Lecture – 09
Bifurcations

Welcome to students. So, today we will be looking into this concept of bifurcations. Now let us again recall the logistic map that we had done previously.

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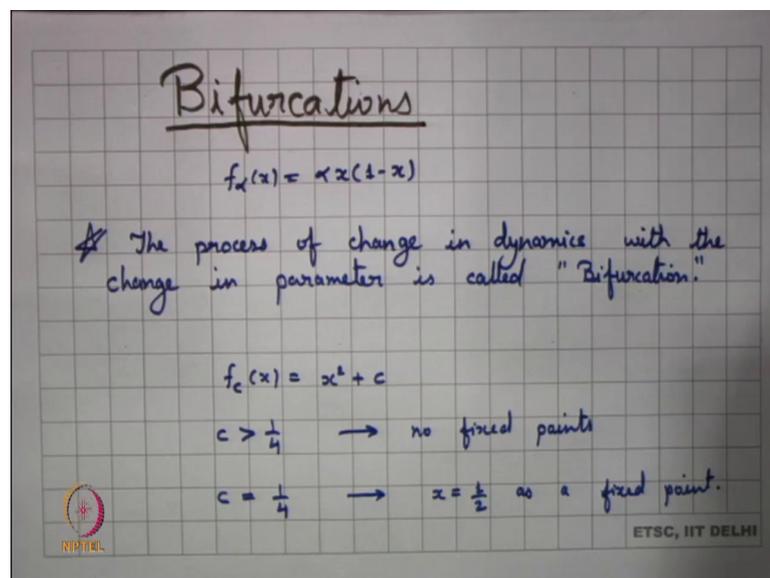
So, we look into the logistic map f alpha x is alpha times x into 1 minus x. Now we had seen that we had looked into a qualitative analysis of this family of functions, and we had observed here that s parameter alpha changes from 0 I mean going up from 0 to 4 right the dynamics changes from being a very simple dynamics without periodic points, with periodic points, simple kind of periodic points and becoming highly chaotic.

So, we had seen that the dynamics changes as the parameter changes, and this aspect of change of dynamics with the parameter is called bifurcations. So, what are bifurcations we can simply defined them as, the process of change in dynamics with the change in parameter. Now in our population model we had studied the logistic map which is again a family of curves, today let us look into another example. So, we take up this example again another family of curves we define fc of x to be equal to x square plus c.

Now, if we look into this particular curve right this again as your c varies right this family varies and with this part we can again try to look into we can look into the qualitative analysis of the dynamics shown by this particular function. So, we observe here that when I am looking into x square plus c right and if my c is greater than 1 by 4, we can have no fixed points is a very simple exercise to check that x square plus 1 by 4 equal to x right will not have any solution; what happens now when c equal to 1 by 4. So, we observe that when c equal to 1 by 4, we have exactly 1 fixed point here right and that fixed point is x equal to 2.

So, you have x equal to 1 half as a fixed point, now what can you say about this particular fixed point.

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So, we see that here, your fc prime of x is 2 x right. So, for this particular point, when c equal to 1 by 4 right we find that f 1 by 4 prime at 1 by 2 happens to be equal to 1. So, you are basically a 1 by 2 happens to be a non hyperbolic fixed point. So, 1 half is a non hyperbolic fixed point and since my double prime, it is not equal to 0 anywhere right this point is unstable. It does not attract me it attacks on one side it does not attract from the other side.

But what happens here as we vary our c. So, we try to see these graphs as c varies. So, we try to see this particular graph.

