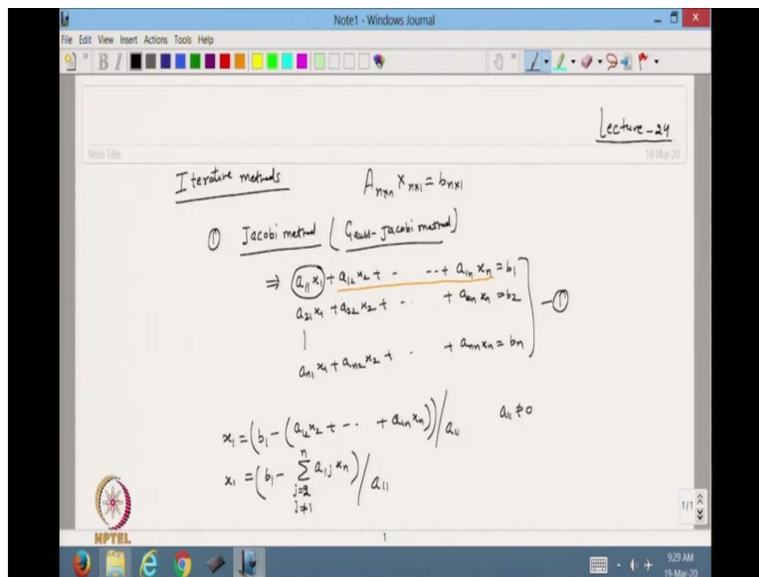


Scientific Computing Using MATLAB
Professor Vivek Aggarwal
Department of Mathematics
Indian Institute of Technology, Delhi/DTU
Lecture - 24

Iterative Method for Solving Linear System of Equations

Hello, viewers. Welcome back to the course on Scientific Computing Using MATLAB. So today we are going to discuss the lecture 24. So in the previous lecture, we have discussed the direct methods like Gauss elimination, LU decomposition, and other one. So today we will discuss the iterative methods for solving a system of equations.

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So, iterative methods. So I have a system of equations that is $n \times n$ system and I want to solve this one with the help of iterative methods. So for example, I start with the first one is the Jacobi method or we also call it the Gauss-Jacobi method. So what we are going to do in this case, like I told you this iterative methods, so suppose, I have a system of equations. So $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$; $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$ and in the last I have b_n . So this is my system of equations.

So in this case, I will reduce this matrix in such a way that I can write a code for that one, and with the help of the initial condition, I should be able to find the solution. So what I do in this case, first I will concentrate on the first row. So in the first row what I will do, I will just keep the term this one and I am taking all the other terms of this system on the right-hand side.

So I will write this one as $x_1 = (b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)) / a_{11}$, $a_{11} \neq 0$

provided my a_{11} is not equal to 0, so that is my condition, because if a_{11} is there and that is 0 then we interchange the rows. So in this case, my a_{11} should not be 0. So this can be written as

$$x_1 = \left(b_1 - \sum_{j=1, j \neq 1}^{nz} a_{1j} x_j \right) / a_{11} \quad . \text{ So that is my } x_1.$$

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Handwritten notes in a Notepad window showing the derivation of the Gauss-Jacobi method. The equations are:

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n)}{a_{22}} \quad a_{22} \neq 0$$

$$= \frac{b_2 - \sum_{j=1, j \neq 2}^n a_{2j} x_j}{a_{22}}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}}$$

$$x_n = \frac{b_n - (a_{n1}x_1 + \dots + a_{n,n-1}x_{n-1})}{a_{nn}} \quad a_{nn} \neq 0$$

Handwritten notes in a Notepad window titled "Jacobi method (Gauss-Jacobi method)". The system of equations is written as:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \quad \text{--- (1)}$$

The derivation for x_1 is shown as:

$$x_1 = \frac{b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}} \quad a_{11} \neq 0$$

$$x_1 = \frac{b_1 - \sum_{j=1, j \neq 1}^n a_{1j} x_j}{a_{11}}$$

The derivation for x_2 is shown as:

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n)}{a_{22}} \quad a_{22} \neq 0$$

$$= \frac{b_2 - \sum_{j=1, j \neq 2}^n a_{2j} x_j}{a_{22}}$$

Now the same way, from the second row, so this is the, my second row. So I will rewrite the second row in the same way

