Chaotic Dynamical Systems
Prof. Anima Nagar
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 23
Introduction to Symbolic Dynamics

Welcome to students. So, today we will be looking into Introduction to Symbolic Dynamics. Now we have already seen many examples involving sequences. We had by infinite sequences, we had infinite sequences also. So, all these examples that we have studied it makes a point to look into the entire theory of the sequences in a bit better detail. The reason why we want to study this sequences in more detail is that it is not only a machinery or an aid to study chaos theory in a better way, but it is also a very important subject on it is own.

Now, it also this theory helps us in using some kind of combinatorial tools to study dynamics, or to study the chaotic properties which would be very, very complicated otherwise.

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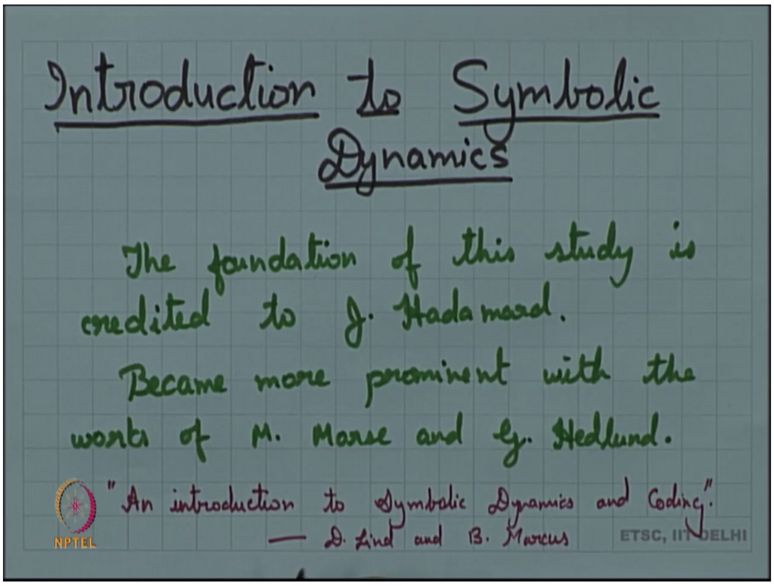

So, we start this theory, and it is it makes sense to mention that the foundation of the study is credited to, this is credited to Hadamard. Basically, he had used this theory or he had invented this theory, to study geodesics flows on negative curvature surfaces of negative curvature.

Now, this theory became more prominent, this became more prominent with the works of M. Morse and G. Hedlund and today this is a full-fledged theory on it is own. We will be looking into some very elementary properties, and very elementary examples connected to this theory. But for those who are interested, I would suggest this book an introduction to symbolic dynamics and coding by d lind and B marcus.

So, we start with this theory the starting point of this theory is we take up a set A. Now this is a finite set.

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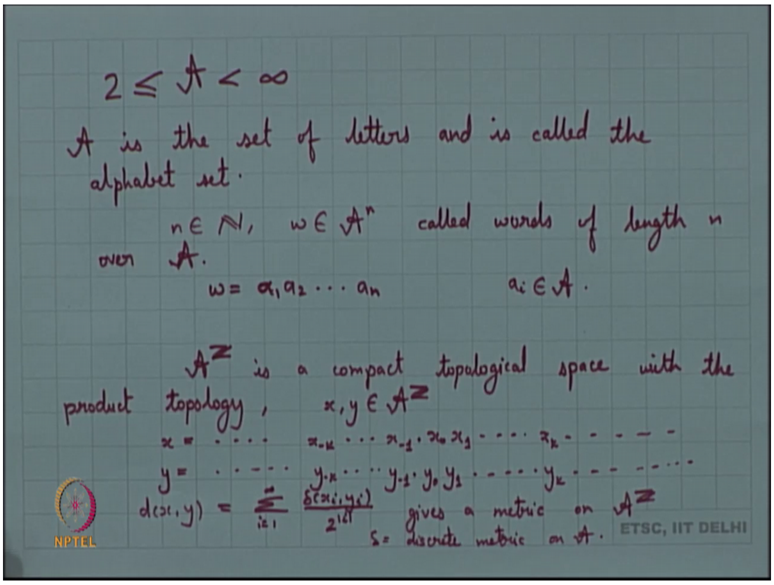

So, we want our A to be a finite set. And of course, a should contain at least 2 elements. So, A is our set 2 less than or equal to mod a less than infinity. So, A is the set of letters. So, the elements of a are called letters, and is called the alphabet set; the elements of A to the power n. So, we pick up an n in N. And you look in to elements of A to the power n. So, if I say w belongs to A to the power n. Then these are basically called words of length n over A. Now what does that basically mean is that w can be written as say I should stray a 1 a 2 a n where each Ai belongs to A. So, these are words of length n on A.