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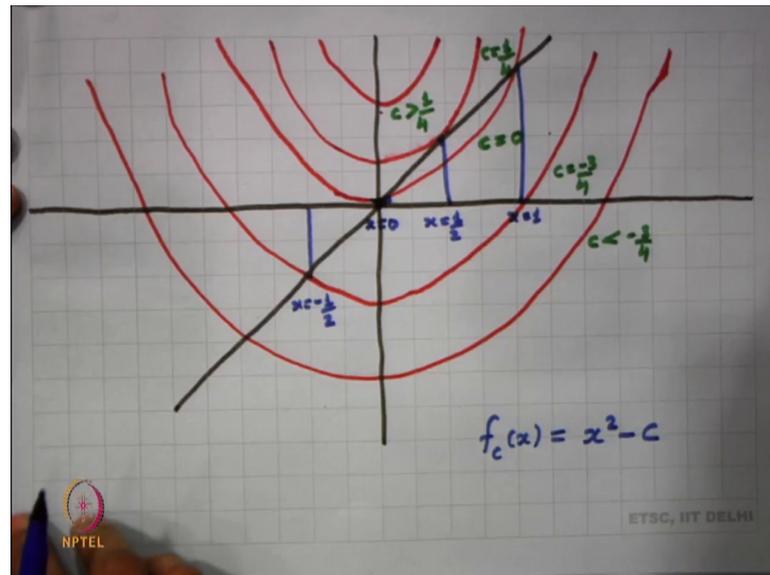
Now, what happens at c greater than 1 by 4, you find that this has no fixed points when c equal to 1 by 4 we find that it exactly intersects the diagonal in 1 half. So, there is only one fixed point here, as c becomes greater than as c becomes less than 1 by 4 we find that there are 2 fixed points here. So, exactly it is equal to 0, we have the function is just x squared and we very well have studied the dynamics of x squared.

So, at c equal to 0 we get 2 points, and then we get 2 particular points 2 fixed points we know that one of the points will be a source and one of the points will be a sink. Now this particular aspect goes on till we reach c equal to minus 3 by 4; what happens at c equal to minus 3 by 4. So, let us start looking into that part. So, what happens when c equal to minus 3 by 4, we have 2 fixed points right and maybe I can put up this 2 fixed points to be say it is minus half and the other point is 3 by 2.

Now, what happens at minus half? So, we find that f minus 3 by 4 prime at minus half right its nothing, but minus 1. So, again here minus half becomes a non hyperbolic fixed point, and what can you say about this particular fixed point this Schwarzian derivative here will be negative its Schwarzian derivative here is negative and hence this is a sink right. Things again change when your c decreases from minus 3 by 4. So, as it goes beyond minus 3 by 4, you find that a periodic orbits it starts with the non hyperbolic fixed point and the other non hyperbolic fixed point we get it at c equal to minus 3 by 4.

And once c goes beyond minus 3 by 4; that means, a decreases from minus 3 by 4, we find that there is a periodic orbit of period 2 coming up. So, we look into this fact what happens when c is less than minus 3 by 4. So, when c is less than minus 3 by 4 of course, it does have 2 fixed points, but then there is a periodic orbit of period 2 coming up. And if we try to look into this aspect right try to look into this further, we find that this again displays the same kind of dynamics that you had observed for the logistic map.

So, we try to look into the bifurcation diagram for this particular function. So, we look into this function fcx equal to x square plus c.

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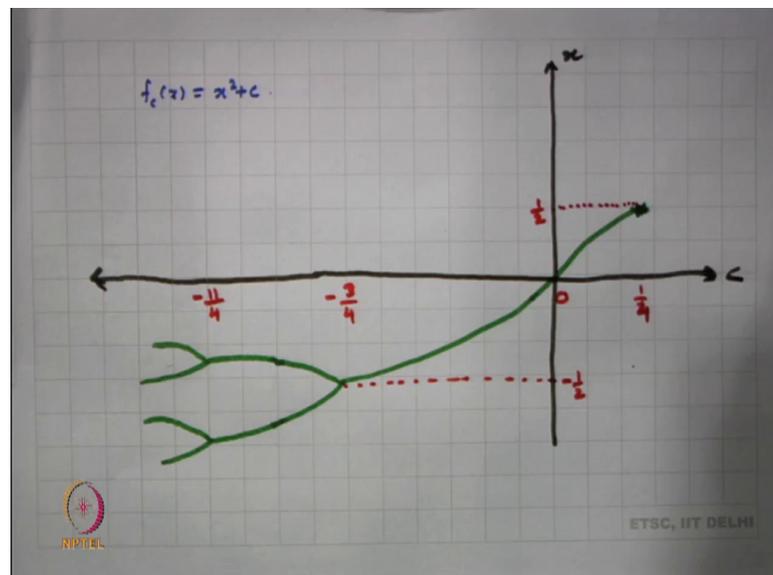
What we find is that there is no periodic point for c greater than half; there is no periodic point here. When c is equal to half you get one particular periodic point right one fixed point basically, either the orbits converge to this point or they diverge right. So, you only find this particular periodic fixed point and that continues till you reach minus c equal to minus 3 by 4, when you have again you have one fixed point minus 3 by 2, you have other fixed point which is not at all basically it is not at all stable here. So, it goes beyond 1 by 4 right this is one fixed point you get 2 fixed points here, but the other fixed point is not at all into picture because it is not at all its a source right.

