Chaotic Dynamical Systems
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Lecture – 07
Sensitive Dependence on Initial Conditions

Welcome to students. Today we will be looking into quantification of stability. The concept of stability and the concept of unstability that was seen earlier. Now this of these observations were made earlier, but this is exactly chaos was not even looked into till maybe the concept of stability was quantified in the late 1800's. And then whole thing had to wait for the advent of computers to give it a particular shape.

So, this story starts with Edward Lorenz, was a meteorologist and also mathematician.

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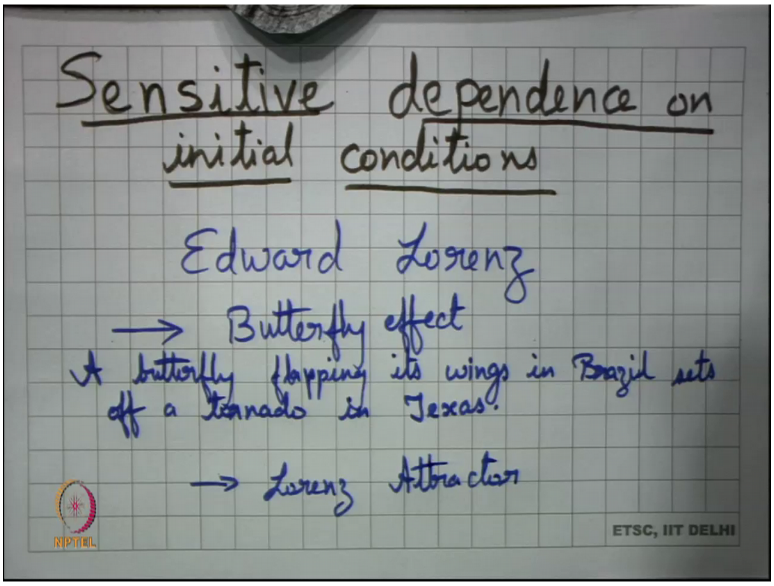

He had used navier stokes equations to model to make a mathematical model for predicting weather. And then this computers helped him a lot in trying to simulate his equations. So, it was one fine morning that he wanted to rerun one of the simulation. So, instead of re running the entire simulation. Once again, he thought of taking a shortcut. So, he had a printout from the previous simulation. And that midway along that he took this data from midway along the ah on in this printout, and he inserted that into the simulation. He inserted that as an initial condition into the equations.

Now when that simulation completed, he found out that the result that he got was completely different from the result of the previous simulation. So, this was a kind of eureka moment. What he observed was that a very small change in the initial conditions. Because the computer gave him the data which had some kind of rounding off for whatever calculations it did. So, this was a very, very minute change in the initial conditions. And this minute change in the initial conditions led to something a very, very sort of a very divergent result.

So, that is something which he named this phenomena as the butterfly effect, that a butterfly flapping it is wings in Brazil sets off a tornado in Texas. So, this is what he observed. And later on, he tried to draw the graph of the orbits, that resulted out of his simulations. So, out of his equations he tried to draw the trajectory of the graph of this orbit. And what he observed was a completely distinct picture.

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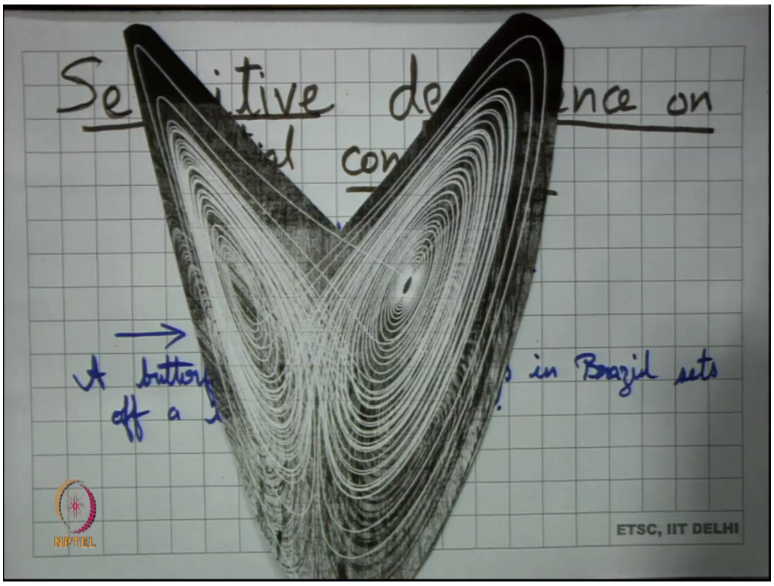

The graph gave was something of the sort where he could find that this trajectory is traversing in a double spiral manner, it is not even intersecting at any point. And yet it is something which is very, very highly unpredictable. So, this is something which came to be known as Lorenz attractor. And unpredictability got a new name the butterfly effect.

So, as we go on there was a need to measure this butterfly effect. So, we now look into this measuring of butterfly effect.

