Lecture – 35
Chaotic Attractors of Henon Maps

Welcome to students. So, today we can formally close our course with this lecture. And for that specifically we will be discussing about the chaotic regime of the henon maps. The chaotic attractors that we get from the henon maps. So, what are chaotic attractors? So, consider say a system f. So, you have a system a maybe a map f from R n to R n.

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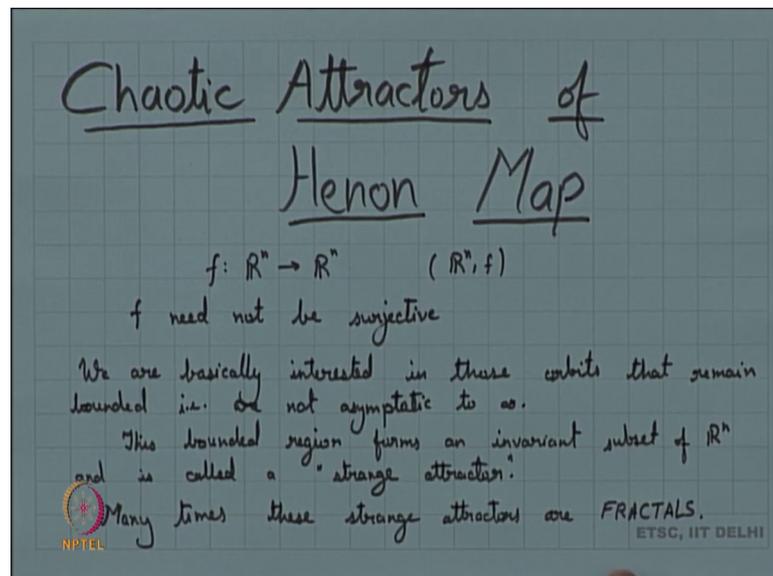
So, we are looking basically into the system R n f. Now here your f need not be surjective. So, what happens here is we are interested in looking into the orbits of the system. And what happens is generally you would find that most of the orbits they escape to infinity. Because if you try to take the one-point compactification here, then your infinity happens to be a fixed point. And possibly that becomes in attracting points sense is. So, some of the orbits the escape here.

So, we are basically interested in those orbits that remain bounded and not asymptotic to infinity. So, what happens here is when they remain bounded. So, these orbits they continuously remain bounded. And we are applying keeping on applying f iteratively again and again. So, the former bounded structure which is a which happens to be

invariant subset of R n. So, this bounded region forms an invariant subset and that is what we call as a strange attractor.

So, what happens is you are continuously applying f to all the points in R n. Most of the orbits or some of the orbits will escape to infinity, and what remains back happens to be a bounded region. And this bounded region and there again continuous application of f again and again. You will find that this contacts to something which is called which remains invariant. So, that is basically called a strange attractor, we call it an attractor of the system it is a strange attractor. And many times these strange attractors are fractals.

So, many times these strange attractors. Now that is another way of looking into attractors or looking into the dynamical system. What are all the fractals they generate? Now fractals are again some geometric structure. So, basically that depends on what is the geometry of the object. So, when you are looking to the strange attractors, the strange attractors by nature analytically are chaotic, but if you look into them geometrically they would be fractals.

So, this is one aspect of study, and definitely we are not going into that particular aspect. So, today we will be discussing about the henon map.

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So, this story starts in 1976 when Michael henon, he suggested a simplified model for the dynamics of the Lorenz system, and which is called now called the henon map. More

significantly we are interested in the strange attractor that this map gives. So, the henon attractor. Now this theory is actually why we need to look into this theory. Because this is one of the cornerstones of chaos theory. Any discussion on chaos theory is incomplete without the discussion on henon map.

So, this theory is one of the cornerstones. And no study on chaos can be complete without discussing the dynamics. Now what is this? So, basically this is a 2-dimensional structure and this is basically a mathematical model 2-dimensional structure which gives rise to many interesting dynamics. This is also a parameterised family with 2 parameters. So, for different values of parameters you get different dynamics. And this basically is believed to be comprising of all kind of chaotic dynamics that you can see in 2 dimension. Hence, I would like to point out at this point that this is not a completely studied structure.

