Introduction to Probability Theory and Stochastic Processes

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Lecture – 02

Now, you are moving in to the definition of probability; definition of probability.

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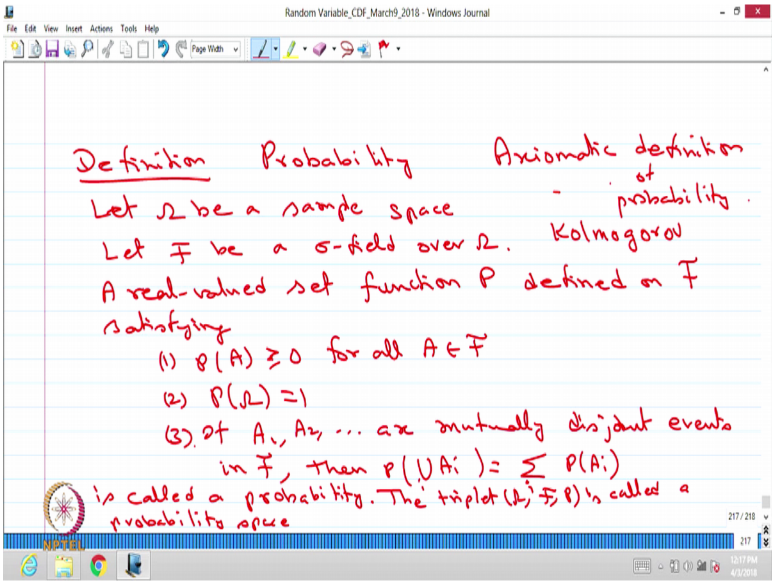

It is a very important definition because the probability theory goes with the definition of probability. Let omega be a sample space, let F; capital F be a sigma field a sigma field over omega. A real valued set function that is P defined on F satisfying the first condition that is P of A greater than or equal to 0; for all A belonging to F.

The first condition is P of A greater than or equal to 0 for all A belonging to F. The second condition P of omega is one, third condition if A 1, A 2 and so, on are mutually disjoint events, I have not said what is the meaning of events; I will discuss events later mutually disjoint events in F, then P of union of Ai's over i that is same as summation P of Ai's over i.

Basically, a real valued set function P defined on F satisfying this 3 conditions that is called a probability, it is called probability. The triplet that is omega F capital P; that is called a probability space this definition is very important we have random experiment you know what is the meaning of random experiment; that means, it is experiment in which the results are not known in advance.

If you collect the all possible outcomes of random experiments that collection all possible outcomes that collection is going to be call it as sample space that is omega. From omega it is a non empty set; obviously, for from there we created the sigma field. So, you create a sigma field one sigma field; it could be trivial or nontrivial you create sigma field overall nonempty set omega.

Now you are going to create set function that is real valued function; that means, it is mapping from F 2 or the P function, it is a set function from F to R, F is sigma field which consisting of subsets of omega satisfying the 3 conditions of the sigma field. Therefore, the elements of F that is nothing, but the subsets of omega subsets since it is a subsets of omega that is called events.

So, here the A when I make the first satisfying the condition P of A greater than or equal to 0 for all A belonging to F. So, each element of F that is called event because that is subsets of omega; that means, few collection of possible outcomes that is going to be the event; even single element is also single possible outcome or any possible result or each sample that is also called a event or subsets of possible outcomes that is also going to be call it as a event.

So, here the element of the F is called event whereas, element of omega that is called samples. Repeating again they elements of omega that is called samples, element of F is called event. We are defining the set function P on F which is the real valued function therefore, it is mapping from F to R; satisfying the 3 condition. See the first condition P of A is greater equal to 0; that means its real valued function and it is greater than or equal to 0. Therefore, the values are possible value of range of P is going to be from 0 to infinity from the first condition. But if you see the second condition, P of omega is equal to 1; if you recall F consisting of element starting from empty set, singleton element and all the possible elements and so on finally you have omega.

