Chaotic Dynamical Systems
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Lecture – 29
Bowen's Definition of Topological Entropy

Welcome to students. So, we again today we will continue our discussion on topological entropy.

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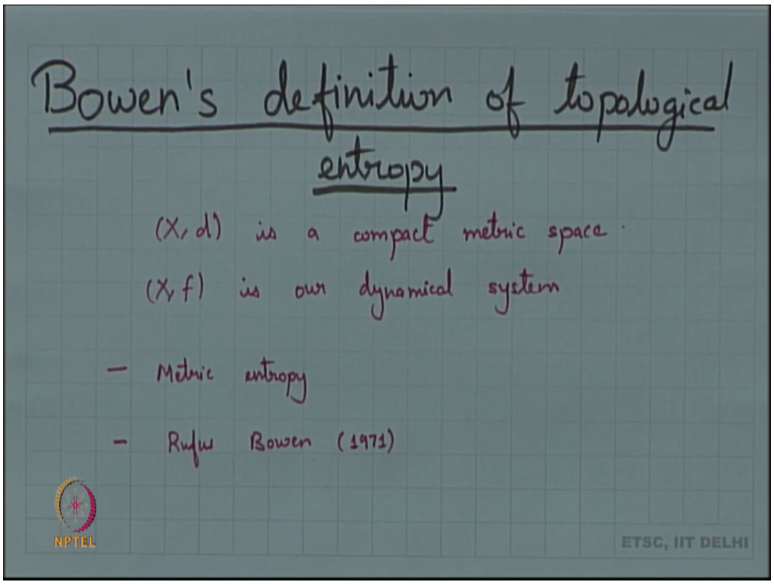

And again, our assumptions will be simply that our xd is a compact metric space, and xf is our dynamical system. Now we have actually discussed this bowens definition of entropy earlier also, when we try to explain what do we mean by the concept of entropy. And today we will actually formally develop it here.

So, this basically definition is a definition of metric and dropping. So, if depend depends on the matric taken, and this is basically credited to rufus Bowen defined in 1971. So, let us try to look into what this basically means. Again, as I said that we had discussed this idea earlier. We have 2 orbits of length n, and we say that these 2 orbits can be separated, if at least there is one iterate between 0 and n, where this the distance between them is say some epsilon apart. So, we say that fine these are distinct orbits, because from the iterate 0 to n. So, they basically at the nth step we can distinguish these 2 orbits, because at the nth step they are at a distance epsilon apart. So, these are different orbits.

Now, what happens? Then we look into how many number of such orbits are there, and then we look into what is a growth rate of those orbits. And then that growth rate helps us in determining what is our entropy. So, we are using the same concept here. So, what we here define is something which is similar to what we had seen when we defined the entropy of a shift space.

So, what we look into is we have some epsilon positive. So, we take a positive epsilon, and we take some natural number n.

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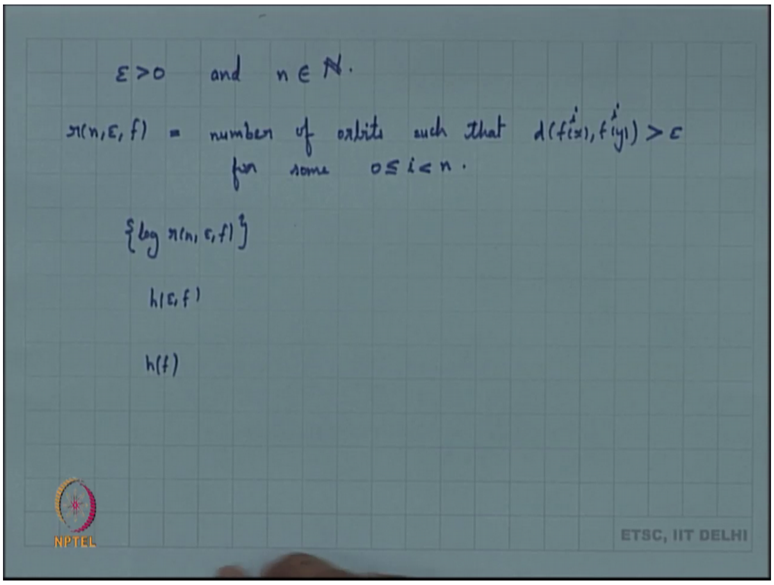

Now, what we want to look into is we want to look into r n epsilon f, which basically turns out to be the number of orbits such that for some I, now we want to look into these orbits and we say that we can distinguish these orbits. So, these are basically distinct orbits when I am looking into the orbit of x orbit of y up to some say step n. And then what we do is we try to look into the growth rate of this one. So, we again look into the sequence log of, right we look into the growth rate of this particular sequence as n tends to infinity. Now the growth rate as n tends to infinity gives us something which is called the entropy the epsilon entropy of f. And from this epsilon entropy of f we calculate we compute the entropy by taking epsilon tending to 0. So, we compute some kind of entropy here.