So, everything diverges away from here there is only one which is attracting something to itself. So, you can see in the bifurcation diagram only one of them coming up over here and one of them coming up over here goes up to here, what happens when c is less

than minus 3 by 4 again you find that the fixed points they become source and what do you find is that a periodic orbit of period 2, that becomes a sink and the orbits are attracted to this periodic points of orbit 2. This goes on till your c reaches minus 11 by 4, once it crosses minus 11 by 4 you again find a bifurcation here because then what happens here is that this periodic orbit of period 2 right it loses its attractive nature, it becomes repelling and then there is another periodic point of period 4 which comes into picture.

So, if we try to look into this and we try to look into this further, what we find here is that we exactly not exactly in a same manner, but we exactly get the same kind of figure that we had seen for the logistic map.

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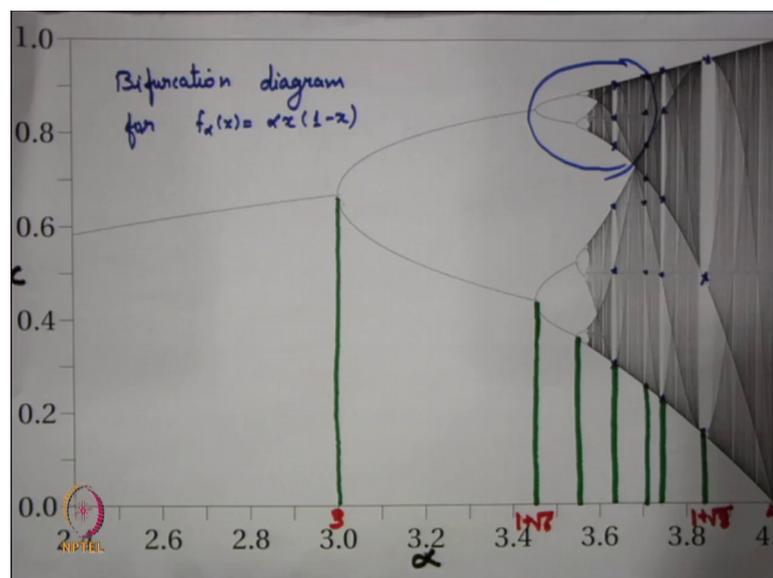
Now, this is the bifurcation diagram for the logistic function. So, I should say if f alpha x equal to alpha x 1 minus x. So, if we will try to look into this bifurcation diagram for, we find that this exactly the picture that you get for this quadratic family is same as what you had got over here.

So, there is a fixed point which is attracting then that loses its attractive nature, and then there is another periodic orbit of period 2 which comes up which becomes attracting and then that loses its nature, and then again there is an attracting period of period 4 coming up. So, what we find here is that there is bifurcation as your parameter changes and when this parameter changes, it basically results into the doubling of period. So, what becomes

repelling and what becomes attracting is exactly the double its the double period of what was attracting earlier and so, this bifurcation is also called the period doubling bifurcation.

So, this is a period this is an example of a period doubling bifurcation. So, we have a period doubling bifurcation here and ultimately we get into the same thing. Now what happens to all such period doubling bifurcations is there a way where we can have some kind of quantitative theory of bifurcations. So, we will let us look into the logistic map once again since we have studied that in detail. So, we look into this logistic map, and we say that we had this sequence alpha ns of parameter, where a bifurcation occurs.

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So, the dynamics changes the dynamics remains actually constant for a long stretch of alpha and then it changes at some particular point.

So, we have this alpha n to be parameters build the sequence of parameters where bifurcations occur, and we are looking into a logistic map. So, let us say that where a bifurcation occurs in the logistic map, in the logistic family. Now what happens to this particular alpha n, is there a way where we can actually try to compute this alpha n? Knowing some parts can we compute the other. So, it was observed by Mitchell Feigenbaum somewhere again in 1980s, that there is a way of quantifying this period doubling bifurcation or quantifying this kind of bifurcation.