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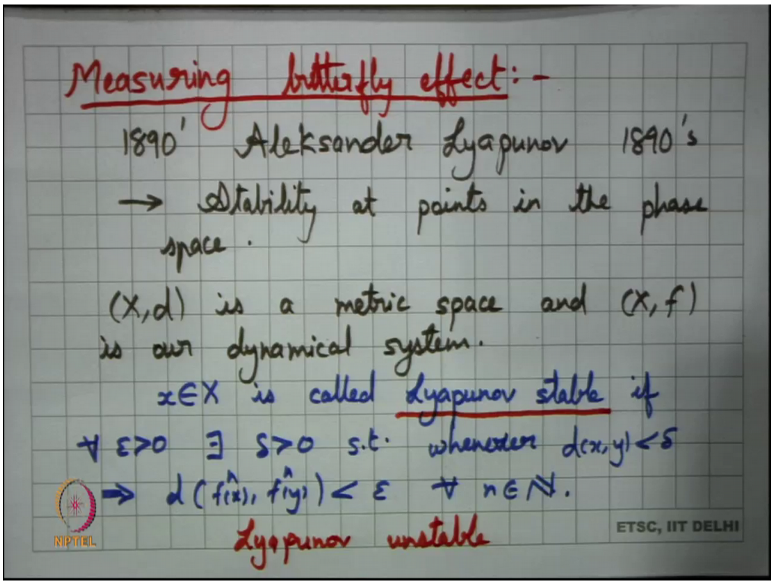

Almost the late 18 or 1890's, this Russian call alexander Lyapunov he wrote his thesis and the stability of orbits. And when he wrote his thesis he defined something called the stability. So, he basically defined the concept of stability and unstability. So, he defined this concept of stability of orbits or stability at points in the phase system in the phase space.

Now, they will exactly try to put up his definition, in a very abstract form in terms of in metric spaces. So, for us throughout this lecture today xd is a metric space, and x with the continuous self-map on x is our system. What did Lyapunov exactly define? So, he said that if we have this x, point x in X, then this point x is called Lyapunov stable; if for every epsilon positive, there exist a delta positive such that such that whenever your x and y, y is delta close to x, then that implies that the orbit of y epsilon close to the orbit of x; that means, fnx will be delta close to f and y for every n in N. So, the orbits are just going to go parallel to each other the orbits with neither intersect, but they will always be very close to each other.

Now, this is what led to the concept. Since this was basically the concept of Lyapunov stable. And so, came up the concept of what do we mean by Lyapunov unstable. Now any point which is not Lyapunov stable is Lyapunov unstable. So, one could guess that Lyapunov stab unstable means that, nearby orbits would diverge at some point of time.

So, Lyapunov wanted to Lyapunov looked into this qualitatively, and he also tried to look into it quantitatively.

So, what Lyapunov defined was something call Lyapunov exponents.

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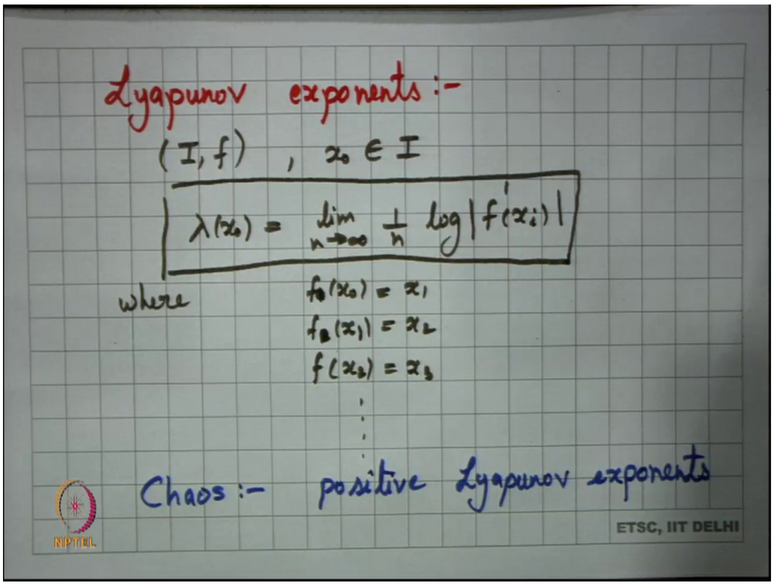

Now, Lyapunov was working with the model of a differential equation. So, he had a differential structure in his mind. And since he was working with this differential structure, his entire his concept is basically defined for a diffeomorphism. What we will try to do is we try to look into Lyapunov exponents from our point of view. So, we will try to look. So, for this for looking into Lyapunov exponents, we take our system to be an interval I and a continuously differentiable function f defined on it. So, our system is if and Lyapunov said that what happens for the system I f we take x not belonging to I.

And now I am looking into lambda x naught to be basically the limit as n tends to infinity of 1 upon n log of mod of f prime of xi; where my xi varies over the orbit of x naught. So, I have f one of x naught to be equal to x 1, right. I have f 2 of x f x 1 to be equal to x 2, I have f of x 2 to be equal to x 3 and so on.

We will looked into so, what Lyapunov exponent measures is take the orbit of a point of x. And try to compute the average of the derivative at each point in the orbit. This gives you a quantification of what kind of divergence you can observe in nearby orbits. So, this is one of this is one of the factors this is one of the quantities that can be associated within orbit. And for many valid reasons Lyapunov exponent turning out to be positive is considered to be chaos.

So, one of the definitions of chaos is having positive Lyapunov exponents. If you observe positive Lyapunov exponents. If you observe positively Lyapunov exponents. It is many times considered chaos. Now this is basically can think of this definition, we can think of this in the more general setting of a manifold. And a diffeomorphism on a manifold. But we tried to look into it from on just an interval trying to understand how this can be how chaos can be quantified, but there was another study taken up in something around 1800’s, where we had this mathematician guckenhiemer and auslandaer and Yorke come up with this concept of sensitive dependence on initial condition.