So, this is the basic idea which we had used when we defined the entropy of when we thought that we are looking into the entropy of the shift space. Where what we had taken is we knew that the number of orbits in that particular case in the case of a shift map happened to be the number of distinct blocks, right. The n blocks that the language admitted. So, we had n blocks that the language admitted. And from that n blocks that the language admitted, we try to look into what is say we looked into the number of n blocks, we looked into the growth rate there. And after taking the growth rate we took the limit as n tends to infinity, we found that that gave us the entropy, right.

So, basically the entropy that we had considered for the shift is actually the bowens definition of entropy. But we still need to look into what is formally the bowens definition of entropy.

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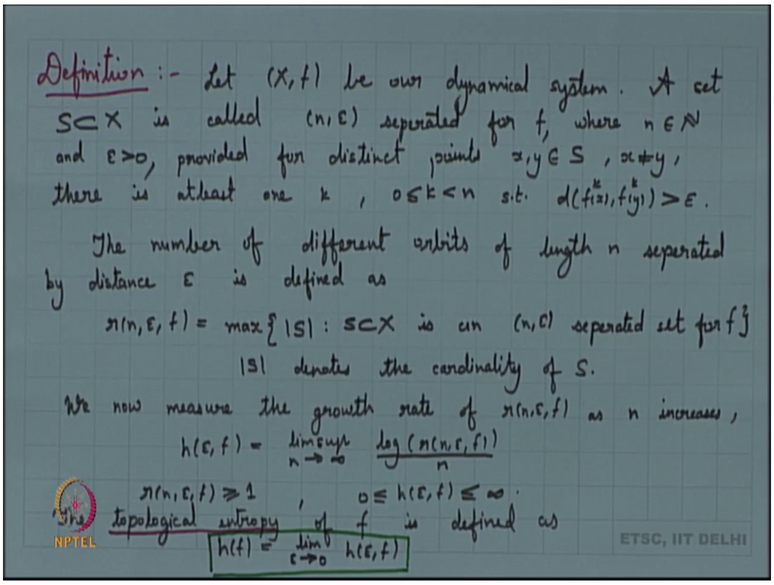

And let us now formally define bowens entropy here. So, we are looking into this particular definition. So, again let xf be our dynamical system. Now A set s subset of x is called n epsilon separated for f; where n happens to be some natural number, and epsilon is something which we are assuming provided. So, we call this s to be an n epsilon set for f provided for distinct points xy in s; that means, I am looking into the points such that x not equal to y. There is at least one k where my k can vary from 0 to n, such that the distance between fkx and fky is greater than epsilon.

So, such a set when you take any 2 distinct points, you find that between one to between 0 to n, right. There is one iterate where they are separated. So, their orbits are basically separated. We call such as a separated set. Now the number of orbits of length n separated by the distance epsilon, now what is this number? So, we are looking into how many different orbits can be of length n can be separated by a distance of epsilon. So, we count this number to be n r n epsilon f to be basically maximum of the cardinality of s such that s subset x is an n epsilon separated set, for f where basically this means the cardinality of denotes the cardinality of s.

Now it is quite possible that you may have different n epsilon sets right. So, n epsilon separated sets we could have distinct many such n epsilon sets. What we do is; we take look into the cardinality of all subsets, take the maximum of them. So, that means, we are looking into the maximum number of orbits which are n epsilon separated v. So, we look into those orbits we take that maximum and we call that number to be r epsilon f.

We now measure the growth. So, basically, we want to look into the growth rate of this part as n increases. So, that growth rate is given as h epsilon f. So, this is our growth rate which happens to be the limsups as n tends to infinity of log of r n epsilon f divided by n. So, we look into this particular limsups, we look into this growth rate here.

And now look into the fact. What is the value of rn epsilon f. It is basically the cardinality of s. And the cardinality of s in any case will be at least 1. Cardinality of s in any case will be at least 1. So, if you look in to this growth rate, right. This growth rate we can say that our r n epsilon f, right. Will definitely be greater than or equal to 1. And so, for this growth rate we can say that this growth rate would be some number between 0 and infinity, right. It could be infinity also because it could be possible that you have we are s happens to be an infinite set.

So, the topological entropy of f; that means, basically the bowens definition is defined as hf equal to now we take the limit as epsilon tends to 0 h epsilon f; that means, the growth rate we look into this definition as n tends to infinity. So, this is basically our topological entropy. So, topological entropy here is defined the topological entropy the bowens definition is defined as the limit as epsilon tends to 0 h epsilon f.

Now, the question is we are simply directly writing it limit. So, we have to make sure that such a limit would always exist right. So, how do we guarantee that such a limit always exists right. So, let us take say sum 0 less than epsilon 2 less than epsilon 1.