So, what he observe that, when whenever you take say let me take talk in terms of delta n. So, let this delta n be say alpha n minus alpha n minus 1 right let us take this ratio alpha ns. So, the difference between the previous 2 changes in bifurcation, where the bifurcation occurred. So, difference between the previous 2 parameters where bifurcation occurred and the difference between the next 2 parameters where the bifurcation occurs. So, look into this particular concept plus 1 minus alpha n. So, try to look into this particular ratio let delta n be this ratio and let delta be the limit as n tends to infinity of this delta n.

Then this limit is a universal constant for all kind of period doubling bifurcation, and this limit turns out to be of course, it is a irrational number it turns out to be something like 4.66920161 and so on. So, this is a universal constant and basically this constant is called the Feigenbaum constant and this shows some kind of universality into the dynamics of all such one parameter families which undergo period doubling bifurcation.

Now, let us look into what is the aspect this Feigenbaum constant has other implications also. So, let us try to look into this bifurcation diagram once again, and for the present thing I just want you to concentrate on 2 aspects. Supposing I forget this part right; I forget this particular portion of the bifurcation diagram and I will just consider this particular portion then we find that this particular portion is exactly similar to the whole forget this particular portion of the bifurcation diagram and forgetting this particular portion of the bifurcation diagram I am looking into this particular portion, we find that this particular portion is also exactly similar to the whole one.

Now, I can do that this at various stages. So, at various stages right I can forget a part of bifurcation and I can look into what happened at particular point, maybe if I try to magnify and look into this particular point, I find that here also I get the same kind of. So, for example, if I look into this aspect I just look into this particular picture right what I find is that going from here to here right I get the same kind of picture. So, what is observed here is that, this particular picture is some sort of has some sort of self similarity into it. So, within chaos there is also an underlying self similarity or something geometric there and the measure of this geometric this kind of geometric figures which are self similar are called fractals.

So, you find that there is some kind of fractal aspect to this also and the Feigenbaum constant is also related to this kind of fractal nature of the bifurcation diagram. So, we have this Feigenbaum constant, and we try to now look into another map. Now this time we are not looking into a family of maps we are just considering one particular map. So, I am looking into this particular map which I call the tent map; the reason of calling it a tent map is because it looks like a tent now look into this tent map. So, this tent map takes the value 2 x between 0 and half, it takes the value twice 1 minus x between half and one.

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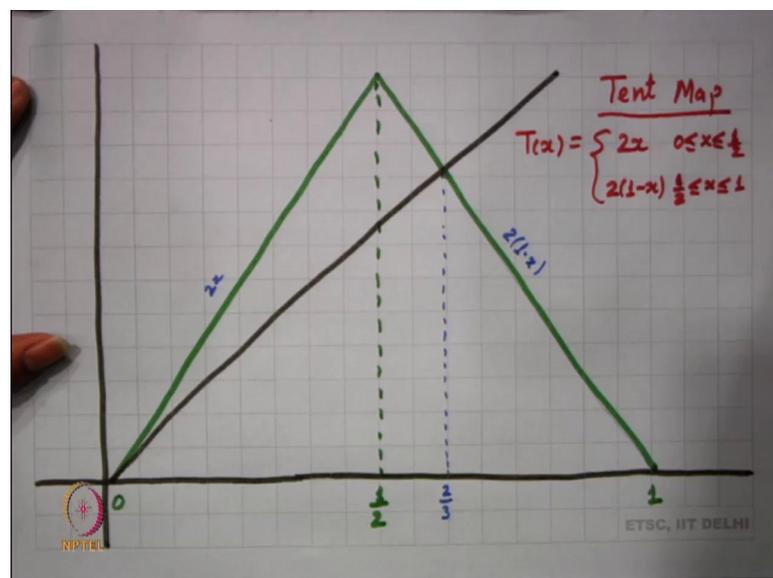
The slope at each and every point is 2 accepting at the point half, where it is not differentiating. So, we find that this particular point is not and this is not a smooth compared to logistic curve, it is on its a smooth curve everywhere, but this particular curve is not smooth everywhere, but this is kind of piecewise I can call this map as piecewise linear. It is linear it is completely linear between 0 and half and it is completely linear between half and 1 and it has a slope 2 almost everywhere, right. So, now, let us try to look into this particular tent map, and try to look into the dynamics of this tent map.