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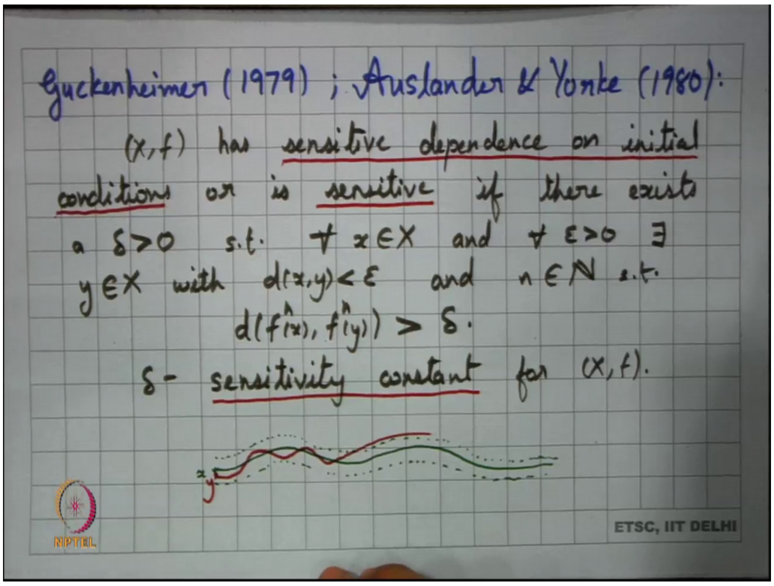

Now, Guckenhiemer had made this study entirely for interval maps, auslandaer and Yorke looked into this study from the point of view of an arbitrary metric spaces. Both this concepts these papers were written at the same time. So, this can be considered as an overlap of ideas. And we can define this as for our system xf. We say that xf has sensitive dependence on initial conditions or in short, we can say that it is sensitive. Maybe, this is what we want to define, if that exist a delta positive. So, for us the delta positive existence of this delta is very important because this is sort of a universal thing for the system.

So, if there exists a delta positive such that for every x in X and every epsilon positive there exist a y in x with dxy less than epsilon and an n in N. So now, what you want is that you have a delta positive, there exist a delta positive such that remember you pick up a point x in X. And whenever you take any epsilon positive, and you there exist a y in the epsilon neighborhood of x, and there exist some n some natural number n which corresponds to the iterate, such that your d of fnx and fny is always greater than delta. And here our delta since this is universal for the system, we call delta to be the sensitivity constant for our system. So, this is the sensitivity constant for the system.

Now, exactly what does sensitivity mean? So, what we try to do is let us take we have a delta. So, our delta is already we do have a delta positive. So, I am taking looking into this point x. And I try to look into the orbit of x. And since we have a delta positive given to us. So, one can think of delta neighborhood, right of x. So, you can have a tube of radius delta, right. Around the orbit of x 1 can consider it. Then there will definitely exists a point y which is very, very close to x, such that if I look into the orbit of y no more wonder how the orbit of y would be after some point it will move out of this delta right. So, this delta neighborhood of the orbit it will move out of the delta neighborhood of the orbit.

Now, we can also look into sensitivity in on a particular point. So, we look into this definition and again say that what happens here. So, I say that take any x in X.

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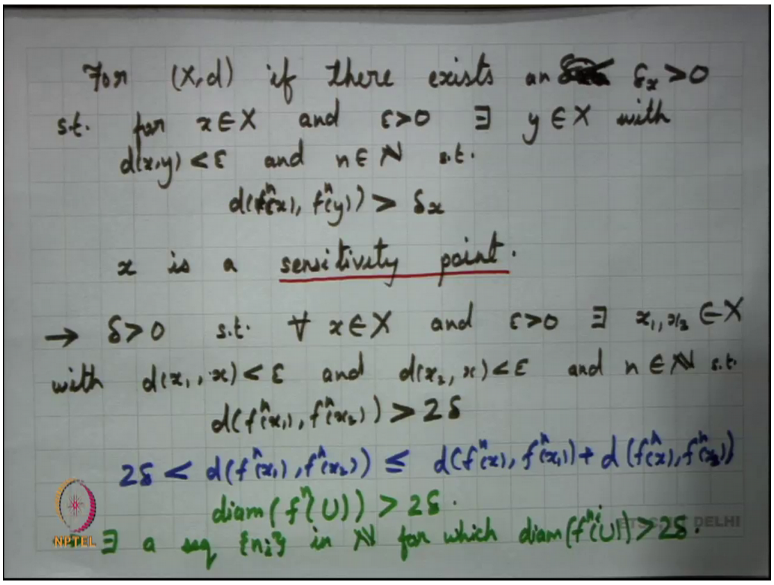

So, for the system xd if there exist a delta x say a delta x positive such that for x in X, right. And an positive epsilon there exist a y in x with dxy less than epsilon, and n in N such that d of fnx and fny is greater than delta x. Then we say that x is called then x is a sensitivity point. Of course, if the system is sensitive every point is a point of sensitivity. But we can have systems which have some points of sensitivity, and maybe some points it is not sensitive. So, this is x happens to be a sensitivity point in that particular case.

We can also think of sensitivity in a different manner. So, what we say thinks we know is now for sensitivity we have a delta positive. So, I am looking into another definition of sensitivity. So, we have a delta positive such that, for every x in X and epsilon positive there exists, I would say x 1 and x 2 belonging to x with d x 1 x less than epsilon and d x 2 x less than epsilon and an integer n such that, d of fnx 1 and fnx 2 is greater than say I am calling it 2 delta.

So, if there exists a delta such that take any point. So, in the neighborhood you find 2 points such that x the iterates of this, right. At some point differ by 2 epsilon then we say that the system is sensitive. Now we can easily see that this is equivalent to the definition of sensitivity that we have taken. Because we simply use our triangular inequality. We get d of f of x 1 fnx 2 is always less than or equal to d of fnx fnx 1 plus d of fnx fnx 1 x 2.

Now, we know that d of fnx fnx 2 by this definition will always be greater than 2 delta, right. That implies that, one of them has to be greater than delta. So, that means, that I find one point in the epsilon neighborhood of x such that, at some iterate it differs by the orbit if it is orbit difference by the orbit of x by a magnitude of delta right. So, this is basically the same definition that we have considered.