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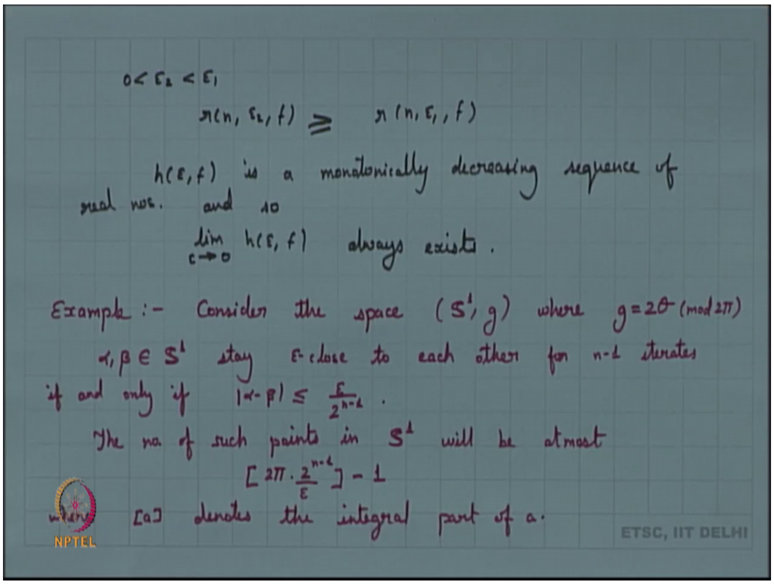

Now, think of this number I have this r n epsilon 2 f, and on the other end I have r n epsilon 1 f. Which number do you think would be bigger? I have an epsilon 1 separated set and n epsilon 2 have separated set right. So, which number would be bigger? Which is?

Student: Epsilon 2.

Epsilon 2, right? So, this number is always greater than this number. It could be equal also. So, what happens here is, when I am looking into h epsilon f, right h epsilon f when we look into h epsilon f. This would be a monotonically decreasing sequence of real numbers, right. We dividing it by n, right. We taking this part. So, this is a monotonically decreasing sequence. And so, limit as epsilon tends to 0 always exists. And as we can see now from the definition. Since, we have this definition we can see from this definition that the entropy for the shift space that we had computed some talks back, right. Lectures back was basically just this bowens definition of entropy.

Now, interesting fact here is to consider some kind of examples. So, let us try to look into some examples here. So, I am looking into the first example here. Consider the space. So, then I am looking into my circle map, and I am looking into g; where g is the argument doubling map, right to theta mod 2 pi. We have already studied the system earlier also. Let us try to compute the topological entropy for this particular system.

Now, think of that. If I take 2 points alpha and beta belonging to s 1, then they will stay say epsilon close to each other for n iterates or for n minus 1 iterates, if now here I can say that if and only if. I know that every time it is angle is being double. So, alpha becomes 2 alpha beta becomes 2 beta and then 2 square alpha 2 square beta and so on.

So, we want that for the n minus 1 iterates, the remaining just epsilon close where we have choosing some epsilon you can choose our epsilon to be very, very small in that case. So, this happens only if and only if you look into the length between alpha and beta, we are claimed between alpha and beta. So, that should be less than or equal to epsilon upon 2 to the power n minus 1. So, after n stage anyway they will be after n stage they will be just between less than epsilon.

So, what is the number of such points in s 1? So, the number of such points will be at most. So, I am looking into this figure, because it should be less than this it happens to be epsilon close. So, the number of such things which you want to take up, we know that the total circumference is 2 pi epsilon right. So, how much, how many such number will be there? So, the number will be 2 pi, right into 2 to the power n minus 1 by epsilon, such points and how many you think of that? Because now we are on a circle. So, points can be closer from the right-hand side also and points can be closer from the left-hand side also. So, you want to take care of that part. So, it will be minus 1.

So, the number of such points in s 1 will be at most this value, where this denotes the integral part of a right. So, this is an integer value. So, what is our r n epsilon g?

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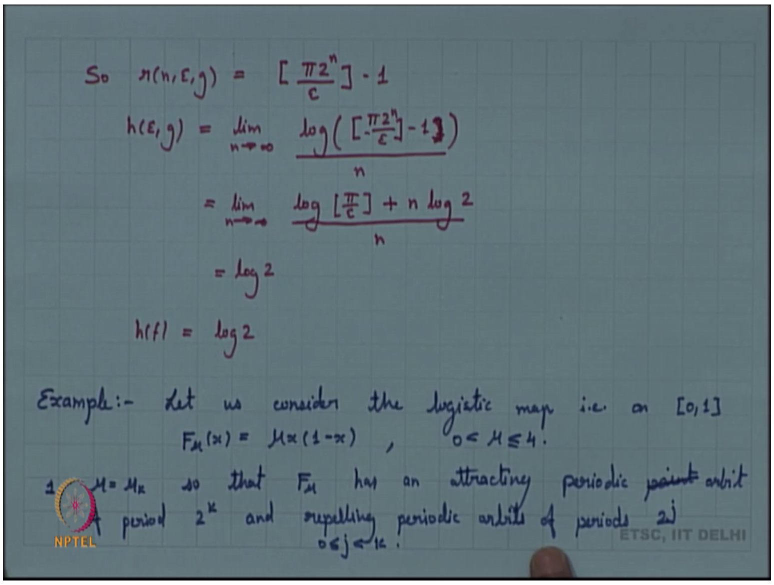