So, 1 thing here is I can easily see this from the picture itself that it has 2 fixed points it has 1 fixed point at 0 and it has another fixed point at this particular point. Now its very easy to check that it is at this particular point. So, basically this will be a fixed point of

this particular equation right and that turns out to be if you can compute that that can turns out to be 2 by 3. So, what happens here is you have 2 fix points here, you have a fixed point at 0 and you have another fixed point at 2 by 3. The question comes up here that for this particular tent map we do have fixed points do we have period 2 points do we have period 3 points.

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So, one can simply observe that look into the tent map take any n in N, and we want to know all those values of x for which T to the power n of x will be equal to 0. Now try to compute this for any n right we want 2 to the power n of x to be equal to 0, and we find that whenever x happens to be equal to twice k upon 2 to the power n actually it should be 2 to the power n minus 1 does not matter, where 0 is less than or equal to k is less than or equal to 2 to the power n minus 1. So, if I look into all these aspects, I find that for all such x my 2 to the power n of x happens to be equal to 0.

Now, I want to also compute it is just kind of computing stuff, that T to the power n of x 1 is it equal to 1. So, looking in to the fact when T to the power nx is equal to 1, we find again something similar. So, this happens this is true for all x equal to 2 to the power k minus 1 upon 2 to the power n, where my k varies now my k varies between 1 and 2 to the power n minus 1. So, one thing is clear I have at all these particular values, I have that in n steps x reaches 0 and for these particular points in n steps x reaches 1.

So, whenever I start with this interval. So, I start with this interval k upon 2 to the power r and k plus 1 upon 2 to the power r, if I start from this interval and I know that in this interval I will always find points such that one of them in n step reaches 0, one of them in n step reaches 1 and hence there will be a fixed point for T to the power n in this interval. So, what we have observed is so, there exists x right such that T to the power nx is equal to x. There are 2 things that we observe from here, first thing is that for every n we have a periodic point of period n and how many such periodic points can we have as we know that this is tending to 0 for all k between. So, these are 2 to the n values here right.

So, the fixed points of T to the power n right will be 2 to the power n. There are 2 to the power n fix points. So, how many periodic points of period n you can have? What you are doing is you are just subtracting on your from 2 to the power n just remove all the fixed points right of T to the power n minus 1,T to the power n depending on because they will be fixed points here in case n happens to be a multiple of k. So, you remove all the periodic points what you get is the periodic points of period n. So, that gives me 1 thing, it says that there do exist for every n there exists periodic points of period n and something more.

Now, think of this interval right I am looking into this interval as k varies I am looking into this interval, we are just partitioning you can vary k from say 2 to the 0 to 2 to the power n minus 1. So, if you vary k in this manner we are just taking a partition of 0 1 right into say partitioning that right with this gap of 2 to the power r right length 2 to the power r, with where we are partitioning that into subintervals of length 2 to the power r and that tells me that an any every such subinterval contains a periodic point.

So, it also tells me that the periodic points will be dense right. So, what we find here is that the periodic points are dense. Now we are interested in in fact, this is a very simple curve. So, we may be interested in looking into the periodic points also. So, this is some kind of numerical results or may I would say in certain some kind of number theory results that you can get.

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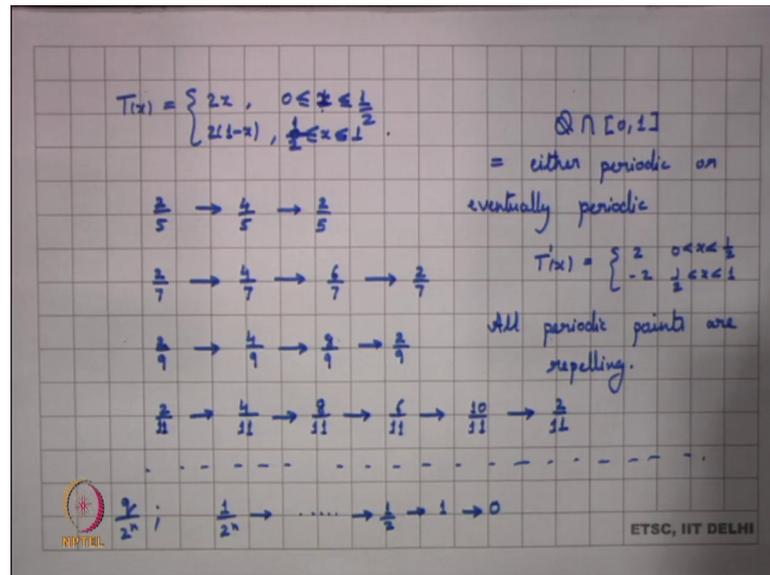
So, I am defining my Tx to be equal to 2 x for 0 less than or equal to half, x less than or equal to half and it is twice 1 minus x, for 0 less for half less than or equal to x less than or equal to 1.