So, people have studied this structure, but for certain values for certain there are certain limitations of the study of this dynamics of and the henon map. And more on that is again yet open to investigate. So, we shall look into what is basically the henon structure. So, this is a 2-dimensional system given by the first order difference equation. So, here we have x at n plus 1 happens to be 1 minus a x at n whole square plus y at n, and y at n plus 1 is given by b of x n.

So, you start with some initial conditions take the initial condition to be say x at 0 y at 0. And then you derive what is x at 1, y at 1. And then further you derive what is x at 2 y at 2 and so on so forth. What you get is you get you are tracing out a dynamics, right based on this difference equations. So, these are the difference equations basically which describe the henon map. And as henon had suggested that these very well model the lorenz systems; the lorenz systems are basically studied in terms of differential equations, but here we are looking this in terms of maps. So, these are difference equations.

So, what happens here is where my a and b are parameters, I am always taking my mod b to be less than 1. So, mod b is something less than 1. Now think of this difference equations. Let me write down this difference equations once again.

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So, we have xn plus 1 equal to 1 minus a times xn square plus y of n, and your y of n plus happens to be equal to b times x of n.

Now, if it try to look into this difference equation. We can also write that your y of n would be b times x of n minus 1. And in that case we can write this henon structure itself, right in terms of a one parameter family here xn plus 1 happens to be equal to 1 minus x square xn square plus b of x n minus 1. So, this gives a second order differential equation difference equation.

Now, let us think of what is the role of a and b here. So, we have 2 parameters here a and b. We take a to be some parameter of course, the dynamics differs for both of them. But our assumption is that mod b is less than 1. So, what is our parameter be meant for? So, this parameter b here is simply the measure of the rate of area contraction. So, the parameter b gives us some kind of an area contraction.

And so, this b basically gives area contraction as we have seen earlier that whenever you think of chaos it is some kind of stretching and folding. So, we are again looking into the same stretching and folding concept here. So, here b gives some kind of area contraction. And what happens when b equal to 0? You imagine b equal to 0 here, you me just get a quadratic equation here, right.

So, your dynamics is reduced to that. So, the dynamics is reduced to that of a quadratic equation. And we have already studied that this would give us the period doubling route to chaos which will be basically conjugate to the logistic function. So, this gives the period doubling route. What is observed till date is that, the solutions of henon map are bounded for a range of values of a and b.

So, these are all computer experiments where people have found that for certain values of a and b your henon map the solutions are basically the attractors, they remain bounded, and they can be studied. The still the theory still is not completely steady.

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So, there still not many things open over here. So, let us try to look into the henon map here. So, let us consider this function. So, we are looking into this function H ab xy, right. Takes the x coordinate to 1 minus a x square plus y and take the y coordinate to bx.

Now, the first thing we observe here is that this is a 1 to 1 function. So, first observation one can say here is of course, when b is not equal to 0, we need we do not need the quadratic structure here. So, when b is not equal to 0 we want it on r square. So, H ab is a 1 to 1 function. And the nice way to see that is supposing your H ab of xy, the same as H ab of say z w, what happens in that case? So, what you have here is that 1 minus ax square plus y and bx, this happens to be same as 1 minus a z square plus w. And again, you have b z. So, this is same as this part.

Now since this point to same as this point, right. We can simply conclude that x is equal to z and if x is equal to z, right. The push that over here, right. We can simply conclude that your y would be equal to w. So, this is a 1 to 1 function on hour square. Now it looks interesting that this is how the function is behaving, but the henon map can be cut off as a composition of 3 different maps.

So, let us look into that aspect. So, although it appears to be just a single map your; henon map that is, actually composed of 3 different transformation.

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So, what are these transformation. So, we define H 1 x y to be equal to x and 1 minus a x square plus y. You look into the second function H 2 x y which is just b x and y as it is. And your H 3 xy is just the inversion or I can say the rotation by 90 degrees it is yx.

So, you can observe that your H ab is basically your H 3, where basically we are moving from this side. So, it is basically H 1, then H 2 and then H 3. Now what is H 1 think of that? So, your H 1 happens to be an area preserving bending map, right. It is area preserving, and it is bending basically it is bending because of this term xy. So, this is an area preserving bending map.