So, r n epsilon g happens to be, now we are looking into how many the maximum number here. So, this happens to be say pi into 2 to the power n upon epsilon minus 1. And so, what is your h epsilon g? Which happens to be limit as n tends to infinity, now you are looking into log of integral value of pi 2 to the power n by epsilon minus 1 sorry, minus 1 upon n.

Now, you could very well think of this minus 1, we could simply because as n increases we could just simply discard this minus 1 here. So, we can think of this to be limit as n tends to infinity log of pi by epsilon, right. Plus, n times log 2 divided by n. This definitely is a constant value pi by epsilon because we are fixing our epsilon here. So, this happens to be a constant value of epsilon is fixed here. So, if I take this by n and take the limit as n tends to infinity this term will be tending to 0. What remains is this part where my n gets cancelled right. So, the value turns out to be log 2. So, what we have here is that for any given epsilon, right. The growth rate h epsilon g itself turns out to be log 2. And so, your entropy is also log 2. So, the entropy of the angle doubling map turns out to be log 2.

Now, the situation we can look into another example here. And maybe we try to look into example on an interval, right. Now and we know the best example we have an interval is the logistic map, right. Where we have for different values, right we have different behaviors for different values of the parameter we have different behaviors. We have already seen the bifurcation diagram for that purp for that example.

So, let us consider here. So, let me take the next example. Let us consider the logistic map; that means, on the interval 0 1, right. We are defining our map f mu mu x times 1 minus x, right. This is our map on the interval, and we especially looking into this for the values of parameter 0 less than mu, right. Less than or equal to 4, we are interested in these values of the parameter.

Now, what happens for this particular map. So, let us first look into the case. We know that what happens is mu increases from 0, right. We first have only one fixed point, right. Then we have 2 fixed point, then the fixed points they become say repelling, right. And there is a period of there is a periodic point of period 2 which becomes attracting, right. And then after that again the period 2 loses all it is attractiveness right. So, then what happens is the period 2 points become repelling and then you have a period 4 map and so on, and then how the period increases is the period increases in terms of period doubling right. So, the attracting point fixed points comes from period doubling, and then the previous periodic points they lose their attractiveness they all become repelling.

So, let us look into now. So, we are going as we go with increasing mu, right. We are going through this period doubling process. So, let us look into the first case we look into here is let mu be equal to mu so that f mu has an attracting periodic point, or not point I should say an orbit attracting periodic orbit of period 2 to the power k. And then it has repelling periodic orbits, 2 to the power j, where your j varies from 0 is less than or equal to j n is less than k right. So, what happens here is that you have repelling periodic orbits of periods 2 to the power j. And you have an attracting periodic orbit of period 2 to the power k.

Now, we are considering that 2 value to be our mu. So, this encompasses all the cases, right. Where you have period 2, period 3, p period 2, period 4, period 8 these are attracting it in compasses all those cases.

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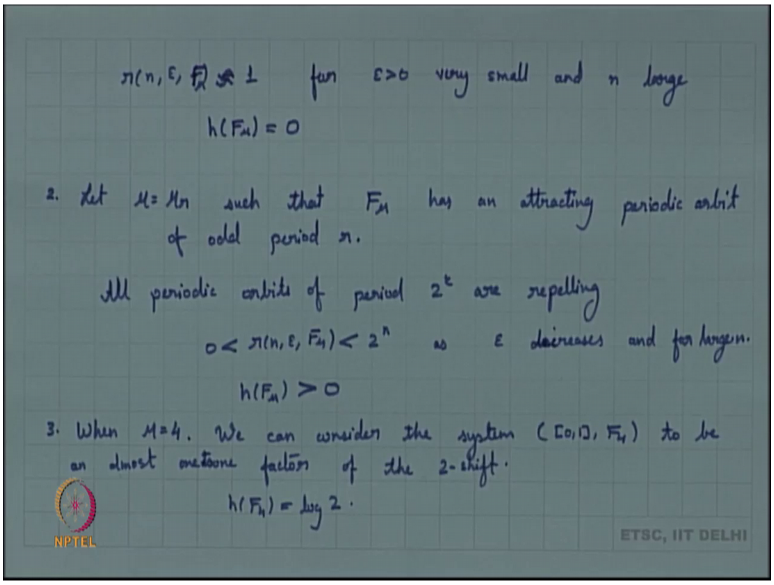

So, what happens in that case? Now you look into your r n epsilon. So, basically, we are looking into r n epsilon f, right. And you are taking this value for a large n. For a large n you write you will be getting some 2 to the power k there right. So, large n 2 to the power k happens to be attracting the rest of all of them are repelling.