Also, we can think of another aspect here. We know that we are for sensitivity we only want that there should exist a point, there should exist one iterate where is moving apart. Now think of that from a larger point of view. Instead of looking into an epsilon neighborhood I can think of any neighborhood over here, right. Now I have 2 points supposing now at nth iterate, right. The orbit there is a point why whose orbit is moving out. I just remove y from that particular neighborhood. And then I can think of what happens and then what happens to still I have a neighborhood there.

And what happens after that is that you will find that for some integer greater than n, right. I will find another point z here in the neighborhood such that, the orbit of z will differ by the from the orbit of x, right by magnitude of delta, right. At some particular point say k will k is greater than n.

In that manner, right I can keep on removing these points right. So, what we get here this we always get a sequence of integers, right. We get the sequence of natural numbers such that, there will exist some point over here, right. Of course, that point depends on the number n, there exist some point over here such that at the orbit of this particular point will def differ from x in at some particular at this iterate n.

So, what we can say here is, now this again I can summarize it by saying that if I look into the diameter of some neighborhood of nu of x, if I look into some neighborhood u of x, then there existed n such that the diameter is greater than 2 delta, right. The diameter becomes greater than 2 delta. And that tells me that and when I am looking into this factor, right. I get. In fact, infinitely many right. So, there exist there exist a sequence I can say that there exist a sequence ni, right in N for which the diameter fni of u, right will be greater than 2 delta. So, this is what our sensitivity is, and now let us try to take some examples.

So, we take simple examples we are only going to look into some of the examples that we have already seen. So, we look into this example.

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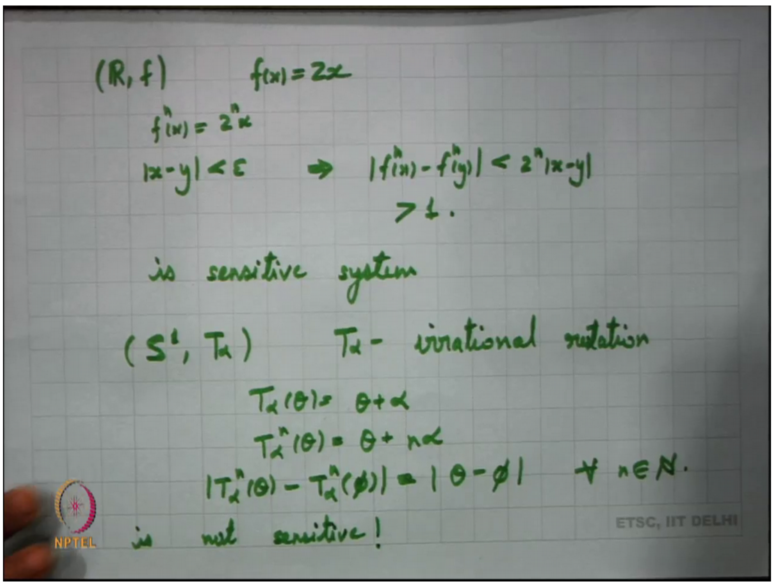

And the first example turns out to be we looking with R and f when I am defining my fx to be equal to twice x. We know that what happens here in this particular case, right. My fnx turns out to be this particular case my fnx turns out to be 2 to the power n times x.

Now, I mean also Euclidean metric. So, whenever I have an epsilon positive such that mod of x minus y is less than epsilon. Then that would imply that what is mod of fnx minus fny? This, right would be basically less than 2 to the power n mod of x minus y, right. Now. So, that means, however, small the epsilon is, right. This epsilon gets multiplied by 2 to the power n, right. And at some stage you get it greater than or of course, I can say that that at some stage this quantity becomes greater than 1. So, this is an example of a sensitive system. So, my system here is sensitive. So, this is sensitive this is a sensitive system.

What happens on the other hand? When I am looking into my circle, and I am looking into where my T alpha is say for example, the irrational rotation. What happens over here? Now we know that irrational rotation, right. We have seen that it has defined as T alpha of theta is theta is theta plus alpha where alpha happens to be an irrational multiple of 2 pi. So, T alpha theta is theta plus alpha, and you can easily observe that T alpha n of theta, right. Happens to be theta plus n alpha.

So, ideally if I look into the arc length, right. Here I have that mod of T alpha n of theta, right. And T alpha n of any point may be phi over here closed by, right. This is ideally going to be equal to same as theta minus phi, right. The arc length is preserved at each and every iterate, right. And this is true for every n in N. So, basically, we cannot think of delta where it could diverge, right. Where the orbits could diverge. So, this system is not sensitive. We can also think of as we had defined that there could be sensitivity at particular points.

So, we can think of this example again on R.

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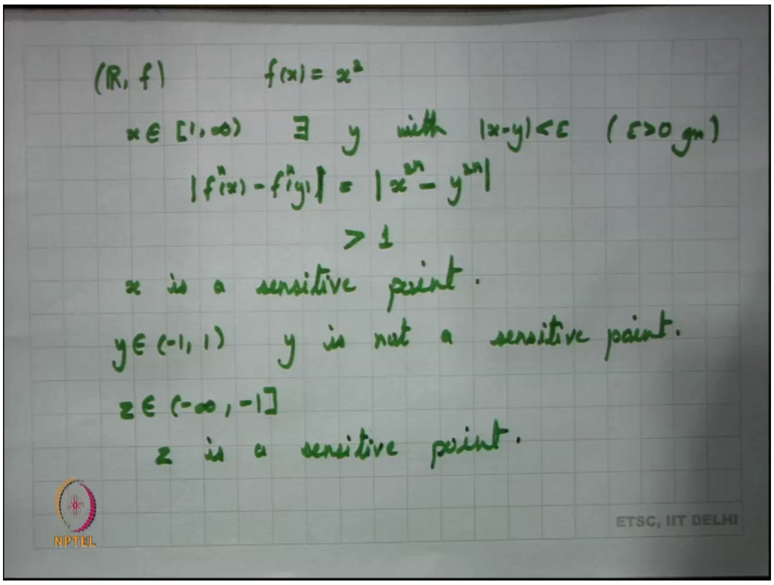

And again, my function f, where my function f happens to be equal to x square. Now we know very well we have seen what happens to these particular functions, right. We have seen the phase portrait of this particular function. So, we observe here that whenever I take any point x belonging to, let me take 1 infinity. You take any x belongs to 1 infinity in the neighborhood you will always find the y right.