So, it is basically represents a parabolic action, right. Basically, turning out to be the region it turns out to be into some kind of a parabolic figure. So, this is a parabolic action. And if you see what is H 2 doing. So, we know that our mod b is always less than

1. So, H 2 is contracting in the x direction. So, this is basically a contraction in the x direction, and what it is H 3 doing this. So, it is just a rotation by angle pi by 2. So, this is just a pi by 2 rotation here. And this is what gives us the henon map.

So, let us try to look into this figure here. Or maybe before that we look into another aspect of the henon map, and that is the henon map is invertible. Now why use the henon map invertible. So, let us try to look into all these maps H 1 H 2 H 3. Look into the Jacobean of H 1. Now H 1 come H 1 comprises of to say coordinate functions. One coordinate function is basically x, the other coordinate function is 1 minus ax squared plus y.

So, if you try to compute it is Jacobean, right. It is Jacobean would turn out to be having determinant 1. So, H 1 happens to be invertible look into H 2, right again for H 2, right. You have 2 coordinate functions b x and y, and if you look into H 2 also, right. It is the determinant of the Jacobean turns out to be equal to 1. So, your H 2 happens to be invertible and if you look into H 3, right. Your H 3 here the determinant happens to be minus 1, but again you would see that the modulus of the determinant is 1.

So, this is also invertible. So, your H 1 H 2 H 3 all the 3 of them are invertible, and hence you can say that your map henon map is also an invertible map. So, your and the inverse of henon map right. So, this happens to be basically H 1 inverse, composite H 2 inverse, composite H 3 inverse. So, this is an invertible system, right. And you this is of course, we had seen that this is 1 1, this is an invertible system, can think of that the reverse process also to be some kind of a henon map.

So, let us try to see how we can geometrically see the henon map. What would be the inverse of the henon map. So, maybe we just roughly can think of that H ab inverse of xy would be just equal to y by b, and here we have minus 1 plus a by b y square plus x. So, this is also some kind of a henon map, where you are getting the structure you getting some kind of expansion, right. In the structure. So, this is again one more variation of looking into the henon map, your mod p is less than 1, right. So, this happens to be some kind of expansion here, but you have a very nice well to do structure here.

So, let us try to see this geometrically. So, let us see look into maybe what happens to an elliptic region.
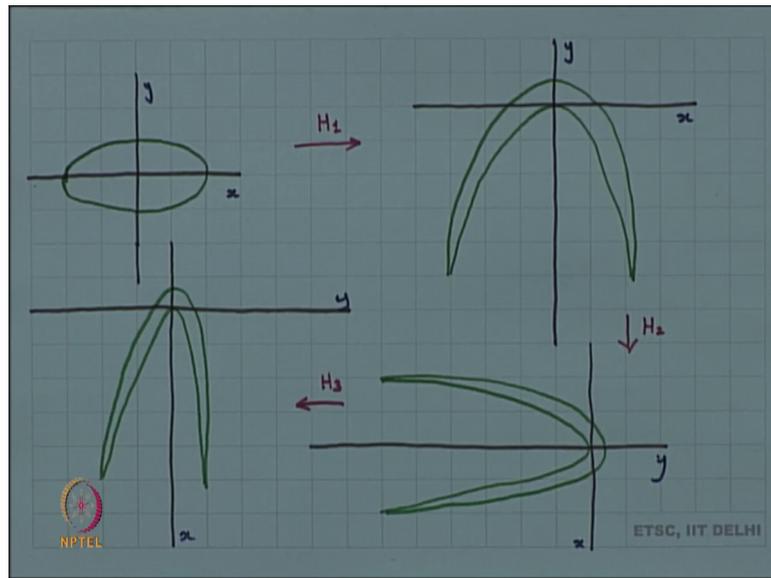
So, we start with let us try to see this elliptic region. And so, this is my direction of x, this is my direction of x, this is my direction of y. And we are interested in seeing how is this elliptic region mapped under H 1. Because our henon map is basically composition of H 1 H 2 H 3. So, how does it map under H 1.