$x_2 = (b_2 - (a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n))/a_{22}$, $a_{22} \neq 0$ So in this case also, I am considering that a_{22} is not equal to 0. In fact, it should not be a very small number because if we divide it with a very small number then we will lose the significant digits.

$$x_2 = \left(b_2 - \sum_{j=1, j \neq 2}^n a_{2j}x_j \right) / a_{22}$$

So this one can be written as . So the same way I can go, keep going, and in the last line in the last equation, I can write as a

$$x_n = \left(b_n - \sum_{j=1, j \neq n}^n a_{nj}x_j \right) / a_{nn}, \quad a_{nn} \neq 0$$

So in this case, if you combine all these equations together you will see that you will get

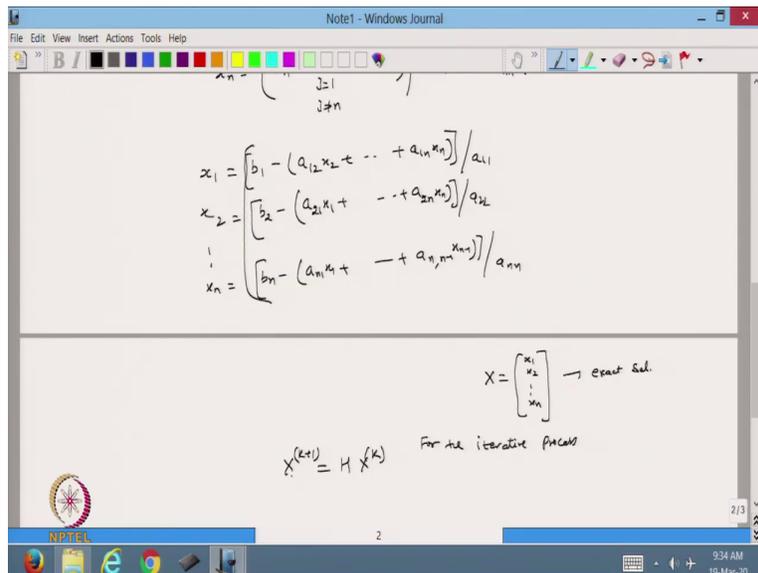
$$x_1 = [b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)]/a_{11}$$

$$x_2 = [b_2 - (a_{21}x_1 + \dots + a_{2n}x_n)]/a_{22}$$

⋮

$$x_n = [b_n - (a_{n1}x_1 + \dots + a_{n,n-1}x_{n-1})]/a_{nn}$$

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Now, if this is a system I am getting, so from here, now I know that in the iterative process I want that this is my iterative process, so suppose I take a capital X, so capital X is a column

vector representing x_1, x_2, \dots, x_n , so suppose this is my exact solution. So in this case, I want that for the iterative process, for the iterative process I want that $X^{(k+1)} = HX^{(k)}$. It means that is the iteration, the solution I am getting at the k th iteration, then I will multiply it because from here, I can see that this is the system we are going to get, so I will multiply by the some system and I will get the new approximation of this one.

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System (2) can be written as

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + a_{13}x_3^k + \dots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + a_{23}x_3^k + \dots + a_{2n}x_n^k)}{a_{22}}$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \dots + a_{nn}x_n^k)}{a_{nn}}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + \dots + a_{2n}x_n)}{a_{22}} \quad \text{--- (2)}$$

$$\vdots$$

$$x_n = \frac{b_n - (a_{n1}x_1 + \dots + a_{nn}x_n)}{a_{nn}}$$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow \text{exact sol.}$

For the iterative process

$$X^{(k+1)} = H X^{(k)}$$

$$X^{(k+1)} - X^{(k)} = \begin{bmatrix} x_1^{k+1} \\ \vdots \\ x_n^{k+1} \end{bmatrix} - \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix} = \begin{bmatrix} x_1^{k+1} - x_1^k \\ \vdots \\ x_n^{k+1} - x_n^k \end{bmatrix}$$

And now, from here I can define the term at $X^{(k+1)} - X^{(k)}$. So that is the difference between $X^{(k+1)}$ minus $X^{(k)}$. So if you see from here, it can be written as I have a

x_1^{k+1} up to x_n^{k+1} so this is one of the vectors minus the other vector, so x_1^k . So this is the difference between the two vectors, so I can write this as

$$X^{(k+1)} - X^{(k)} = \begin{bmatrix} x_1^{k+1} - x_1^k \\ \vdots \\ x_n^{k+1} - x_n^k \end{bmatrix}$$

So this is I am getting the, another vector and that is the component-wise difference I am finding between the k th approximation and the $(k+1)$ th approximation.