Now, what is what does this mean it attracting? It is attracting everything to itself, right. We look into attracting what does attracting mean? 2 to the power k now we are fixing a k 2 to the power k is attracting here, right and when we are increasing our n beyond that, right. What happens here is all the orbits are tending towards this periodic orbit of period to 2 to the power k. The rest of them are repelling right. So, they are anyway sending away. So, these are all. So, if you look into this separated set n epsilon separated sets, how many points would that n epsilon separated set contain my, n goes very, very large, right. Such a point will be singleton because the rest of all orbits will be tending towards it, right. It is attracting right. So, rest of all the points will be tending towards it.

So, my this rn epsilon f turns out to be equal to 1 for epsilon very small and n large. So, as I take my n to be very large, and I as I take my epsilon to be very small, this number is almost equal to 1. What happens in that case? So, I should say that this is almost equal to 1 maybe it is almost equal to 1. So, what happens in that case? What can you say about the entropy here, right. Then your growth rate turns out to be equal to 0, right because it tends to be almost one singleton. So, growth rate turns out to be equal to 0. Since the growth rate turns out to be equal to 0, what can you say about your entropy? Say if you think of your entropy entropy of f mu turns out to 0. So, if you have an attacking periodic orbit in the logistic map there is an attracting periodic orbit of period 2 to the power k, then definitely the entropy turns out to be equal to 0.

But now we know what happens is as you increasing your mu, right. All power of 2, right all periodic orbits of powers of 2, they lose their attractiveness. They all become repelling, and what happens here is you get some point written say an odd number, right or multiple of an odd number right. So, such things come into picture. If we still go further what happens is you get a periodic point of an odd period right. So, periodic point of odd period is say attracting rest of all of them are repelling especially the periodic orbits of period 2 are repelling.

So, let us look into the second case. So, let your mu be equal to mu r, right. Such that f mu has an attracting periodic point periodic orbit of period of odd period r. Now we have an odd period r. So, we know that all periodic points periodic orbits of period 2 to the power k are repelling.

Now, since these are repelling, right. What can you say about r n epsilon, and f mu sorry I should have f mu here. What happens to this factor? This is certainly greater than 0, right. This is certainly greater than 0, but then these are all repelling right. And so, I can say that when I am looking into the nth orbit, right. The nth orbit, right these n periodic orbits will be separate, right when I am taking epsilon to be very, very small. So, this is less than 2 to the power n.

So, what we get here is that this is your number of nth the number of elements in any n epsilon separated set turns out to be some number between 0 and 2 to the power n. As epsilon increases, it cannot go beyond that right, as epsilon increases sorry, as epsilon decreases, as epsilon decreases and large n for large n.

So, what can you say about your entropy in this particular case, right? We do not know what the entropy would exactly be equal to, but we are definitely sure that the entropy here would be, something positive we are not sure exactly what number it would be because this depends, right. It is some number which is greater than 0, certainly greater than 0, but it cannot exit 2 to the power n. And it is it will be some number less than 2 to the power n. So, what happens is if you look into this entropy? This entropy we are not able to exactly compute what it would be, but definitely we can say that this entropy is greater than 0. So, if you try to look into a logistic map once you attend an attacking periodic point which is odd period, or in in that case anyway if you have an odd periodic point bond, right. Your map turns out to be chaotic because the entropy that turns out to be positive.

And then the third case we can look into is what happens when mu equal to 4. So, when mu equal to 4 we can consider the case. Now this is something which you also we have already discussed earlier. So, we can consider the system 0 1 and f 4 to be an almost 1 to 1 factor, of the 2 shift. In fact, what we had discussed was that you look into the 2 shift you to look into the tent map, right. And you can realize that tent map as an almost 1 to 1 factor of the 2 shift, for mean by almost 1 to 1 factor is in this case I can just explain that, what we know is that only there is only one point. There are just 2 points which are mapping to say the shift, right. There are 2 points in the shift which is mapping to one point here, right. Rest all it is a 1 to 1 correspondence, right. It is excepting for finitely many there is a 1 to 1 correspondence that is 1 to 1 almost 1 to 1 factor.

So, what we have here is that this system, because that we had considered for the tent map, but we know that the tent map is conjugate to the topologically conjugate to the logistic map when your mu is equal to 4. So, what you can think of again this logistic map with mu equal to 4 is an almost 1 to 1 factor of the 2 shift.

Since it is almost 1 to 1 factor of the 2 shift, right look into the growth rate look into the number of forbids, look into the growth rate of the orbits. That would be almost the same. So, if we are trying to look into the limiting case, right. Of the growth rate if you are trying to measure the growth rate, right that measure also would be the same. And we had already seen for the 2 shift, right. The entropy happens to be log 2, right.