Now, we can think of more and more words on A, but more than that we are interested in constructing sequences out of the letters from A. So, we look into A to the power z. Now thinking of A to be a topological space, you can think of a to be a discrete space, right. We can give a discrete metric here, discrete space here. So, considering a to be discrete space, we define A to the power z, that gives us the set of all by infinite sequences over

A, and peixotos theorem guarantees that this A to the power z is a compact topological space with the product topology. So, we have a compact topological space with a product topology, and if I look into any arbitrary xy in A to the power z, right. We can write x to be equal to x minus k x minus 1. We have already seen this earlier. And y can be written as y minus k.

So, these are typical elements of A to the power z, and think of this function basically an A to the power z cross A to the power z which is defined as summation i going from 1 to infinity delta xi yi upon 2 to the power mod i. So, this gives a metric, right. That gives the product topology on A to the power z, where my delta I happens to be the discrete metric on A. So, we know now that A to the power z happens to be a compact metric space and now we define some kind of a function on this A to the power z.

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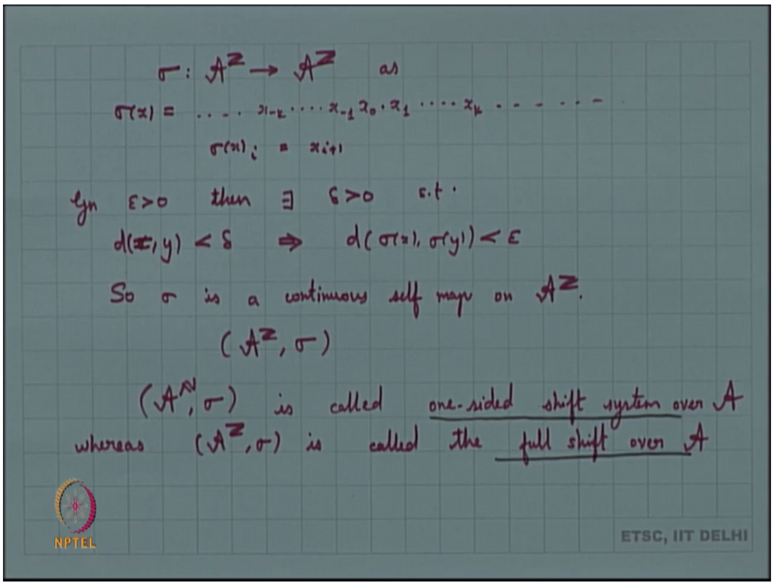

So, let us define this function sigma on A to the power z to A to the power z as; now we have already given x. So, what is our sigma x our sigma x happens to be equal to; so, that means, we are bringing the sequences or we are taking this dot towards the right side one step in the right side or we have to bringing the sequences towards the this side.

So, this is our sigma x which we can basically define as sigma x at i is xi plus 1. So, think of this function sigma and it is very, very easy to note that; if an epsilon is given, right then such that then it is easy. There will exist a delta positive such that if I am using my d f sorry, x y is less than delta will imply that d of sigma x sigma y is less than epsilon.

Now, this is a very easy exercise to check up this part. Because sigma x sigma y is less than epsilon; that means, now I am going into this then take the definition of metric again, then there will be a large portion on which there will be agreeing right. So, it will be easy to find a delta because sigma is just shifting on one side. So, it will be easy to find a delta such that this holds.