Now, from the above this one I can write my system. So this is my, I can write it as 2. So from here, I can write my system as, so 2 can be written as

$$x_1^{k+1} = (b_1 - (a_{12}x_2^k + a_{13}x_3^k + \dots + a_{1n}x_n^k))/a_{11}$$

So that is my first equation.

Second equation will be as this one,

$$x_2^{k+1} = (b_2 - (a_{21}x_1^k + a_{23}x_3^k + \dots + a_{2,n}x_n^k))/a_{22}$$

And in the last I will get

$$x_n^{k+1} = (b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \dots + a_{n,n-1}x_{n-1}^k))/a_{nn}$$

So this is my system.

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The screenshot shows a Windows Journal window with the following handwritten content:

- How to Start: For the system with initial approximation $X^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$. Suppose $X^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ or $X^0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.
- From iterative process: we use input X^0 and we get $X^{(1)}$. Then input $X^{(1)} \rightarrow X^{(2)}$. Input $X^{(k)} \rightarrow X^{(k+1)}$.
- How to Stop: $error = \frac{1}{2} \|X^{(k+1)} - X^{(k)}\| = \sqrt{(x_1^{k+1} - x_1^k)^2 + (x_2^{k+1} - x_2^k)^2 + \dots + (x_n^{k+1} - x_n^k)^2}$.

The window title is "Note1 - Windows Journal" and the system tray shows the time as 9:45 AM on 19-Mar-20.

System (2) can be written as

$$\begin{cases} x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + a_{13}x_3^k + \dots + a_{1n}x_n^k)}{a_{11}} \\ x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + a_{23}x_3^k + \dots + a_{2n}x_n^k)}{a_{22}} \\ \vdots \\ x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \dots + a_{nn}x_n^k)}{a_{nn}} \end{cases} \quad (2)$$

How to Start For the start with initial approximation

$$X^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

Suppose $X^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ or $X^0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$X^{(k+1)} = H X^{(k)}$ For the iterative process

$$\underline{X^{(k+1)}} - X^{(k)} = \begin{bmatrix} x_1^{k+1} \\ \vdots \\ x_n^{k+1} \end{bmatrix} - \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix} = \begin{bmatrix} x_1^{k+1} - x_1^k \\ \vdots \\ x_n^{k+1} - x_n^k \end{bmatrix}$$

System (2) can be written as

$$\begin{cases} x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + a_{13}x_3^k + \dots + a_{1n}x_n^k)}{a_{11}} \\ x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + a_{23}x_3^k + \dots + a_{2n}x_n^k)}{a_{22}} \\ \vdots \\ x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \dots + a_{nn}x_n^k)}{a_{nn}} \end{cases} \quad (2)$$

How to Start For the start with initial approximation

$$X^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

Now, how to start. How to start, so how to start means first I will choose, first we start with initial approximation. So in this case I will start with x_0 . So I should write it as a capital X_0 . So capital X_0 means I am having the x_{10} x_{20} and x_{n0} . So this value I have to choose. So I can start with any number.

So maybe suppose I take X_0 is equal to, maybe I will take 0 0 0 all 0, or I can take my X_0 is equal to 1 1 1 all 1, so any number I can start with. So this is my initial approximation and then what I will do is that I will put this value here and I will find the new value of X_1 . So from here, this is my iterative process. So from iterative process 3, we will input X naught and we get X_1 . So this is my X_1 I will get.

Then I will input and I will give, I will get X^2 as output and keep 1. So it means that I am going to input, so we are going to input X^k and get in $X^{(k+1)}$ approximation.