So, you find that this is your tent map, and for this particular tent map you want to see what happens to the what are all the periodic points. So, a simple computation right some kind of observation gives me that if I look into this particular point say 2 to the power 5, 2 by 5 supposing I take this point to 2 by 5 then what happens to 2 by 5? Now 2 by 5 basically lies in the first half and in the first half my or my map the value of the function is 2 x. So, it is being mapped to 4 by 5.

Now, 4 by 5 lies in the second half. So, here my function is twice 1 minus x. So, where is 4 by 5 being mapped to?

Student: 2.

It is again being mapped to 2 by 5. So, we find out 2 by 5 happens to be a periodic point of period 2 right. 2 by 5, 4 by 5 these are periodic points of period 2. Now I go a little bit further right and I take this observation of looking into 2 by 7, where is 2 by 7 being mapped? To its mapped to 4 by 7, where is 4 by 7 being mapped to 6 by 7, where is 6 by 7 being mapped to? Again 2 by 7.

So, we find that 2 by 7, 4 by 7, 6 by 7 these are periodic points of period 3. These are all elementary function because the function is very simple we can have make such elementary observations, let me now look into 2 by 9 where is this being mapped to? Obviously, 4 by 9 where is 4 by 9 being mapped to.

Student: (Refer Time: 27:02).

So, this is being mapped to 8 by 9, where is 8 by 9 being mapped to?

What did we observe here again periodic orbit of period 3, what happens to 2 by 11 going a little bit further 6 by 11, next 10 by 11.

Again 2 by 11 what did we get? A periodic point of period 5 very simple deductions we can see that all points of the form even number divided by an odd number right 10 to be periodic points even by all right into 10 to be a periodic points 4 some particular period here. Now we can take this theory, you can just make this observation here the other thing which we again want to look into what happens to the dyadic rationals. So, we are looking into the form all rationals of the form q upon 2 to the power n, what happens to this particular say rationals.

So, let us start with 1 by 2 to the power n right and we know that in n steps in n minus 1 steps this is going to reach 1 by 2 right and then 1 by 2 is mapped to 1 and then 1 is mapped to 0, we know that 0 is a fixed point. So, you find on this point 1 by n 1 by 1 upon 2 to the power n happens to be eventually fixed, what can you say about all dyadic rationals in this particular way? They all turn out to be somewhere eventually fixed. Now let us look into those rationals right may be of the form odd or may be try to let us try to summarize this for all rationals.

Now, I am looking into all rationals between 0 and 1, can we say that all these rationals are either periodic points or eventually periodic points. The dyadic ones are definitely there. So, a very nice elementary observation about this tent map is that all this rationals right. So, if I take Q intersection 0 1, all these rationals right they are either periodic or eventually periodic. Let us look into the next observation this is my map T we know that this map T has a slope 2 everywhere right module as 2 everywhere.

So, what we find here is let T prime of x this is either 2 or minus 2 everywhere. So, either 2 or minus 2 its 2 basically when your 0 is less than x and pushing this up less than x less than half, we are not looking into the end points and half is less than x is less than 1 right we find that T prime x is always 2 or minus 2. That let tells me that each of this periodic point is always going to be a source right they are all repelling periodic points; that means, if I look into the dynamics of tent map. So, very interesting dynamics we find that all it has periodic points of all orders right. In fact, it has a dense set of periodic points, what we have seen is that every rational is either periodic or eventually periodic.