So, basically this particular sigma happens to be a continues self-map. So, sigma is a continues self-map on A to the power z. And hence we can consider this system, dynamical system A to the power z sigma. Now what we had seen is we had considered all the by infinite sequences on A. We can similarly consider infinite sequences on A. So, basically, we are considering this set A to the power n, right. We are just considering one side infinite sequences on A. So, this is A to the power n. And when we think of A to the power n again the same theory says that this would be a compact metric space, and when once it is a compact metric space, right. We can think of a shift map on this particular system also A to the power z on this space also. And then A to the power sigma this is also a dynamical system.

The differences that A to the power n sigma is called a 1-sided shift, whereas your A to the power z sigma is called the full shift system over a of course, our alphabet set is A. And this is called the full shift over A. This is one sided shift and this is full shift.

The only difference that comes up between infinite sequence and by infinite sequence is that by infinite sequence is an indication that my shift map happens to be a homomorphism. Because then I can also think of shifting the shift map I am taking the inverse of the shift map, and that also turns out to be continuous mapping and this becomes A 1 1 this, this shift is a homeomorphism where is this shift is not right. So, this shift is not A 1 1 mapping in general on one sided shifts.

So, that is a basic difference between these 2. So now, we have a full sided shift system, and we have a 1-sided shift system. We will be looking into the general theory, right. Without assuming whether we are looking into the full shift or shift because the general theory happens to be the same in both the cases.

Now, we want to like to see why we find this theory to be more simpler than the earlier theories.

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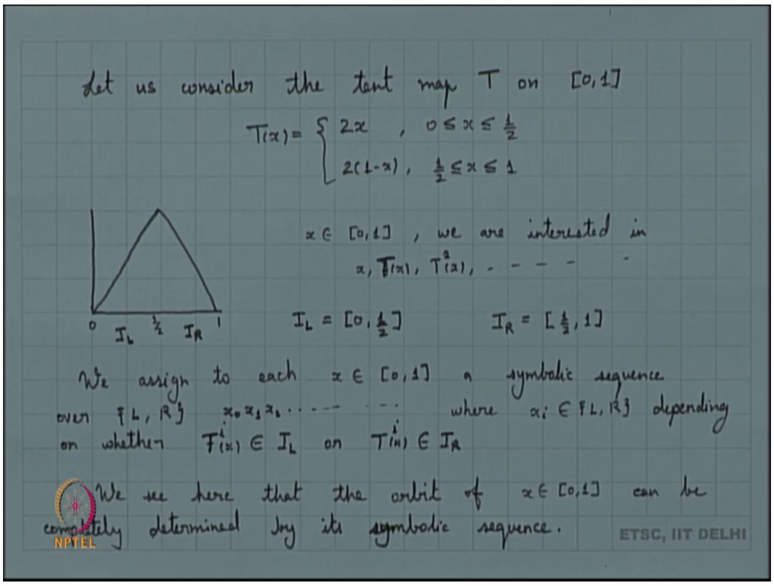

So, we will look into a very known example, and let us consider the example of the tent map. So, we consider the tent map. So, we are considering the tent map T on 0 1. And you define your tx to be equal to twice x. We have seen this many times earlier we have considered this examples many times earlier, and this is twice 1 minus x where half is less than or equal to x is less than or equal to 1. So, these map goes something and think of this map going something like this right. So, this is your half this is one this is 0.

Now, what we are doing here is; we are interested now given any x in 0 1 we are basically interested the orbit x tx T square x and so on. We are interested in this particular orbit. Now what we try to do is we divide this 0 1 into 2 subintervals. So, we divide this into the left subinterval I R which is 0 half and the, right. Sub interval which is I R and we call it as half one. So, we dividing this into 2 parts. So, this is your I R and this is your I R.

Now, dividing this into 2, we assign each x belonging to 0 1 to each x in 0 1 we assign a symbolic sequence over this alphabet I R depending on whether. So, we define the symbolic sequence. So, we say that fine we are assigning the symbolic sequence. So, the sequence is x naught x 1 x 2 and so on. Where each xi belongs to this set L or r depending on whether your Ti of x, right belongs to what phase right. So, Ti of x belongs to I R or Ti of x belongs to I R. So, we put your xi to be equal to L if Ti x belongs to I R. If put your xi to be equal to r if Ti x belongs to I R. So, we have that the orbit of what we see here is; so, we see here that the orbit of any x in X in x in 0 1, right. Can be completely determined by it is symbolic sequence.