So, what we find here is that, now if we try to look into what happens here as we have seen that the x direction is giving us a parabolic structures. So, this basically H 1 happens to be an area preserving parabolic structure. So, what happens here is that the map turns out to be something of a parabolic structure here. So, under H 1 the ellipse is basically forming a parabolic structure, and again this is our x direction. This is our y direction, and we are now interested in how is this be in map under H 2. So, what happens under H 2?

So, there is some kind of contraction in this is contacting by the b right. So, what you find is, you find some kind of a contraction here. So, your x direction is basically being contacted. And what you find here is; so, your x direction gets contacted here. And your y direction remains as it is your x direction is contracted here. So, that is what happens under H 2 and what happens now under H 3 is what we get is a reversal here.

So, here what we get is the same picture right, but. So, there is an area preserving bending over here. So, here this is your x direction again let me, right point it out. That this is basically your x direction. This is your y direction. Because this was just

contacting in x. And are this is your x direction, I have just come back here. So, this is your x direction. This happens to be your y direction, and this is your x direction right. So, this is how it basically forms kind of a (Refer Time: 27:28).

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Now, we look into one observation here. The Jacobean for the henon map. So, we looking to the Jacobean the Jacobean is DH ab xy, and the point xy happens to be equal to minus 2 ax 1 b 0 with determinant of the Jacobean to be minus p, for fixed real numbers. Now here we are fixed real numbers for fixed real numbers. So, this is one of the features that you see for the henon map that the determinant of the henon map once you fixed your a and b, right. The determinant happens to be minus b. And that is what gives it is the contracting nature, right. Because we have always assumed that our mod b is less than 1.

Furthermore, if you have a square x square plus b square is greater than or equal to 0, then the eigenvalues are real numbers what are these real numbers you can say that the eigenvalues happen to be lambda equal to minus ax plus or minus root of a square x square plus b square. For the henon map the determinant of the Jacobean is always minus b, and furthermore if my a and b are chosen. So, on well. So, that a square x square plus b square is greater than equal to 0. So, whenever this is greater than equal to 0, right. What we find is that this Jacobean has real eigenvalues, and the real eigenvalues are given in this term.

Now, we very well know that the Jacobean gives us the action of, right. What is the action of the map when we are looking into 2 dimension. It is just like looking at basically the differential right. So, it is basically the differential map fine. So, the differential helps us and looking into what happens to the what is basically the structure of what is a dynamics at that particular point, right. And that is what gives us. So, the eigenvalues they are very important. Because then the eigenvalues helps us and looking into directions where you have like basically the action is preserved as it is right.

So, this gives us real eigenvalues for this particular case. And to see the proof is like basically this is just some kind of calculation. So, we know that the coordinate functions of H ab. And that is what uses the Jacobean to be equal to minus 2 ax, right. I have a one here then I have a 0 I have b here and I have 0 here. And so, the determinant of DH ab xy happens to be equal to minus b.

What is the characteristic polynomial here?

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The characteristic polynomial; and so, the eigenvalues, but then that is same as saying that this is minus ax plus or minus root of a square x square plus. And hence the eigenvalues are real if a square x square plus b is greater than equal to 0.

So, let us now look into what is the dynamics of the henon map. So, we are now interested in what is the dynamics of henon map. Of course, we cannot study that

completely. So, let us try to look into whatever we can say about the dynamics of henon map. We are just looking into some preliminary observation here.

My H ab of xy happens to be equal to x y; that means, I am looking out for what could be the possible fixed points right. So, we are basically looking out for the fixed points of the henon map. So, I am looking out for those points x and y for which H ab of xy is same as xy. So, that would basically imply that my 1 minus ax square plus y and v x should be equal to x y. Now this readily gives me the answer of what is a relation between x and y.

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So, my y should be same as b x. So, y is same as b x, and in that case my x would be same as 1 minus ax square plus bx.

Now, this gives us a polynomial function in x. So, we get ax square plus 1 minus bx minus 1 equal to 0. This is basically a quadratic equation, and we know that this will have 2 solutions. So, you have the solution here x equal to 1 by 2 a b minus 1 plus or minus root of 1 minus b whole square plus 4 a. So, this gives you the fixed point of the henon map. Of course, given the fact that we have fixed the value of a and b we get this fixed point.