So, there exists y with mod of x minus y less than epsilon for given of course, my epsilon positive is given. Such that, what happens here now? If I look into say fn of x minus fn of y right. So, this is basically mod of x to the power 2 n, right. Minus y to the power 2 n, and this quantity becomes much, much greater, right. Becomes much, much greater I can say that that with some n for which; however, small mod of x minus y be this quantity is always going to be say something greater than 1.

So, this particularly at all these particular points x, right. We find sensitivity and hence all these x, x is the sensitive point now what happens to points which are say between minus 1 and one. So, if my y belongs to say minus 1 and 1, now we know that for all the points between minus 1 and one happens to be the stable set of 0 right. So, if you take any point in minus 1 and one it is orbit converges to 0 right. So, over here this point is not sensitive. Because nowhere the orbits are going to diverge, right. They are all going to they are all converging to 0 right. So, y if y belongs to minus 1 1 y is not a sensitive point.

Now, what happens when I am taking my z in minus infinity minus 1? What happens here? So, it would be easy for you to see that any z over here, right. In the next iterate very next iterate is coming back to this interval, right. So, I will always find a point in the neighborhood of the z, right. Such that the orbit, right. At some point becomes greater than 1. So, these are also if z belongs to this then z happens to be z is a sensitive point. So, we can have sensitivity for some particular points. We may have we may not have sensitivity for some particular points.

So, it is quite possible that there are non-sensitive points also. And now we will look into another example. We take up the example, say I am looking into my x and taking this example of xf, right. When I am considering my x to be minus 1 by n.

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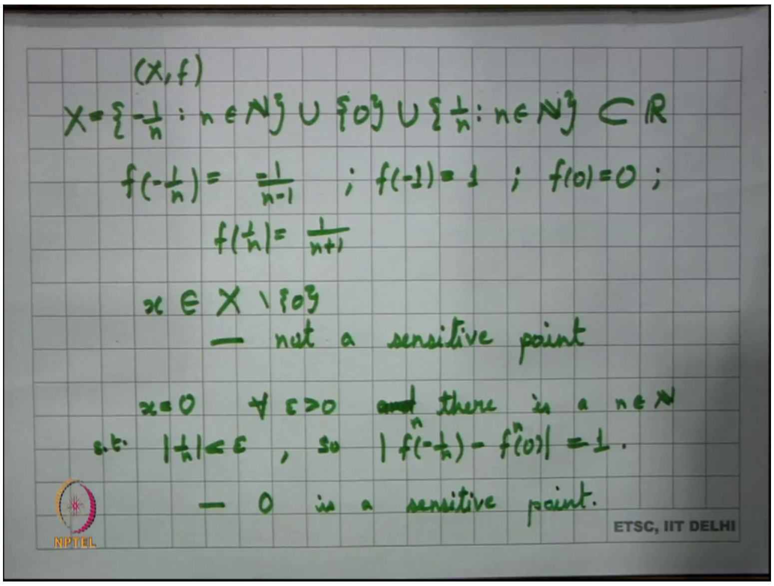

So, step n belongs to N union I have my special point here my special point is 0 union 1 by n such that n belongs to N, and I am considering this as a subspace of R.

Now, when we look into this particular space, we know that our x is a compact space. And on this compact space x we define our f to be. So, I am defining f at minus 1 upon n to be minus 1 upon n minus 1. Now that takes care of every n accepting n equal to 1. So, n minus 1. So, we define f at minus 1 to be equal to 1. My 0 is a special point I define f of 0 to be equal to 0, and that remains defining f on the positive side. So, we define f of 1 by n to be equal to 1 upon n plus 1.

Now, if I look into this particular f this particular f is actually a homeomorphism on x. And now let us try to look into the dynamics here. So, what happens to all these points? So, what happens to all these points other than 0. So, if my x is a point in x minus singleton 0, since I take any non0 point in x, then we find that each of this point is isolated. Since each of this point is isolated by definition we can say that the function is not sensitive here. So, this is not a sensitive point.

Now, what happens when x is equal to 0? What happens to the point 0? So, we observe that, right for every epsilon positive, and say every epsilon positive, there is an n such that mod of 1 by n is less than epsilon, right. Now what happens here now? So, I am looking into what is the dynamics I am looking into what happens in that particular case. So, we observe that here mod of f of minus 1 by n, right. Minus 0. So, basically fn 0 right. So, I am taking fn of minus 1 by n, right. Minus fn of 0 if I look into that aspect. Now that fn of minus 1 by n will be equal to minus 1 right. So, this particular mod turns out to be equal to 1.

So, what we find is that whatever epsilon we start with, right. You always find some quantity in the epsilon in the epsilon neighborhood you find a 1 upon n minus 1 by n in the epsilon neighborhood; such that, nth stage it is reaching to 1 minus 1, right. And then the orbit here varies by 1. So, we say that here 0 is a sensitive point. So, we can have systems where we have only one point of sensitivity. Now sensibility if we have seen if you have understood the definition by now is very much metric dependent.

So, we start with sensitivity here. We find that it is we try to look into one example again. So, again I am look into this example of the real line in my function f with I am defining my function f to be equal to x plus 1.