So, here also the entropy here for f 4 is log 2. Really how does this factor business matter may be?

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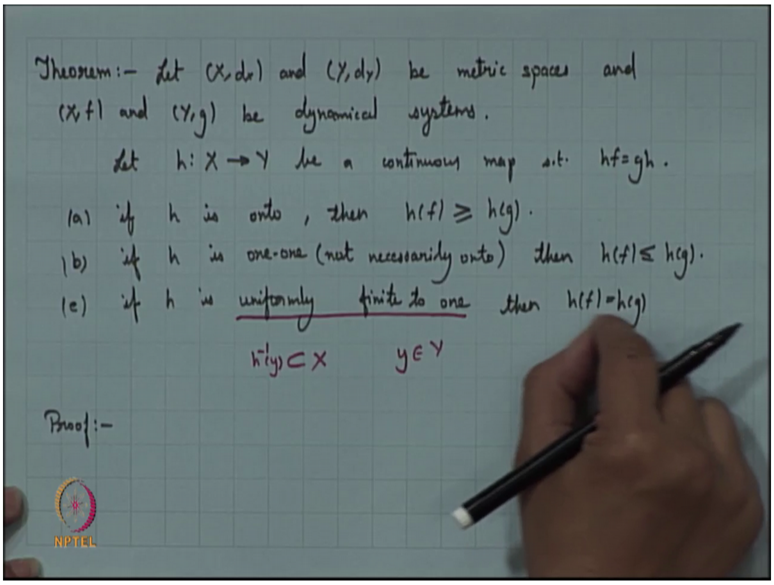

Let us try to look into that as a theorem, we can understand it better. So, let us look into this particular theorem. We just want to define a map h from x to y, right. Let this be a continuous map which implies the action of f and g. So, it is such that h of f is g of h. Supposing this holds true, then you have the first theorem. The first case if h is on 2 then the entropy of f is greater than or equal to the entropy of g. The second case says that if h is 1 to 1.

So, basically not necessarily grand onto, then your entropy of f is less than or equal to the entropy of g. And the third case says that if h is uniformly finite to 1. Then the 2 entropies are equal. What do we mean by uniformly finite to one think of this part. We looking into what do we mean by uniformly finite to 1.

Now, think of this part you take any y in y, then h inverse y will be a subset of x. So, for every y in y I can consider this y bar h inverse y is a subset of x. What we mean by uniformly finite to one is that uniformly the cardinality of each h inverse y will be finite and it will be the same. So, h inverse the cardinality of each h inverse y is the same it is finite and it is the same right. So, that is what we mean by uniformly finite to 1. And if my map is such that it is uniformly finite to 1 then the 2 entropies are equal.

Now, that is what happened if you look into our this case, right? Of the tent map what we had here was that everywhere it was 1, 1 accepting at one place where it was 2, right.

Does not matter then we can almost neglect that part right. And so, the 2 entropies are equal.

So, let us now look into the proof of this we just shortly take a proof of this part now think of that my h happens to be on 2 right. So, if I look into all n separated points of y, N epsilon separated points of y, right. Then 3 images would be n separated n epsilon separated in x, right. This map is on 2. So, each on n separated sets gives n separated here, but then there could be more n separated here in x, quite possible. Because x could have some more sets which are n separated, right. And hence we say that the entropy this entropy would be larger. Because we can have larger n separated sets here right. So, this entropy will be larger.

What happens if h is 1 1? Now we know that a h is 1 1, right. This h is 1 1 we do not know whether it is on 2 it need not be on 2, right. We know that h is 1 1. So, n separated sets in x will all be n separated y. Now n separated sets in x or n separated y right, but then the map is not on 2. So, y could have more n separated sets.

So, in that case your entropy turns out to be less than that. Now what happens when it is uniformly finite to 1, right. When it is uniformly finite to 1 we know that the cardinality of n separated sets, right. In y and in x you think of that. Would be the same thing? The growth rate would be the same here, which gives that the entropy would be same, right.

So, the proof as such as very simple it depending on we will just looking into what could be the possibility of n epsilon sets, right. And that leads us to very interesting case here, that if you look into our dynamical system xf, right. Or maybe let me write it over here say if you look into our dynamical system xf, right. We know that our f square is also continuous map on x. So, h x f square is also dynamical system right.