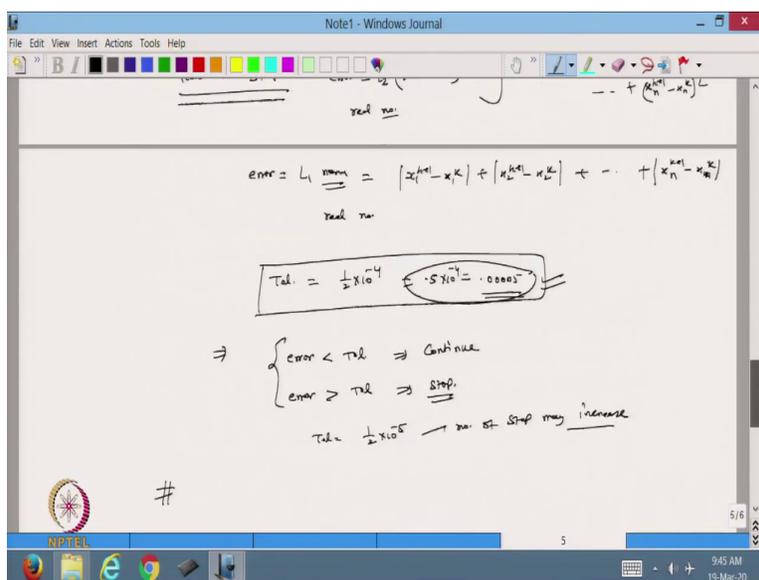
Now the question is how to stop. So in this case, we will find out the error. So, the error is how we will be finding out? So just now, I just told you that the difference between X^0 and X^{k+1} and X^k , so in this case what I will do is I will find out $X^{k+1} - X^k$. So this difference I will find and if you will see that this difference is again a vector.

So what I will do after getting this value, I will take, this is a vector so I will take l2 norm, l2 norm means then I will do that it is the same as Euclidean norm. So this is a vector so then I will do

$$\text{error} = L_2(x^{k+1} - x^k) = \sqrt{(x_1^{k+1} - x_1^k)^2 + (x_2^{k+1} - x_2^k)^2 + \dots + (x_n^{k+1} - x_n^k)^2}$$

So this is called the l2 norm.

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Or maybe in other books, we can also take error as l1 norm, or l1 norm means that I am taking $\text{error} = L_1 \text{ norm} = |x_1^{k+1} - x_1^k| + |x_2^{k+1} - x_2^k| + \dots + |x_n^{k+1} - x_n^k|$. So this is my l1 norm.

So from here, you will see that I am getting a number from here, a unique number. So this is a

real number I am getting, this is a real number. So from here what I will do? I will take this error and if this error is, I will define the tolerance or the accuracy, so I will define the tolerance.

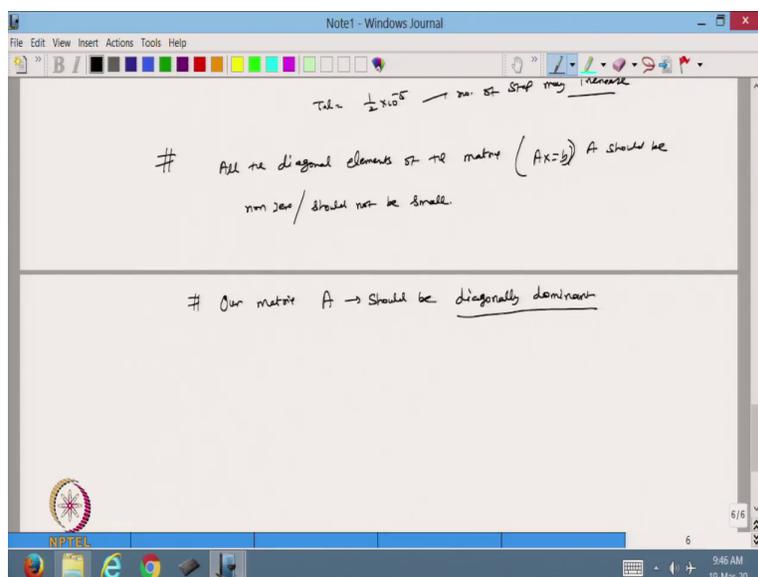
So this tolerance is suppose I am giving $(1/2)*10^{-4}$, so it is 0.0005. So this one, from here I am getting, suppose I am taking this tolerance and then I am just writing this one because I am taking it, so it is $0.5 * 10^{-4}$ and if I take this one as well it will give you 00005. So that is my tolerance.

So I am getting this tolerance and then I will find out my error and if error is greater than tolerance then continue and if the error becomes less than tolerance then we stop. So it depends upon how much accuracy you will need it, you are interested in. If you want more steps then suppose I take tolerance= $(1/2)*10^{-5}$, so in this case, the number of steps may increase.