Now, what is interesting to look here is; what are all the symbolic sequences that we can that appear in this particular scheme.

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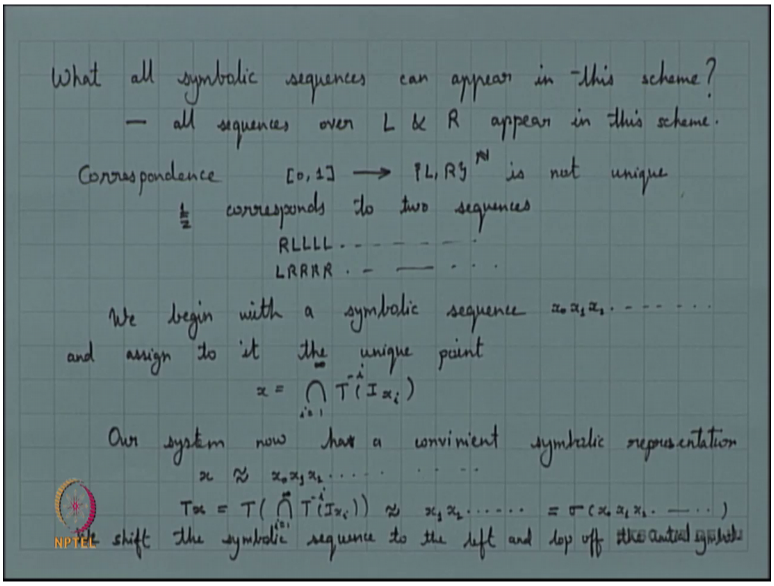

So, let us try to look into this aspect, what all symbolic sequences? Now if you try to again look into tent map, right. You have something from here, right at some particular phase going here something from here at some particular phase going here and because of the dynamics of tent map which we have already studied. We can say that all sequences over L and r appear in the scheme.

So, all sequences what they find here is that the correspondence if I look into this correspondence I am looking into this correspondence 0 1 to L r to the power n. This correspondence is not unique. The reason is that 1 by 2, right. The point 1 by 2 here corresponds to 2 sequences. And what are those 2 sequences? So, my 2 sequences are I have I can first think of it to be a point on I R, right. And then once it is on I R it is always on left hand side. So, this is one sequence, and the other sequence I could think of it as a point in I R right. So, this would the other sequence would be Ln.

So, basically if I find this sequence this is not like I cannot say this is 1 to 1 correspondence here right. So, this correspondence is not 1 to 1. So, in order to bring it back, what we try to do is we begin with a symbolic sequence. So, we begin with a symbolic sequence. Say, we start with a symbolic sequence x naught x 1 x 2 and so on. And assign to it the unique point, and what is that unique point that unique point? Is x equal to intersection I going from 1 to infinity, right. T minus i of I, right. I am looking into xi think of this part we find that there is a unique point here which can be assigned to this part, and with this correspondence. So, that becomes a unique point x, and with this correspondence our system now has a convenient symbolic representation.

So, our system and what is that representation? So, we say that if I am looking into x which is a point of 0 1, and I am saying that x is corresponding to this point x naught x 1 x 2 and so on. Then the correspondence here turns out to be tells me that your T of x it is basically T of x what is that? So, that would be basically T of intersection I going from 1 to infinity. This corresponds to the symbol x 1 x 2 and so on, but x 1 x 2 and so on I can think of this in terms of x, right. To be sigma of x, right; the sigma of basically this original sequence. So, any point x and 0 1, now can be uniquely correspondent in such a manner that your tx happens to be just the shift.

So, basically, we are now looking into the tent map on 0 1, right. And we are corresponding it with the shift system, on the alphabet set L R. So, we shift the symbolic sequence to the left and lop off the initial symbol right. So, what we are doing here is that in order to study that tent map, what we have done is that we are looking into now symbolic sequences. And we just shifted them to the left and loped off the initial symbol.