The P of omega is equal to 1 makes the real valued set function defined from F to R which has the range from 0 to 1; it is real valued function that is means it could be minus infinity to infinity. Since the P of A is equal to greater or equal to 0 that restrict from 0 to infinity. The second condition P of omega that is equal to 1; that is makes lies between 0 to 1; therefore, these domain is F range of P is between 0 to 1 close to interval, it can start from 0 it land up 1 therefore, it lies between 0 to 1.

Now, we will come to the third condition though F consisting of many elements; starting from empty set with the many elements, the last element is omega. So, if you take mutually disjoint events what is the meaning of mutually disjoint? If you take any 2 elements, if you go for intersection that is going to be the empty set, that is called mutually disjoint event if you take any 2 elements in the F, make intersection that is going to be the empty set.

So, if you take such events mutually disjoint events; it could be finite element or countable infinite elements, if you take those elements and you verify P of union of Ai's make an union of those elements; that is also belonging to F because of the sigma field. And the right hand side if you make summation of P of Ai's if both the values are going to be same if this condition is satisfied by taking any mutually disjoint events in the F, then you can conclude the said function P which has a domain F, range is 0 to 1 that real valued function we call it as probability.

That is a probability it is very important because the omega be a sample space for random experiment whereas, you can have more than one sigma field on the same omega. For fixed F, we are creating set function that is one set function satisfying this 3 condition therefore, it is a probability; that means, we can always create some other set function on the same omega and F, it may satisfy all the 3 conditions then we can have another probability for the same omega and F. So, many more probability can be defined by satisfying the, this 3 conditions for as fixed omega and F.

Therefore, this triplet; the omega, the fixed F and the fixed P this triplet is called the probability space. Because you can have many probability for a same omega and F therefore, you can have may probability space can be created for the same random experiment; that is very important concept in probability. Whenever we come across whenever we make real valued problem using the probability theory and we are trying to solve it; we should have one probability space to solve that problem.

So, that probability space comes from collection of all possible outcomes for fixed F then for fixed P, you can have different F you can have different P therefore, you will have different probability space. That is means the same problem can be solved in different probability space; that is going to be very important topic in a stochastic calculus or in a financial mathematics, people will solve the problem in different probability space therefore, there results are going to be different based on different probability space.

So, here we have omega and fixed F fixed P therefore, its one probability space. There is a one more observation since when the P is set function satisfying the first condition and the third condition therefore, these P is called measure. You can always define measure on non empty set with sigma field we should have non empty set omega and you create sigma field on and on empty set; then you can define measures satisfying the first condition and the third condition; then that is going to be call it is measure.

Any set function satisfying the first condition and the third condition is going to be call it is measure and here the P is going to be measure because its satisfies it coincides the definition of measure. Therefore the probability the word P is the probability we say the P is measure also.

Not only that by seeing the second condition it is additional condition while seeing the concept of measure; it has the value it cannot cross more than 1 therefore, it is called normed measure. Whenever the measure has finite value then it is called normed measure. So, here the finite value is 1 therefore, it is special case of measure which is a normed measure. So, the probability is measure probability is normed measure because of the second condition.

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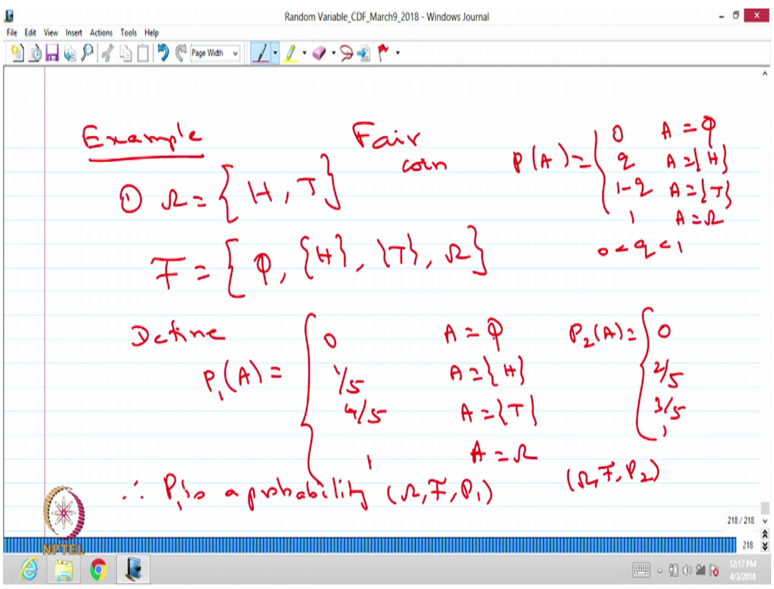