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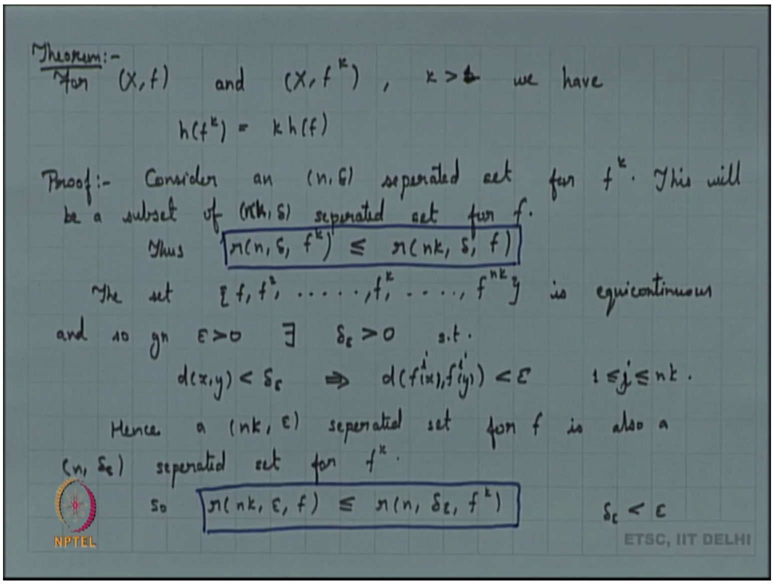

In fact, h to the power x to the power k f to the power k, this is also a dynamical system for any given k. So, for the systems for say k is greater than 0 or k is greater than 1 I should take up, right. K some number greater than 1, right.

We have entropy of f to the power k is k times the entropy of and we can think of this to be our theorem. So, if I am taking iterates, right. Iterates of f basically n increasing the entropy. So, the proof here is something simple here. So, let me consider an n delta separated set for f to the power k. Now this is n delta separated set for f to the power k; that means, under the iterates f to the power k, right. When I am looking up to k steps, right. The orbits are at least distance delta part.

So, this will be a subset of n delta or I should say yes, it is nk delta not n delta, because here we are going up to n here we are going up to n k. So, this will be a subset of n k delta separated set for f because we are looking into f, right. Going up to the power n k, and then we are looking into those orbits which are separated by distance delta part, right. So, this particular n delta separated set for f to the power k will be a subset of nk delta separated set for f. And then we can say that.

So, because of this we can say that if I am looking into r n delta f to the power k, right. This is less than or equal to r n k delta f. You again look into what is f to the power k it is basically n composition of f with itself k times and we all know that we are working in a compact metric space. So, all continuous mappings are uniformly continuous. And if you take any finite collection of continuous mappings, that becomes equicontinuous because we just a finite collection, right. It is the for each of this finite you can find some common delta right. So, each of this becomes equicontinuous.

So, the set now we have the set f, f square we go up to nk. So, we have f k here somewhere in between and we go up to f to the power n k, we look into this set. This set is equi continuous and so, given epsilon positive. There will exist a delta now this delta depends on this epsilon. So, I am calling it delta epsilon. There exists a delta epsilon positive, such that whenever your distance between x and y is less than delta epsilon, right. This implies that the distance between fi x and f I y is less than epsilon for all 1 less than or equal to I, less than or equal to nk for each of this iterates, right. This distance will be less than epsilon.

So, we can say that if I am looking into an n k epsilon separated set for f is also a n delta epsilon separated set for f to the power k. So, if I looking into an nk epsilon separated set for f, right. I can also say that it will be an n delta epsilon because I can always consider fk xf k y. So, less than delta epsilon, this is also an n eps delta epsilon separated set for f to the power k. And so, what we have is rnk epsilon f is less than or equal to rn delta epsilon f to the power k, right.

And since we are choosing our delta epsilon here, we can always choose our delta epsilon less than epsilon we choosing delta epsilon, right. We can always choose our delta epsilon to be less than epsilon. So, what we have here is we have 2 results here. So, our one result says that, this relation holds true. And our second results says that we have this relation holding 2. So, we combine these 2 results right. So, combining these 2 results, we can think of this result and then this result.

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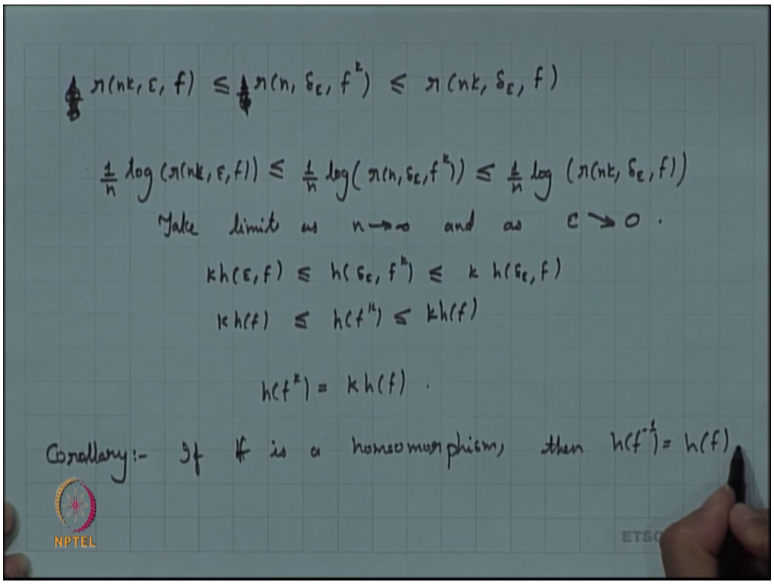

So, what we have here is rn k epsilon f is less than or equal to rn delta epsilon f to the power k. And clapping it up with a previous results. This is less than or equal to r n k delta epsilon f right. So, we have this result here.