The key of utility of symbolic dynamics is that the dynamics now is given in terms of a very simple coordinate shift. So now, we have just some kind of a shift system which gives you the dynamics, and many of your systems can be converted into such symbolic systems, and it can be studied in a more simpler manner.

So, the benefit of the representation for example, I am trying to look into this benefit here.

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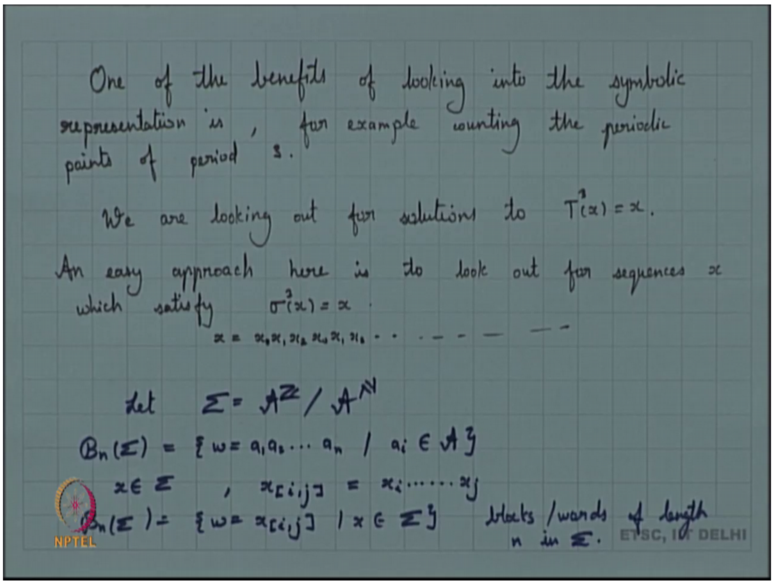

So, one of the benefits of this system representation; for example, counting the periodic points of period say let me say 3. So, we want to look into whether there exists a periodic point of period 3, and if there exists how many periodic points of period 3 can be there. So, we are trying to look into that particular kind of problems. And so, what are we looking out for us. We look out for solutions basically we are looking out for solutions to this equation T cube x equal to x. And the easy approach here, look out for sequences I am again using the same word x here.

We are now we will trying to look out for sequences x, right. Which satisfy sigma cube x equal to x. And we can easily see that since we now have just 2 letters in our alphabet then we will have 8 such points which satisfy this with which have the symbolic representation. So, we will have this x to be of the form x naught x 1 x 2. Then again, I have x naught x 1 x 2 and so on. And there will be 8 such points, right with this kind of representation over lnr, right which satisfy this equation

So, how many what are all the periodic points of the tent map, right. Can be easily constructed can be easily seen via looking into what are in the periodic points of this particular shift system. So, this is one of the benefits of getting into the symbolic case for studying our case further or studying our system right. In fact, there are more and more complicated things which have easy representation here. That is one of the basic point of starting of this particular frame.

So, let us now try to see this particular filled in more details. So, we see that. So, we now assume that our sigma is the space A to the power z or A to the power n since whatever we are going to study like holds the same thing whether we look into by infinite sequences or infinite sequences. So, we are using just this symbol sigma which represents both the cases. Only fact is that the shift map happens to be a homeomorphism, in case of by infinite sequences it is not a homeomorphism in the other case, but in the other case definitely it is a continuous mapping.

Now, what is our B n sigma now I am defining a word I am defining something here B n sigma. So, B n sigma happens to be basically the set of all words I am looking into a 1 a 2 an such that ai belongs to A. Now think of that part, we are trying to look into this factor let us take a typical element x of sigma. Then we know that our x can be written either in a by infinite manner or an infinite manner. So, we define a block xi comma j, we defined this block to be everything going from. So, we have xi up to xj. So, this is our block ij.

And now if I am looking into these words, but what is B n sigma; so it is not just the words of course, here it will be the words of n symbols. But what happens here is that B n sigma happens to be the set of all words which form a block in any sequence or by a sequence in your sigma.

So, we can say we can redefine B n sigma to be the set of all words which is equal to x ij it is equal to this block x ij, right such that x belongs to sigma. And now we can say that what is your B n sigma. So, this is basically I can say that these are blocks or words of length n in sigma. And now once we have defined words, right we can define languages. So, what is the language of sigma?