So, we know the dynamics of a density here, but for the other half, we do not know what happens to the dynamics because everything here right all the periodic points here none of the periodic points here is attracting, all the periodic points here are repelling right. So, you can say that all periodic points are repelling here. Now this gives an interesting observation what happens to the non periodic orbits. So, if you look into the non periodic orbits they have no way to settle.

So, they will just keep on moving throughout 0. Now with this observation we try to see something more on tent maps, but before that we look into some more definitions. So, I want to define few things here and for that again we get back to the very abstract concept of metric spaces. So, I am looking into my metric space x. So, I have a metric space say Xdx.

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And I have another metric space say Y dy and now I am looking into this dynamical system I am looking into 2 different dynamical systems. So, I have a dynamical system here xf and I have another dynamical system here y say g.

Now, I have this 2 different dynamical systems and I am looking into the fact that if there exist a homeomorphism h from X to Y, such that hf is same as gh; that means, these 2 functions are the same at each and every point x of x right, then the dynamical systems xf and yg are topologically conjugate. Now exactly what do we mean by saying that these are topologically conjugate. So, let us draw this picture here, say I have this x here right my space x here, my space y here, there is a map here from x to y, x to x there is a map there is a map from y to y, there are maps from x to y, here and we know that f takes X to X g takes y, to y h takes X to Y and again h takes X to Y; that means, h inverse will be taking y to x and we find that this basically this diagram right happens to be commutative.

So, I can say that I have hf equal to gh or I can write my g as hf h inverse same as g or I can say that my f is same as h inverse gh that is what we mean by topologically conjugate or we say that this is what is called a conjugacy. So, we call this a conjugacy, what are the properties of topologically conjugate systems. So, we look into this topologically conjugate system.

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Now if h from defined from X to Y; now x is simply a map from x to y. So, if h from X to Y is a conjugacy. So, between my xf and yg, then I can take any power of f. So, I can take any iterate of f it will also be a conjugacy between that. F to the power n and y g to the power n right where every n in N, I can take any power right.

It is going to be a conjugacy if my f and g are homeomorphisms, we can basically take n belonging to z because then it will be a conjugacy for the other aspect also. So, fetches if my f and g are homeomorphisms, we can look into n belonging to z. Now what does essentially what is the meaning of this aspect? Think of that, if I have my f of x to be equal to x. So, I take my f of x to be equal to x now what is h of fx? Simply h of x, but I know that my hf map hf is same as gh. So, what we get here is gh of x is same as hx which basically is same as saying well since we are looking into composition the g of hx is same as hx, which tells me that every fixed point of f in x corresponds to a fixed point of g in y right the converse is also similar every fixed point will be corresponding to the fixed point in x.

So; that means, that the number of fixed points that f has is same as the number of fixed points that g has right and both these fixed points correspond can be corresponded with each other. There is another aspect now; as I said that it is also conjugacy you can also see that it is a conjugacy between f to the power n g to the power n, what happens when f to the power n x equal to x. I look into again h of f to the power nx same as hx, which means that g to the power n h of x same as hx which basically means that g to the power n of hx is same as hx.

So, the periodic points in x correspond to the periodic points in y, the 2 systems are distinct right my 2 systems I have the dynamical system xf and I have a dynamical system yg. So, the map acting on y is g, the map acting on x is f, but since there is a conjugacy between them we can say that they have the same orbits. The same number of orbits same type of orbits you can always relate 1 to the other. So, this is something which topological conjugacy is something which is sort of giving us equivalence in the dynamical systems.

So, we call that 2 dynamical systems to be equivalent, if they are topological conjugate. So, let us write it down