So, that definitely whether the henon map has a fixed point or not, right. Depends on what is the relation between a and b. So, it has fixed points for certain relation, and when

we look into the fact that this when a equal to 0. So, if a is not equal to 0. So, if a equal to 0, right. There is nothing here the henon map fine. So, if a is not equal to 0, the fixed points are real, if and only if now since we have this quadratic equation, they are real if and only if. I can write this part this should be greater than 0, greater than equal to 0 or my a should be the value of a should be greater than equal to minus 1 upon 4 times 1 minus b whole square.

So that means, unless your a is greater than equal to this pa you may not come up to any fixed point. So, let pk and q be 2 fixed points. So, let p and q be to fixed point of H ab. We have 2 fixed points here. So, we have 2 points here call them fixed points. Now once we have fixed points. When you are looking into dynamics we are basically interested in whether the are attracting or repelling. So, whether they are sink or a source. So, all we need to do now is look into what are the values of p, right. And what happens exactly at p.

So, we said that p is attracting the fixed-point p is attracting. So, we are taking the values our p is greater than q right. So, we are looking into those values of p and q of 2 fixed points. So, what p is attracting or sorry, I should say not p here not p greater than 2. May be if I can write my p as p 1 p 2 and my q as q 1 q 2, then all I want is my p 1 to be greater than q 1.

So, my p is attracting is the eigenvalues lambda 1 and lambda 2 of the Jacobean of H ab are such that; of course, I want to that p, right. Mod lambda 1 is less than 1 and mod lambda 2 is less than 1. So, that means, that if my eigenvalues, right high modulus less than 1 at p then I say that this p will be attracting. So, as we have started, right. Let pb equal to pn q.

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So, let p b equal to this having coordinates p 1 and p 2. Then we know that your p 1 happens to be 1 by 2 a b minus 1 plus or minus root of 1 minus b whole square plus 4 a. Your 2 ap 1 should be same as b minus 1, sorry we are taking this to be plus not the minus 1. So, plus, right root of 1 minus b plus 4 a. And that would give us that your 2 ap 1 would always be greater than b minus 1 or your 2 a p 1 my plus 1, right is greater than b.

So, this is basically what you need, right. For your p for your p 1, right. Your 2 a p 1 plus 1 should be greater than b. And we recall here the eigenvalues. So, what happens at p? Now at p the eigenvalue we want the eigenvalue to be less than 1. So, what we have here is minus a times p 1 plus or minus root of a square p 1 square plus b square. We want this modulus to be less than 1. At p we want this eigenvalue to be having the modulus less than 1, because we want our p to be attracting.

So, for p to be attracting, this eigenvalue should be this less than 1; that means, both my eigenvalues should be less than 1, right. And this is true, but has been found out that this is true, if your a belongs to see there was one thing we had seen about our a, that our a should be greater than or equal to minus 1 by 4 1 minus b whole square. And we find that whenever you are a belongs to this range. So, this range is 1 by 4, 1 minus b whole square up to 3 by 4 1 minus b whole square. We find that this eigenvalues are less than 1. And so, your p happens to be in attracting x point.

So, whenever this happens to be true for this particular purpose.

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Thus $\left| \det DH_{ab}(p) \right| < 1$ and so p is an attracting fixed point