So it all depends upon how much accuracy is needed. So based on the accuracy, we will do the iteration process and with each iteration, my approximation will be better as compared to the previous one.

If the system is going to converge and then within the number of iterations we will be able to find our solution with the accuracy whatever we have defined. So this accuracy is needed or is achieved with the number of iterations. So that is the way we can stop the method that is called the Jacobi method.

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Note1 - Windows Journal

File Edit View Insert Actions Tools Help

$$\Rightarrow \begin{cases} a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \quad (1)$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}} \quad a_{11} \neq 0$$

$$x_1 = \frac{b_1 - \sum_{\substack{j=1 \\ j \neq 1}}^n a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n)}{a_{22}} \quad a_{22} \neq 0$$

$$= \frac{b_2 - \sum_{\substack{j=1 \\ j \neq 2}}^n a_{2j}x_j}{a_{22}}$$

$$\vdots$$

$$x_n = \frac{b_n - \sum_{\substack{j=1 \\ j \neq n}}^n a_{nj}x_j}{a_{nn}} \quad a_{nn} \neq 0$$

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9:45 AM 19-Mar-20

Note1 - Windows Journal

File Edit View Insert Actions Tools Help

$$x_n = \frac{b_n - \sum_{\substack{j=1 \\ j \neq n}}^n a_{nj}x_j}{a_{nn}} \quad a_{nn} \neq 0$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + \dots + a_{2n}x_n)}{a_{22}} \quad (2)$$

$$\vdots$$

$$x_n = \frac{b_n - (a_{n1}x_1 + \dots + a_{n,n-1}x_{n-1})}{a_{nn}}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow \text{exact sol.}$$

$$X^{(k+1)} = H X^{(k)}$$
 For the iterative process

$$x_1^{k+1} \quad x_2^{k+1} \quad \dots \quad x_n^{k+1}$$

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9:45 AM 19-Mar-20

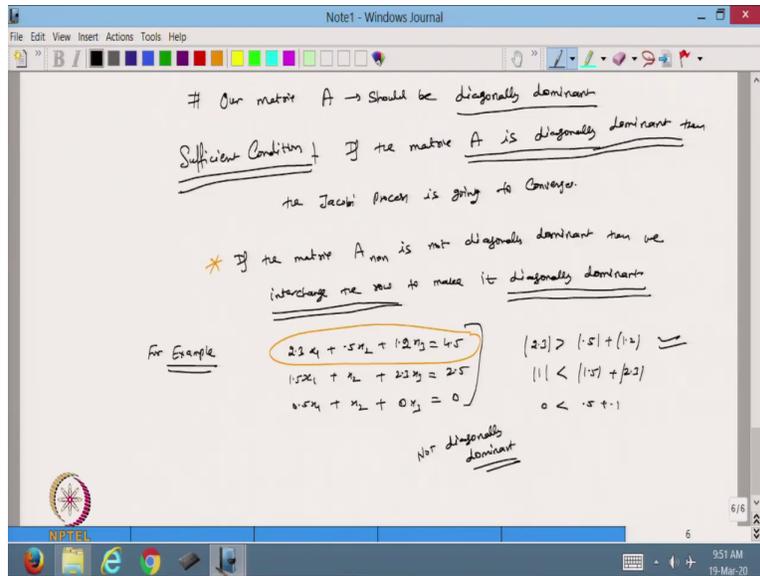
Now, the thing is that in the Jacobi method, we have seen that all the diagonal elements, so whatever the diagonal elements we have defined, this diagonal, all the diagonal elements so that should be non-zero.

So we know that all the diagonal elements of the system $Ax=b$ matrix A should be non-zero or should not be small because we know division by a small number is avoided, we should avoid that one. So in this case, what you will do is this, so to eliminate this type of possibility we want that our matrix A , so whatever the matrix we are choosing should be diagonally dominant.

So the same condition was needed when we were dealing with the Gauss elimination process or

the LU decomposition process. So in that case also the diagonally dominant matrix was needed.

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So this is and we know that this is a sufficient condition. So from here, we can say that if the matrix A is diagonally dominant then the Jacobi process is going to converge. So in this case, we do not know that if the matrix is not diagonally dominant then what will happen whether it is going to converge or not, so that is an unnecessary condition but sufficient condition, that if A is diagonally dominant matrix then definitely the Jacobi process which is going to converge and we are going to get the approximate solution.