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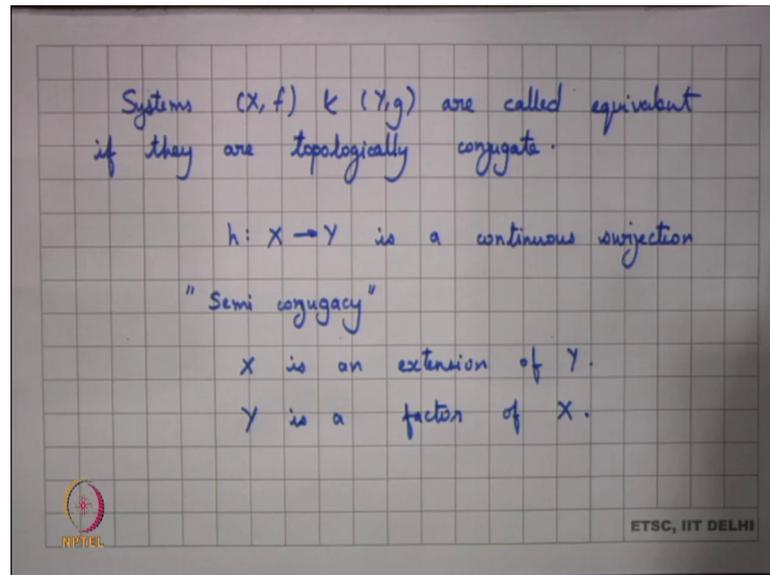
So, the systems xf and yg are called equivalent if they are topologically conjugate. This can be thought of something like a homeomorphism between 2 dynamical systems right. So, basically this is a morphism. So, conjugacy is nothing, but its basically a morphism between 2 systems. Now similar aspect here is which we can also look into, but not going into details here, but we will be needing this concepts on the time later.

What happens when you are looking into your map h right this map h from x to y is just a continuous surjection? So, if x h from x to y is simply a continuous surjection,   I   do not get these are not homeomorphic, but I can say that periodic points here right they are mapped to periodic points here may not be the same period. In case of conjugacy its the same period right here it may not be the same period, but periodic points here are mapped to periodic points here. So, ultimately we are carrying out this property from x to y, and this is something which is called a semi conjugacy

Or we say that x is an extension of y or I can say that y is a factor of x. So, we can think of and then my h happens to be a factor map, we can say that fine y is a factor of x I always have a continuous surjection which takes my x to y. So, their systems x and y are related in the sense that their whatever property that there are properties of x which are being carried forward to y. The entire if we look into the abstract theory of dynamics it always looks into what are the properties which can be extended right what are the properties which can be taken as a factor which are satisfied by factors. So, this is

another aspect of looking into systems, but why we had mentioned this here was we again want to go back to our system logistic map.

So, we go back to our logistic map here. So, I have my f 4 of x.

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Now, I am looking into my parameter being equal to 4. So, f 4 of x was 4 x times 1 minus x and I had my tent map here Tx equal to right twice x when I have 0 is less than or equal to x less than or equal to half and I have twice 1 minus x where my 1 by my half is less than or equal to x is less than or equal to 1 we defined h from 0 1 to 0 1 is h of 6 equal to sin square pi by 2 x. So, this we can consider at half of 1 minus cos of pi x, what is. So, we start with. So, 0 less than or equal to x, less than or equal to half.

Now, my T here happens to be equal to 2 x. So, if I look into my h of T,x my h of Tx is nothing, but half of 1 minus cos of 2 pi x. Now 1 minus half of 1 minus cos of 2 pi x I can write it as 1 minus cos square pi x now 1 minus cos square pi x I can write it again as 4 times half of 1 plus cos of pi by.

Student: Pi x.

Pi x into 1 minus cos of pi x and I need a 1 by 2 again here multiplying by 4. So, this is half of 1 plus cos pi x into half of 1 minus cos pi x, and we find that this is nothing, but I have 4 here this is nothing, but my h of x. So, this is h of x and we find that this particular term right turns out to be nothing, but this is 1 minus h of x. So, this particular

term turns out to be f 4 of h of x. So, this is f 4 of h one can calculate what happens between half and 1 my Tx changes here.

But again we find the same we can have the same calculation we can say that here my h of Tx happens to be f 4 of hx. So, what they find here is that you take the interval 0 1 go to the interval 0 1 right. So, your f 4 is taking 0 1 to 0 1, again I have this interval 0 1 here my h takes 0 1 to 0 1 again in interval 0 1 here my h takes 0 1 to 0 1 and I have my tent map T taking 0 1 to 0 1, this diagram commutes and hence we can say that the dynamics of the tent map is same as the dynamics of the logistic function when alpha equal to 4.

Now, we may have already seen what is the dynamics of the logistic function when alpha equal to 4 right its highly unpredictable there, and hence we can say that the tent map displays the same dynamics there the a orbits of the tent map basically correspond to the orbits of the logistic map when alpha equal to 4, and they have the same dynamics we have. So, we stop here today.