If $a > \frac{3}{4}(1-b)^2$ then the fixed points become a source and there is an evidence of the existence of period 2 orbits that are attracting.
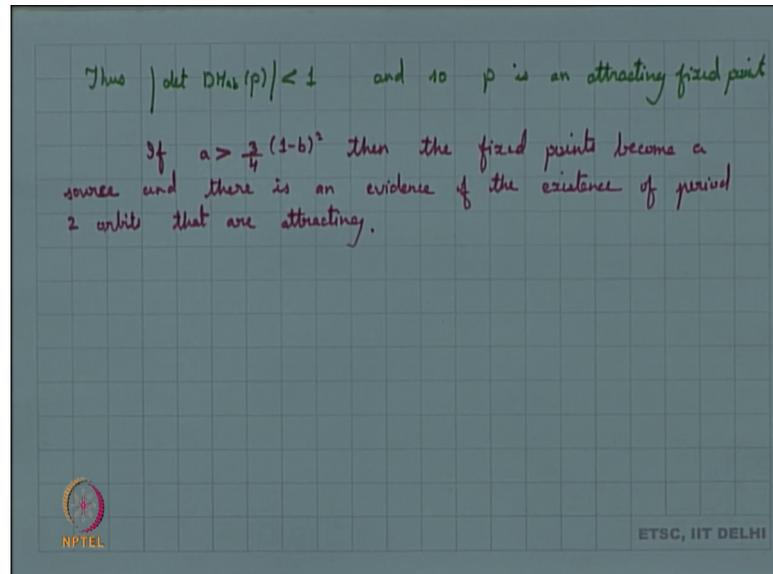
And so, what we have here is that modulus of the determinant of DH ab at the point p is less than 1. And so, your p isn't attracting, now see here we have not even touched b at all, right. We only assume that b some value less than 1, we have not touched b at all. So, what we find is that, given we are fixed b and for that particular b, right. We are taking the range of a and that gives us whether p is and attracting point.

Now, if you are a turns out to be greater than this part, if you are a turns out to be greater than 3 by 4 1 minus b square, then the fixed point they become a source, and there is an existence of there is an evidence of the existence of period to orbits, orbits that are attracting. So, for all values of a and b. So, we need to basically specify what is the value of a and b, and for these specific values of a and b, we find that we do at times get a period doubling route to chaos. And we will look into this example here for an henon attractor for so, we look into this example of henon attractor. So, we are starting here from the point.
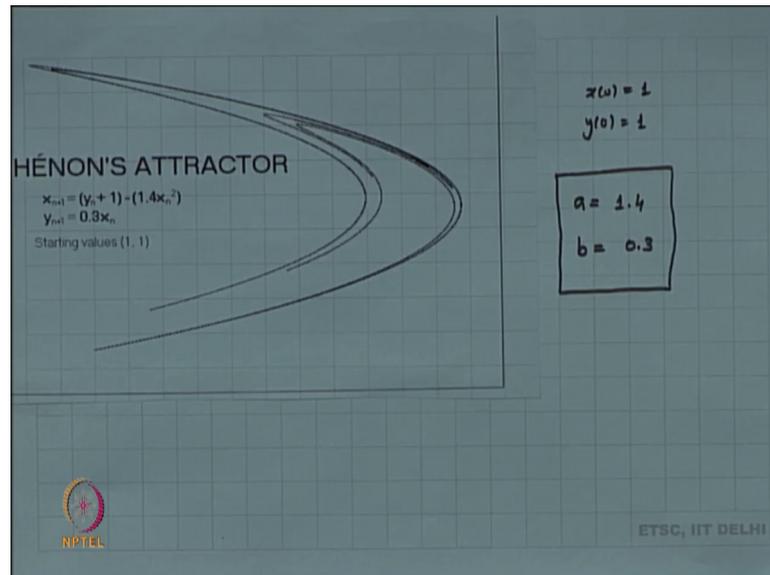
So, our initial point x naught is one our final point initial point y naught is one. And from this we start with this henon attractor, when my a turns out to be 1.4 and b turns out to be 0.3.

So, for these particular values we find that the henon attractor is essentially like a horseshoe attractor. So, it behaves exactly like a horseshoe attractor. And we very well know the dynamics of the horseshoe attractor. So, basically for these 2 values of the parameter the henon map has been studied very well, and this gives a kind of chaotic attractor. Because this is basically the dynamics is similar to that of a horseshoe attractor. And you can think of that to be like either conjugate to a shift of over full shift over 2 symbols. What happens when you have deviating the values of a and b?

Now, most studies have been done by fixed b to b equal to 0.3. Like people have not gone beyond b equal to 0.3. They have not studied something more rather than b equal to 0.3. The some strange reason by the study this system only for be equal to 0.3. But still for what are the other values of a, what are the other values of b, what could the dynamics lead to, right.

Is still quite open and this could be an interesting investigation. But as I said that chaos theories incomplete without study of this particular system, because this particular system is again still untraceable, right. We are not even sure of all the values of a and b

which will give us bounded attractors. So, I think we end this course with the study there is something led there is a lot more to do here, but we are not going to cover that up.