We are going to see different probability for the different omega and F as examples. As a first example, we will take the easiest example that is omega consisting of H T H or T; F will go for the largest one that is empty set, singleton element and the whole set. I have to create function set function P such that it satisfies the all the 3 condition of the probability, then I can conclude that set function is probability define P of A such that when A is going to be empty set; I am going to make it 0 and when A is going to be H, I am going for 1 by 5, when A is going to be tail it is going to be 4 by 5.

When A is going to be whole set; 1, you can verify whether all the 3 conditions of probability definition satisfies. It is always greater than or equal to 0; that is satisfied for all elements A belonging to F that is first condition, second condition P of omega is equal to 1 that is satisfied.

If I take a mutually exclusive events in F, then the union of events then finding the P; P over the events that is same as summation of P of Ai's. Suppose I take the element H and T; so, if make a union. So, that union is going to be omega and P of omega is 1; in the right hand side if I take P of H that is 1 by 5, P of tail that is 4 by 5 if make an addition that is going to be 1.

Not only this, if I take any 2 elements if I make union that is going to be same as P of union is same as some of P of Ai's. For any 2 or 3 or any collection of the mutual exclusive events; it satisfies. So, since it satisfies all the 3 conditions P is probability. Now, we can think of is there any other function also satisfies the same 3 conditions therefore, will have one more probability. So, here this omega F P that is a probability.

So, let me go for making notation called suffix one; that means, P 1 is probability therefore, omega F P 1 it is probability space. By seeing this numbers, we can think of, we can change the numbers; still you can have a probability that is I will go for immediately P 2 of A; that is going to take the value 0 2 by 5 3 by 5 1; the same way A is equal to H, A is equal to tail A is equal to 1; that means, this is also satisfying the probability conditions.

Therefore this is also probability space; is there only these 2 we can go for many combination in which I can go for P of A such that I takes the values 0 with some q; 1 minus q and 1 where q is lies between. So, this is for A is equal to empty set and A is equal to H, A is equal to tail and A is equal to omega where q is open interval 0 to 1, where q is open interval 0 to 1; that means, I can have a many probability.

Therefore, I can have a many probability space for this example; whatever we discuss in the elementary level, we always go for fair coin or unbiased coin. In that case the probability of getting head or probability of getting tail or P of A, when A is could H or A is equal to tail; this values are going to be 1 by 2 and 1 by 2 then it is corresponding to the random experiment of tossing a fair coin or unbiased coin.

So, whatever we discuss in the school days are the elementary level having a probability 1 by 2 and 1 by 2 that is one of the probability space which we have discussed, but in general if the coin is not fair or bias coin; we can have a many more probability space for the same example for a by going for different P. And one more observation by different F also we can have a different P's then you can go for different probability space.

I have fixed a F here I am going for different P, but you can change the sigma field then also you will have different P satisfying the condition therefore, you will have different probability space. I forgot to mention this definition is called axiomatic definition of probability this definition is called axiomatic definition of probability.

This is developed by the probabilistic person name Kolmogorov is Russian mathematician who contributed quiet a lot for probability theory. Therefore, this definition is called Kolmogorov axiomatic definition of probability and this definition is valid for whatever be the situation you have for random experiment and this definition is valid.

Whereas, there is classical definition of probability which is the special case of axiomatic definition of probability that we are going to discuss after few more examples this axiomatic definition of probability does not have any functions on random experiment on omega and F and so, on.

Whereas, the classical definition of probability which is a special case of axiomatic definition of probability that we are going to discuss after few more examples this axiomatic definition of probability does not have any assumptions on a random experiment on omega and F and so on whereas the classical definition of probability which has some assumptions with that we will go for that easiest definition of probability that is the classical one whereas, this one does not have any assumptions therefore, this is in generally it is true for any set of situations where we have going to apply the probability.