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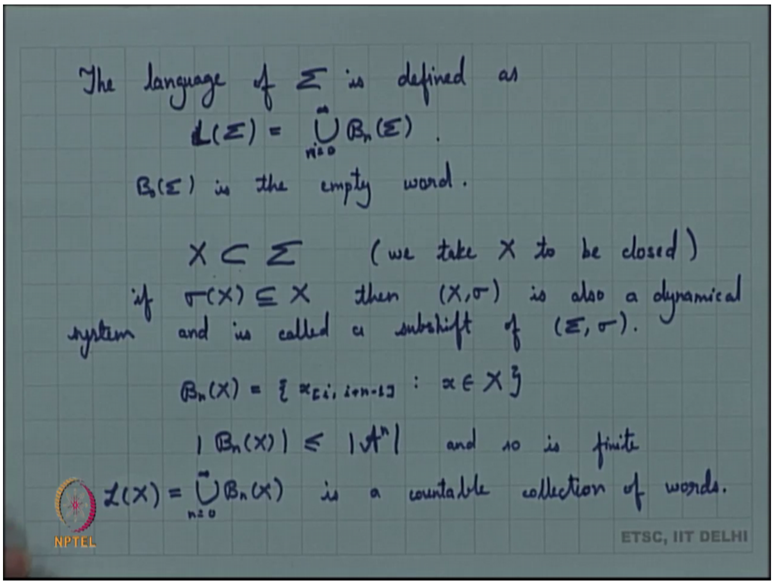

The language of sigma now I am looking into let me say use the script word here L sigma. So, L sigma happens to be union of all blocks of length n over sigma, and where i goes from n goes from 0 to infinity. So, I am looking into all possible blocks. So, then the question comes up what is B 0. So, this is basically your language of sigma. And we are interested in looking into what is your B 0. So, how do we define B 0. So, your B 0 sigma is the empty word.

Now, one thing one can say about what happens to a language here is because language is just say a collection of words. Then this language can also be thought of as a semigroup, with the operation of concatenation. So, you concatenate 2 words you get another word, right. And since the word is in a language, right. Basically, your language is characterized that you can concatenate 2 words, right. And get another word also which is there in the language.

So, under that this forms A. So, under the operation of concatenation this language forms a semigroup. And empty word here when we talk think of the empty word here. So, this is the empty word here it turns out to be something like an identity right. So, this is some kind of identity here right.

Now, let me take x to be any subset of sigma. So, our x is any subset of sigma and now we want x to be closed. So, disclosed in sigma if sigma is invariant under x sorry, x is invariant under sigma, then we say that is also dynamical system. Now this is also a dynamical system and this is called a sub shift. This is a sub shift of the full shift and the sub shift of the original shift system, say so this is a sub shift.

Now, when we think of a sub shift we can again think of what are all the words that we can define on the sub shift. So, on this sub shift we define the words B n x again we know that B n x will be the set of all words. So, I can think of B n x to be the set of all xi comma here it should be I plus n minus 1, right. I am looking into B n x this is i comma i xi i plus n minus 1, right which such that x belongs to x, right. We are looking into all words of length n that appear in some sequence in x.

So, these are my words. And if I look into how many such words would be there. We know that the bigger system is sigma. So, this is always an for sigma we know that words of length n are A to the power n, right. Basically, how many elements are there in A to the power n. So, this number of words is always less than or equal to the number of words in A to the power n. And so, this is a finite set. And this happens to be a finite set.

And hence when we look into the language of x, which would essentially be n going from 0 to infinity, this is a countable set right. So, this is a countable collection of words. Now this countable collection of words gives us another view of looking into the sub shift. So, let us look into this another view.