Now, since we have this result here. I can always divide it by 1 upon n. We are looking into the growth rate here. Maybe not dividing the upon in, right. Now, but we know that this wholes true, right. Now since this wholes true, right. What happens now when I am looking into say 1 upon n? Now we think of this part log of r n k epsilon f, right. It is less than or equal to 1 by n log of r n epsilon f to the power k. And this is less than or equal to 1 by n log of rnk delta epsilon f.

Now, we are looking into these particular results, this particular equation. We note that this gives us the growth rate here. And then we can take the limit as n tends to infinity, and as epsilon is strictly decreasing to 0, right. We can think of this result here.

Now, if we try to look into this factor. This is what is going to give us the entropy of f this gives us the entropy of f, right. Think of that part what do we get over here. So, this this gives k times the entropy of f. So, what do we get here is basically the first factor, what we observe here is that k times h because this is nk. So, this is k times h epsilon f, right. Happen to be less than or equal to here what we get here is h delta epsilon f to the power k. And again, here this is same as less than or equal to k times h delta epsilon f. This is what is the result that you get from here.

Now, when we take n tending to when we take this epsilon strictly tending to 0, epsilon strictly tending to 0, we know that our delta epsilon is less than epsilon. So, when this is tending to 0 and delta epsilon is also tending to 0. So, when we take this limit what we get here is k times the entropy of f is less than or equal to the entropy of f to the power k, and this is less than or equal to k times entropy of f. And that gives us that the entropy of f to the power k happens to be k times the entropy of f.

And we can simply derive as a corollary here, a result which is discussed last time which can be proved using the open cover definition also, that if h is a homeomorphism if f is a homeomorphism, then the entropy of f inverse is same as the entropy of f. And there is something more to it. So, I am just stating the theorem here I am not going to prove that because the prophetic really very long to do, but it is available in books and you can look into those books.

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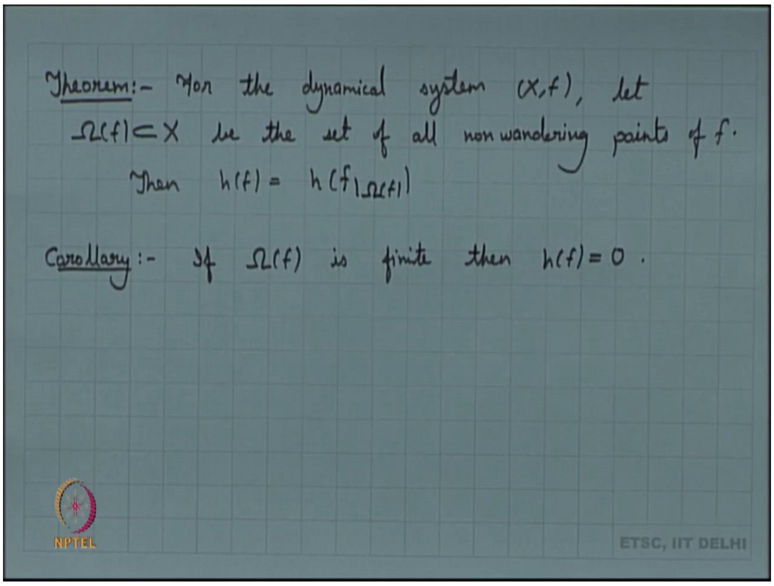

So, the interesting observation here is this really, really very, very important, but if you take a dynamical system xf, now we know that; we have the set of non-wandering points here. Then the entropy of f is same as the entropy of f computed on just the set of non-wandering points. So, this is one of the achievements of bowens definition, and of course, bowens ideas here, that instead of computing the entropy on each and everything just forget all the wandering points, just look into the non-wandering set. And on the non-wandering set try to look into what is the entropy there. That is exactly going to give you what is the entropy of f. And this has a very nice corollary, that if the set of non-wandering points is finite. Then the entropy of f is 0, and this is what we had used for the case, right. For our logistic map when we had finite number of periodic orbits, the non-wandering points where finite, right. And entropy would be 0.

So, as long as we have finite non-wandering points entropy turns out to be equal to 0. So, this is a very nice result, and this helps us a lot when you actually want to compute the entropy I would like to stop here. And we will continue this discussion again in the next lecture.