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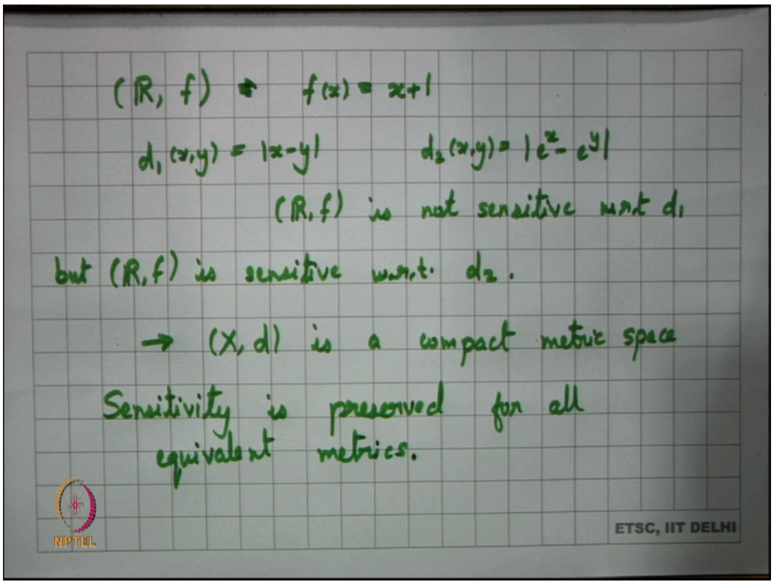

Now I am looking into R, and on R I can define some equivalent matrices. So, I take my R I am taking the metrics on R as d 1 of xy to be equal to mod of x minus y. And b 2 of xy to be equal to mod of e to the power x minus e to the power y. I am looking into these 2 metrics on these 2 on R.

What happens what can you say about this particular system? So now, I am looking into the system rf, right. With respect to d 1 we find that f happens to be an isometry, right. F is an isometry with respective of f one the metric d 1. And hence with respect to the metric d 1, right. This is not sensitive. So, Rf is not sensitive. But what happens with respect to d 2, right? There is an exponential increase whatever be the distance between x and y, there is an exponential increase in that, right. So, but we can say that rf is sensitive with respect to metric d 2.

So, sensitivity turns out to be a much metric dependent concept. So, how do we take care of this aspect? We need to take care of this aspect also. And so, for that mostly will be talk of sensitivity, we talk of sensitive in a compact metric space. So, what happens if may xd is a compact metric space? What happens here? Now we know that in the compact metric space, right. Whatever your d basically you are thinking of d right, but whatever your d your function f always is uniformly continuous. So, the function is always uniformly continuous. And hence irrespective of looking into different metrics, right. For any equivalent metric sensitivity would be the same. So, sensitivity is preserved for all equivalent metrics right.

So, here sensitivity is preserved. And so, sensitivity is always talked about in the compact space. Because in a non-compact space when the space is not compact, right. Sensitivity becomes like highly depending on what metric you are taking. And so, it becomes sort of absurd to study that. So now, let us look into some more properties of sensitivity. What happens if my system is not sensitive? What happens if we are not looking into sensitive system? So, as we said that sensivity sensitivity is basically looking into Lyapunov unstability right. So, we can again go to what do we mean by Lyapunov stability. And maybe what we had defined Lyapunov stability right. So, we recall what is Lyapunov stable.

So, the Lyapunov stability means that you take any point in the neighborhood of x, right. It is orbit is always going to stay close to the orbit of x. So, we start with Lyapunov unstability. And we try to look into this aspect once again. So, looking into this once again we observe that Lyapunov stability is nothing but looking into this family of functions.

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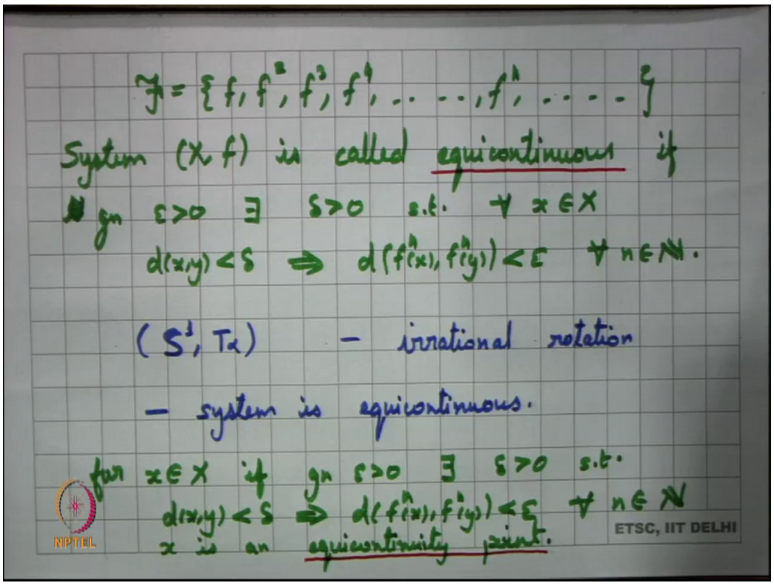

If I look into this family of functions f f square f cube that is the iterates of f, right. And looking into all iterates of f, and Lyapunov stability only means that this particular family of functions should be equicontinuous. So, this family of functions is equicontinuous, and that is what we call that to be an equicontinuous system.

So, my system xf is called equicontinuous if or I can write details in a different manner if given epsilon positive, there exists a delta greater than 0 such that for every x in X, right. Dxy less than delta implies d of fnx in fny is less than epsilon for every n in N, right. Which is same as saying that this family is equicontinuous. So, this is what we mean by saying that the system is equicontinuous.

And now if I try to look into an example here, then we can think of this example. We will already seen this example. Where this is my example of irrational rotation. And irrational rotation, right is an example of an isometry. You have already seen that right. So, this basically is an example of an equicontinuous system. So, this system is equicontinuous.