So, this is a very important definition called axiomatic definition of probability; this 3 axioms are called Kolmogorov axioms. This 3 axioms or condition are called Kolmogorov axioms and the definition of called axiomatic definition of probability.

So, this is easiest example in which I have introduced a many probability; therefore, we get the many probability space for simple example; its. When it is fair coin then the P of A is going to be 0 1 by 2 1 by 2 1; for this element empty set H, tail and omega.

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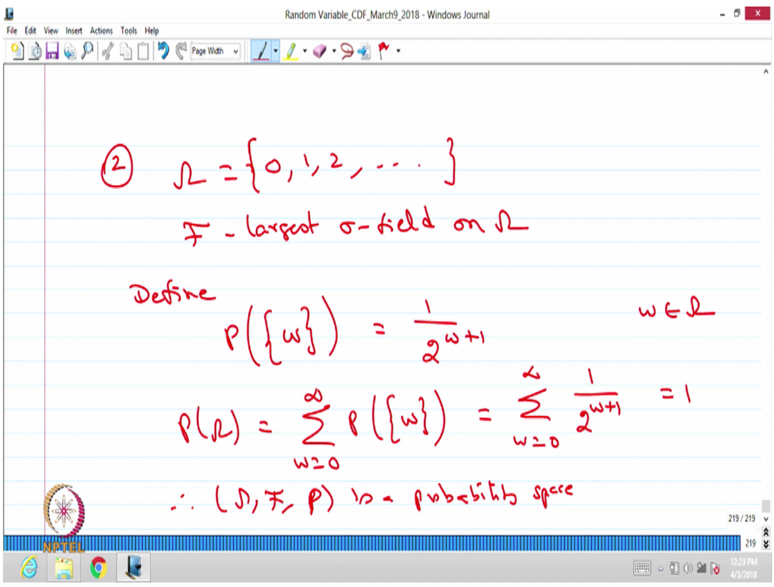

Now, we will go for the second example in which we will have a omega consisting of countably infinite elements. So, omega is going to have 0 1, 2 and so on countably infinite elements. F is the largest sigma field on R over omega. I am not going to list out the elements; that means, it started with a empty sets and all single elements; any 2 elements, any 3 elements and so, on. So, that is the largest sigma field on omega.

Now, we are going to define P of singleton element of samples in the omega itself that is going to be 1 divided by 2 power w plus 1, where w is belonging to omega. Even though the set function is defined from F 2 0 to 1; I am defining for this each singleton element in the omega that for any element in the F nothing, but the subsets of omega that is a union of a few elements of omega that is going to be the element of F therefore, the same set function can be define it further; P of A where a is subsets of F. So, this is going to be 1 divided by 2 power w plus 1.

Let us have a example; suppose w is equal to 0 then the P of 0 is 1 divided by 2 power 1; 1 by 2 suppose w is equal to 1, then P of 1 is 1 divided by 2 power 2; that is 1 by 4. Suppose w is going to be 3, then it is going to be 3 then it is going to be 1 divided by 2 power 4. We can verify whether this is going to satisfy all the 3 axioms of axiomatic definition of probability.

So, the first condition is going to be greater than or equal to 0. So, whatever be the element of w belonging to omega; this value 1 divided by 2 power w plus 1 that is going to be greater than or equal to 0. And P of omega you can verify the second condition P of omega that is going to be same as summation of P of singleton elements of a w's, where w is equal to 0 to infinity. That is same as P of omega because P of omega is nothing, but the union of all the elements that is same as summation of w is equal to 0 to infinity 1 divided by 2 power w plus 1.

So, the first element is 1 by 2, second element is 1 by 4, third element is 1 by 8 and so, on; if you add all the elements it is going to be 1. If you take any disjoint mutually disjoint events nothing, but disjoint samples and find out the P of union that is same as a summation of P of singleton elements. That is same concept I used it in finding the P of omega also, I have used the P of omega is nothing, but the summation of P of individual elements I am using the third property; so, third property also satisfied.