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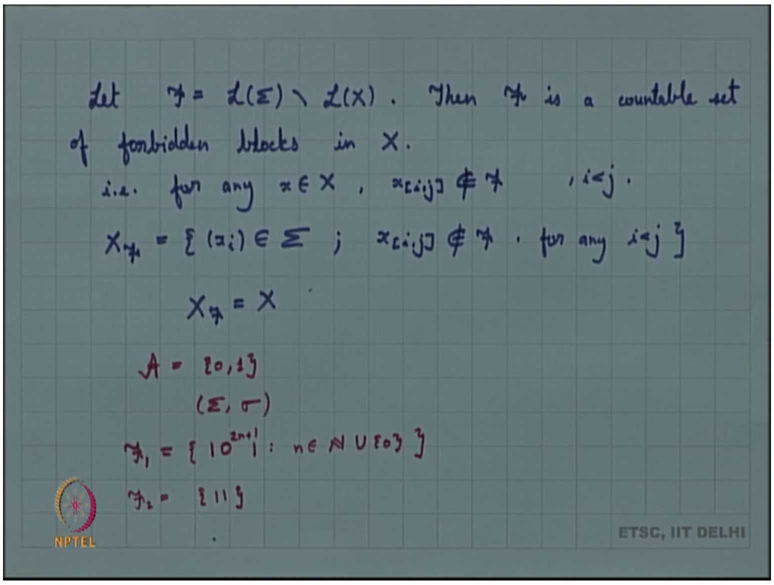

Let f be the set of language of sigma minus language of x. Then we know that this is basically this is also a countable set and this is also a countable set. The resultant set f will be countable, then f is basically the set now if we look into this countable set f this countable set f is the set of forbidden blocks. So, basically, I am looking into those words, right. Which are words over A, but they are not a sub word they do not form any block in any sequence of x.

So, we find that here f happens to be is a countable set A forbidden blocks in x. What does that mean? I can say that; that means, that for any x in X if I am looking into this if this is my block belonging to f. So, basically x belongs to x; so for any inj, right. This block will not belong to. So, I can looking into the set f, I can define a set x f to be the set of all sequences xi in sigma. I am looking into all such sequences such that no block in that appears in f, right. You can think of this part for any I less than j. Here also we have I less than j. So, for any I less than j this does not this block does not appear in f. And all such sequences we denote it as x f.

Now, the way we have defined f, right. We can easily say that x f is turns out to be same as x. So, what does that mean? We now have a sub shift x and our sub shift can be given in terms of some kind of a countable set of forbidden blocks. So, any shift can be defined over forbidden blocks. So, it can be defined in terms of a language, right in terms of forbidden blocks. So, any shift is defined over forbidden block. And so, we can say that this is a shift basically we can define the sub shift over f.

Now, we are more interested in this particular sub shift. So, let us try to look into some examples. So, let us take our alphabet set A. So, I am looking into this example. So, we think of our alphabet set A to be equal to 0 1. And we could think of it to be in terms of z or in terms of n, right. I am only looking into my system sigma. And here now I am defining 2 sets of words in A. So, the first set of words which I define here is f 1 and I think of f 1 to be all such words such that let me take 0 also here. So, I am looking into all sub words here my f 1 consists of the blocks 1 0 to the power 2 n plus 1, 1 where n is an n union 0. So, it could be 0 also. And I take another set here f 2 to be equal to I am using just one element here 1 1; the block 1 1.

Now, let us look into what is your x f 1 and x f 2. So, define your x 1 and define your x 2 to be equal to x f 2.

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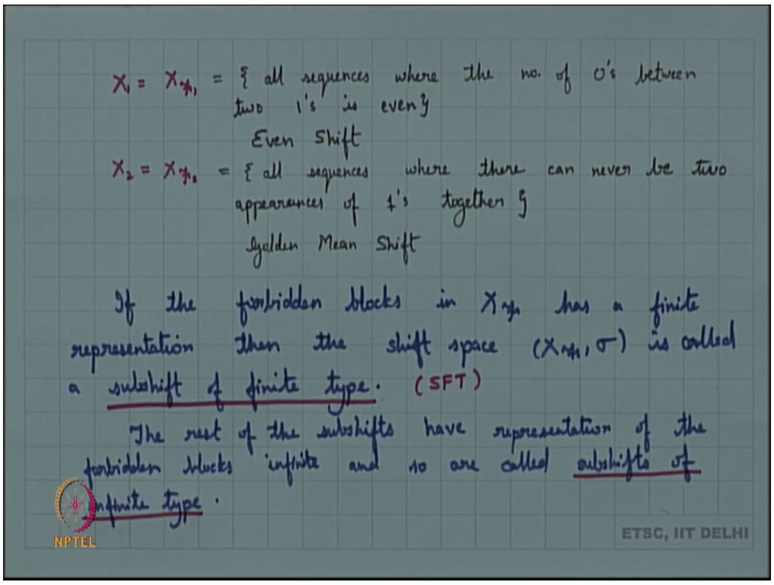