If, so this is the condition if condition, if the matrix A is not diagonally dominant, so if it is, this matrix is not diagonally dominant then we interchange the rows to make it diagonally dominant. So what do we do that we interchange the rows as we have done in the partial pivoting. So we do the interchanging rows and with the interchanging of the rows then it may happen that will get the diagonally dominant matrix. So that is the sufficient condition for the Jacobi method.

So in the, for example, I have a system like $2.3 \cdot x_1 + 0.5 \cdot x_2 + 1.2 \cdot x_3 = b_1 = 4.5$, then I will get $x_1 \cdot 1.5 + x_2 + 2.3 \cdot x_3 = 2.5$, then $0.5 \cdot x_1 + x_2 = 0$, suppose I take this one. So in this case if you see from here, then the first row is okay, it looks like a diagonally dominant. So how can we check? So from here, we can check that to 2.3 is greater than 0.5+1.2. So that is okay.

What about the second one? Second one is 1 because this coefficient and this is 1.5+2.3, so this is less than this one and the third one is the element is 0, so here it is 0. So in this case, this value is definitely less than 0.5+0.1. So in this case, I can say that the whole matrix is not diagonally dominant. So in this case I cannot apply the Jacobi method.

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interchange the rows to make it diagonally dominant

For Example

$$\begin{cases} 2.3x_1 + 0.5x_2 + 1.2x_3 = 4.5 \\ 1.5x_1 + x_2 + 2.3x_3 = 2.5 \\ 0.5x_1 + x_2 + 0x_3 = 0 \end{cases}$$

Not diagonally dominant

$$\begin{aligned} |2.3| &> |0.5| + |1.2| \\ |1.1| &< |1.5| + |2.3| \\ 0 &< 0.5 + 1 \end{aligned}$$

Not diagonally dominant

$$\begin{cases} 2.3x_1 + 0.5x_2 + 1.2x_3 = 4.5 \rightarrow \\ 0.5x_1 + x_2 + 0x_3 = 0 \Rightarrow |1.1| > |1.5| \\ 1.5x_1 + x_2 + 2.3x_3 = 2.5 \Rightarrow |2.3| < |1.5| + |1| \end{cases}$$

Now if I want to apply the Jacobi method what I will do is that, so if I take this matrix and I apply my Gauss-Jacobi method, then the method will not converge. So what I do is exchange this one. So in this case, maybe I will take it, this is okay, now I can interchange this one so this will take it $0.5x_1 + x_2 + 0x_3 = 0$ and the next is $1.5x_1 + x_2 + 2.3x_3 = 2.5$.

So in this case what is happening? This is okay, now in this case I will see that now this is my diagonal element, so from here I can say that 10.5. From here I can say that 2.3 is greater than $1.5 + 1$. So it is not greater so it is in fact, so I can maybe just to make this diagonally dominant I will 1.5 and 1, so maybe I just change little bit or I can, should I change this one or if I take the same one so this is less than this. So this matrix is not diagonally dominant.

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$$\Rightarrow \begin{cases} 2.5x_1 + 0.5x_2 + 1.2x_3 = 4.5 \rightarrow \\ 0.5x_1 + x_2 + 0x_3 = 0 \Rightarrow |1.1| > |0.5| \\ 1.5x_1 + 0.5x_2 + 2.3x_3 = 2.5 \Rightarrow |2.3| < |1.5| + |1| \end{cases}$$

$$\Rightarrow |2.3| > |1.5| + |0.5|$$

$$\Rightarrow \text{diagonally dominant}$$

But in this case, if I change little value here because this is just a random value I have taken so if I take the some small value, instead of this one I changed the value to, maybe I can take here 0.5 and then, in this case, I can say that okay, so if I change this one now, so my this will become now 2.3 is greater than 1.5+0.5. Then I can say that this matrix is diagonally dominant.

So it is not always possible that we will find the diagonally dominant after interchanging the row like we have seen here. But if we, it is possible to make them, the system diagonally dominant then and then we apply the Jacobi method, then we are sure that our Jacobi method is going to converge and it is going to give you the solution. So this is why we should stop here and this is all about the Gauss-Jacobi method. So in the next class, we will go further from this. So thanks for viewing this, thanks very much.