Now, similarly we can define equicontinuity points here. So, what we do is we start with saying that if for x in X if given epsilon positive, there exist a delta greater than 0, such that dxy is less than delta, implies d of fnx and fny is less than epsilon, for every n in N. Then we say that x is an equicontinuity point. So, we find this x is an equicontinuity point, and we can also have examples where the system the entire system will not be continuous, but there exists equicontinuity points.

And let me like go back to one of the examples, that we have already considered. So, we look into this example. So, we go back to this example once again. Now we had seen that our y is not a sensitive point right, but here y turns out to be an equicontinuity point. Each point here turns out to be an equicontinuity point, because for every y here you have a neighborhood such that all points in this neighborhood, right. The iterate s will always remain less than epsilon, right. They will always remain close right. So, these are equicontinuity these equicontinuity points.

So, if we one tries to see equicontinuity, one can think that in continuity and sensitivity are somewhat contrast to each other. Is it always true? That the system, if there is an equicontinuity point, right then if the system is equicontinuous, is it sensitive? Is it is not sensitive? What kind of can we have some kind of dichotomy here, right? Some kind of complete contrast between equicontinuity as such we can have sensitive points also existing in a system and equicontinuous point also existing in a system, but what can we say about the system being equicontinuous in the system being sensitive is there any contrast in the system?

So, for that we have this result. And this result is very new in the sense that it was formulated only in the 1995 or something. So, we have the system right. So, this is basically due to akim Aulander and berg.

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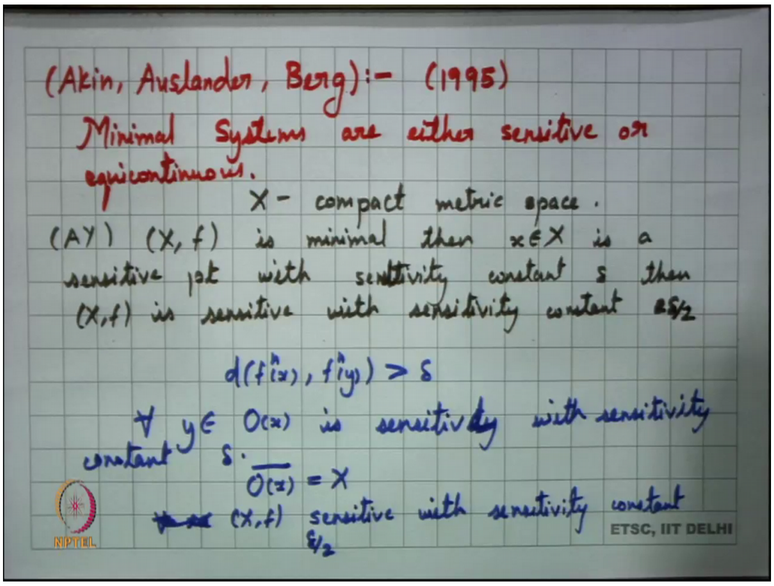

And this was done somewhere in 1995s, perhaps this says that minimal systems are either sensitive or equicontinuous. Now before we go do this we can look into something else which was done earlier by Auslander and Yorke, right the same 80’s.

But what happens for a minimal system? So, if my system xf is minimal, and here I am let taking my x now and looking into my x to be a compact metrics space because we are considering sensitivity here right. So, a x is a compact metric space. So, if I look into my system xf xf, if xf is minimal if x belongs to x is a sensitive point with sensitivity constant say, delta then xf is sensitive constant 2 delta. We look into the proof of this small bit first. And this is a very remarkable observation here.

So, my x is sensitive now. What is the meaning of x being sensitive? I am just writing it down here, right. What they find is that d of fnx, right there is a point x is a sensitive point. So, there is a point y in the neighborhood of x and in iterate n such that d of fnx fny, right is greater than delta d of fnx and fny is greater than delta for some n what can you say about fx now the point fx can I say that fx is sensitive.

So, here if x is a point of sensitivity, right. Every y belonging to the orbit of x, right. Is sensitive I am not using minimality here at all, right. Sorry is sensitive, every y in x is sensitive with sensitivity constant. What happens for minimality? For minimality we know that for every x in X orbit x closure is x right. So, if every point in the orbit, right is sensitive it is sensitivity constant delta. Every point in the orbit closure will be sensitive with sensitivity constant 2 delta, right. We can think of that.

So, every point every point in the orbit, right. That is also every point in the orbit is sensitive, right. I can say that sensitivity constant or maybe let me put not 2 delta, I think it should be delta by 2 I can think delta by 2 I think that will be safe because we are looking into all the points of sensitivity of delta also. So, every 5 here becomes sensitive of with sensitive. So, every x in X or I should say that the system xf itself becomes sensitive sensitivity constant delta by 2. Because 2 points are close by delta by 2 at least x happens to be sensitivity sensitive with sensitivity constant delta.

So, what we have here is for a minimal system, if I have one point of sensitivity; that means, the entire system is sensitive. We take another observation here. And that other observation here comes up to the definition. So, I am defining something call negatively invariant set.