So, all the 3 properties are satisfied all the 3 conditions or all the axioms are satisfied therefore; the P is going to be probabilities. Therefore, omega F and capital P is probability space; again the way I define the P of omega is equal to 1 divided by 2 power w plus 1, someone can think why I has to be this number can I chose some other number so, that I can have all the 3 conditions are satisfied? Yes, you can always create another real valued function so, that all the 3 conditions satisfied; we can have probability.

So, I have just created a easiest one that is 1 divided by 2 power w; I plus w plus one because I know that the summation of this is going to be convergent series that summation is going to be 1 therefore, I have created probability over this. We will go for the third example that is omega consisting of uncountably many elements that is 0 to infinity.

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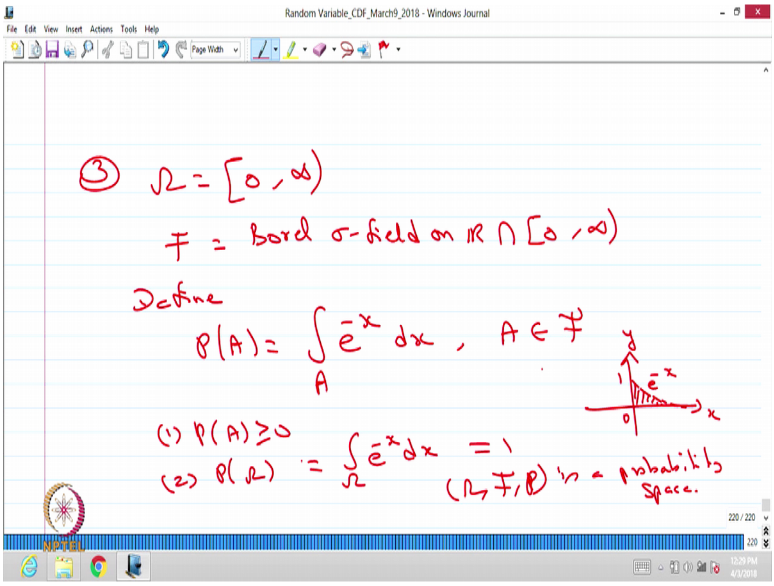

So, the sigma field that is Borel; Borel sigma field on the real line which intersect 0 to infinity correct creating Borel sigma field on real line intersecting with 0 to infinity that is going to be the largest sigma field on omega, where omega is 0 to infinity.

I am going to define a set function P such that this is going to be the probability. The easiest to one that is integration over A e power minus x d x, where A is belonging to F; A is belonging to F. The F consisting of singleton element; sorry empty set singleton element and many intervals between 0 to infinity which is in the same form of Borel sigma field on real line.

Therefore the integration over A that is Riemann integral; you know that if A is going to be a singleton element that is going to be 0. If A is going to be closed interval or open interval because it is a Riemann integral; the answer is going to be integration over that interval e power minus x d x. So, always the first condition is going to be satisfied that is P of A is always going to be greater or equal to 0. The second condition we can verify P of omega that is nothing, but integration over omega e power minus x d x that is nothing, but x y e power minus x.

So, it start from 1, it start from 1; so this is e power minus x integration over omega, omega is from 0 to infinity. So, 0 to infinity e power minus x d x is nothing, but area below the curve I power minus x between the interval 0 to infinity, it is asymptotically 0 at infinity. So, if you find the area that is going to be 1; the third condition if you take a mutually disjoint events in F that is nothing, but if you take non overlapping intervals and find the area in those intervals and find out the intervals separately and sum it up; both are going to be same.

Whether you go for union of non overlapping intervals, find out the P or finding out individually and sum it up both are going to be same; for any mutually disjoint events that is nothing, but the intervals. So, the third condition also satisfied therefore, we can conclude this P is going to be a probability. We can have many more P for the same omega and F. So, this is the easiest one omega F and P that is probability space.

With this 3 examples one is with finite, other one with countably infinity elements and the third one with uncountably many elements of omega; we have created different function P and we got the probability space.