So, what is your x f 1; now again come back to this factor. What are all the blocks which are forbidden this are like we are looking into all words of length. So, we are looking into sequences over 0s and ones right. So, for sequences over 0s and ones, we only know that what the forbidden blocks are ones 2 ones separated by a odd number of 0s right. So, these are 2 ones separated by odd number of 0s; that means, my allowed blocks are only those words which have ones separated by even number of 0s. So, that means, any number of ones can appear, right. Any number of 0s can appear, but whenever there are 2 ones, right. They have to be separated by even number of 0s, right.

So, these are I can say I can characterize this as basically all sequences, where the number of 0s this is even. And we know that now we have known the language. So, we can know the shift space also. Now is it that shift space for it to be a shift space, right. It should be invariant under. The shift map, but again the shift map is not going to change this characteristic right. So, this happens to be a sub shift. So, this is a proper sub shift here. And we call this shift to be even shift. And the word even comes up because of this 0s being even in between 2 ones we call it an even shift.

Now, let us look into what is our next. Now our next forbidden set says that only the block 1 1 is forbidden now if I say 1 1 is forbidden; that means, between any 2 0s there can exist only 1 1. And in fact, I cannot have a sequence of ones at all, right. Because 1 1 is forbidden there is no sequence of ones. So, 0s can appear as many times as it can, but whenever I have a 1 it should be only single 1. So, it should only be one followed by a 0 and proceeded also by a 0; so one whose successor is 0.

And whose presider is also 0. So, we have this is basically the sequences all sequences 2 appearances of one together. And if I look into all such sequences, right. Shifting is not going to change this concern right. So, what we have here is we say that yes this happens to be a shift propers sub shift. And this is basically called a golden mean shift, why we should look into it later, should look into that concept later, but this is called the golden mean shift. And we find that these are all sub shifts. And the sub shifts can be defined in terms of forbidden blocks.

Now the interesting fact here is that if we come compare the 2 forbidden blocks here, right. The forbidden block f 1 this is an infinite set. So, that means, now whenever we are looking into forbidden blocks, right. We do not know what is the forbidden. So, we are we are basically just trying to push up the set which is forbidden.

Now, if I look into the forbidden blocks in the second shift also. The forbidden block is infinite, right. The forbidden blocks are infinite because all possible one, right one to the power n for all n greater than 2, right. That is forbidden. So, the blocks are infinite, but the representation of the forbidden block, right that is finite.

In the first case I cannot have a finite representation of this block, right. Because all I am saying all odds right. So, we are avoiding all kind of odds num odd number of 0s between 2 ones. So, the concept here is that the forbidden this f 1 cannot have a finite representation, but my f 2 has a finite representation.

So, we look into this distinguishing and we say that if the forbidden blocks whenever I am taking a sub shift, x f the forbidden blocks has a finite representation, then the shift space. So, I am looking into the shift space xf which is my sub shift basically xf sigma is called a sub shift of finite type. So, here we have a definition this is called a sub shift of finite type. And the sub shift of finite type can also be written as SFT, right. In short form we can write it as SFT.

But then all sub shift need not be sub shift of finite type. So, what are those shifts? So, the rest of the sub shifts have representation of the forbidden blocks infinite, right. So, this is like infinite representation of the finite blocks. And so, are called sub shift of infinite type.

Now in general sub shifts of infinite type can occur in some way from the sub shift of finite type. So, we are not basically going to look into the theory of sub shift of infinite type. We can only say that yes, this forms, and it gives some kind of characterization or some kind of properties can be studied using sub shift of infinite type. So, all we should be concentrating will be on sub shift of infinite type.

And today we stop here. And we shall be discussing more on this concept in some of the next lectures. I hope this is clear to all of you.