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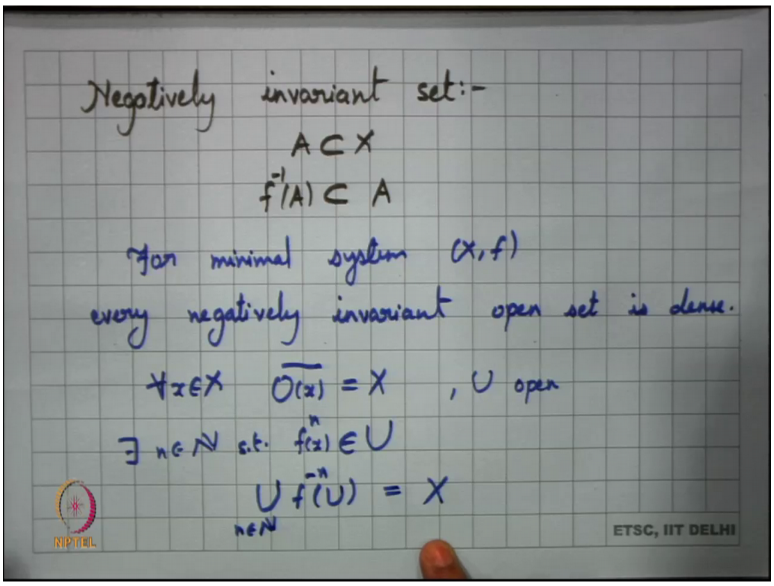

Now, what do we mean by a set to be negatively invariant? So, a let a be a subset of x, right. If f inverse of a is contained in a, right. Then we say that a is negatively invariant.

What happens in case of a minimal system now? We want to observe this for minimal system. So, we said that for minimal system. In fact, this can be considered as one of the equivalent properties of (Refer Time: 51:26) that every negatively invariant open set happens to be dense. And we can observe this over here, right. I am leaving this is an exercise for you to see this, but we look into one of the hints over here. So, what happens here is for minimal systems we know that orbit of x is dense right.

So, start with any open set u. Now since orbit of x is dense, then there will be some iterate of x belonging to u, right. So, that means, that there exist in n. So, my u here is open fn of x belongs to u, right. Now what happens for a minimal system for a minimal system? We know that orbit of x is dense for every x in X right. So, for every x in X there exists an n, right. Maybe I can say nnx depending on x such that the iterate of x belongs to u.

So now if I look into u, right. If I looking to collect all the pre-images of u, right. Say I am looking into union f minus n of u, right for all n in N, then I can say that this is basically the whole x this covers x. This is the whole of x. So, this is the whole of x, right. And this simple observation you can see that, if I have a negatively invariant open set; that means, my open set is such that f inverse of open set is contained in this open set, right. Then such an open set has to be dense.

Let us go back to the proof of this result. So, this dichotomy result we are just looking into the proof of this result now.

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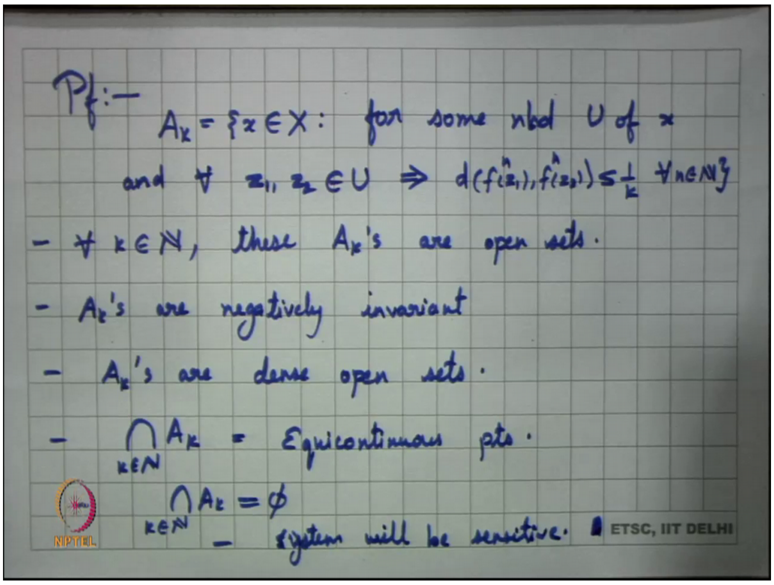

And the proof here turns out to be very, very simple. So, we know that our system is minimal, and we try to work out with the set Ak which is the set of all x in X, such that for some neighborhood u of x, right. And every z 1 z 2 belonging to u, I have d of f n z 1 and fnz 2 is less than or equal to 1 by k for every n in N.

So, whenever is z 1 z 2 belongs to u, for every point x I am looking into collecting those points x, such that there is some neighborhood of x for which they take any 2 points in this neighborhood, right. Their orbits are at a distance always remain 1 by k close right. So, we call this sets as Ak, now for every k in n it is simple to observe that this Ak's are open sets right. So, these Ak's are open sets. Is it also simple to observe that these Ak's are negatively invariant? Take any y, right such that f of 5 is x right.

So, what are all the points that come to Ak, right? What are all the points that basically come to ak? They are something from Ak itself, right. Because then their iterates are all going to be less than 1 upon k. So, Ak's are negatively invariant. And now we can use this our observation, that if Ak's are negatively invariant they are open sets they are negatively invariant open sets. So, they will have to be dense right.

So, Ak's are dense open sets, the next observation this Ak's are dense open sets. The next observation goes that, what happens if I take the intersection of all Ak? The intersection of all Ak are precisely my equicontinuous points right. So, these are basically my equicontinuous points. So now, one thing is sure if I have my equicontinious points, right. If I say that this equicontinuous points if there exists at least n one equicontinuous point, then because it will belong to some Ak Ak's are negatively invariant right. So, open dense it becomes the whole space x right.

So, in that sense I can say that all the system becomes equicontinuous. What happens if the system is not equicontinuous? Then what we have is that intersection of Ak is empty. Intersection of Ak is empty. What does that mean intersection of Ak is empty means that I have a dense set of open sets, right. Such that the total intersection is empty.

Now, what I have is what I know is if I have a dense set of open sets. Then barer category theorem says, that the total intersection is also dense and open, but I have dense set of open whose intersection is empty; that means, that there is no equicontinuity point which means that this system there will exist at least one sensitive point. And since we are in a minimal system there exists at least one sensitive point means that the system is sensitive right. So, this gives us that xf, right that the system will be. We end up here, we will see some more properties later.