Chaotic Dynamical Systems
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Lecture – 19
Chaotic Properties of Mixing Systems

Welcome to students. So, today we will be looking into the chaotic properties of mixing systems. Now what we are going to start with our presumption again is that X d is a compact perfect metric space and xf is our dynamical system.

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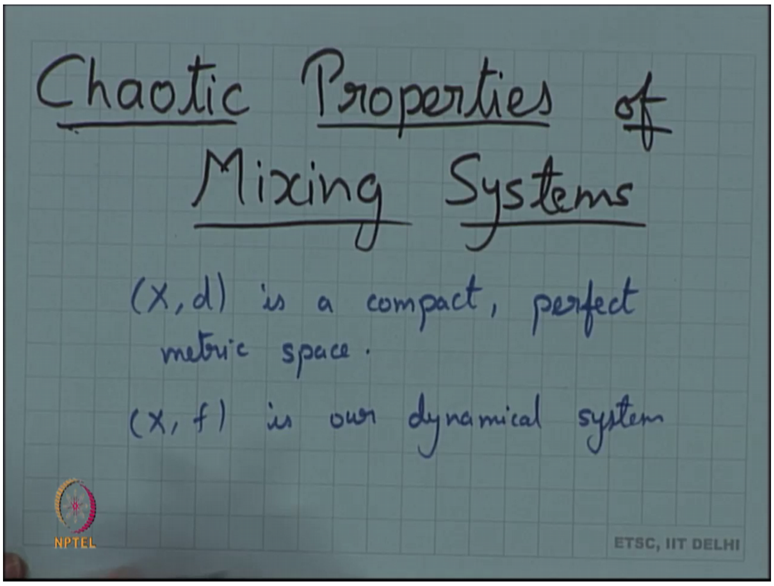

So, let us recall the definition of mixing systems once again.

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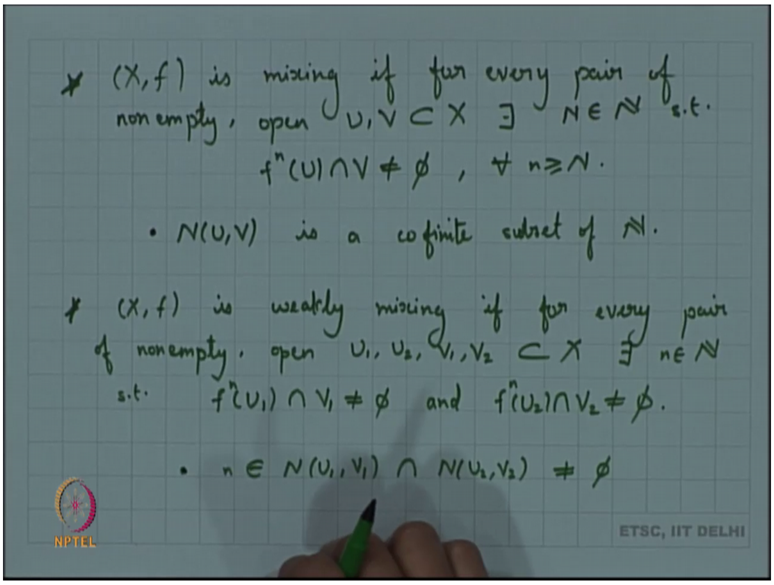

So, we say that xf is mixing if for every pair of non-empty open subset x there exists an n in N such that f n U intersection V is non-empty for every n greater than or equal to.

Now if I try to see what basically this means, then I am looking into the hitting time sets N U V this hitting time set should contain everything from n onwards all natural numbers from n onwards. So, basically I can say that this would be a co finite set, co finite subset of N and what do we mean by co finite subset of N basically this is a subset of n whose complement is finite right. So, only for finitely many things it does not happen, the hitting time does not take place otherwise you get the hitting time for almost all n right. So, this is a co finite subset of n. So, this is our interesting property of mixing systems.

Now let us again recall; what do we mean by weak mixing systems. So, let me take x f is weak mixing or weakly mixing if for every pair of non-empty open, I am taking U 1, U 2 V 1, V 2 right subset of X there exists an n in N right such that your fn of U 1 right intersection V 1 as non-empty and fn of U 2 intersection V 2 is non-empty what does it mean in terms of a hitting set that. If I take this n U 1 V 1 if I look into N U 1 V 1 and I look into N U 2, V 2 then basically I have this n belonging to this intersection of this hitting time between U 1 V 1 and U 2 V 2 so; that means, that this hitting time is non empty.

So, I take any 4 pair of open sets right non-empty open sets, we find that the hitting time is. So, this is what is meant by weak mixing system and as usual one can look into the general properties of this system.

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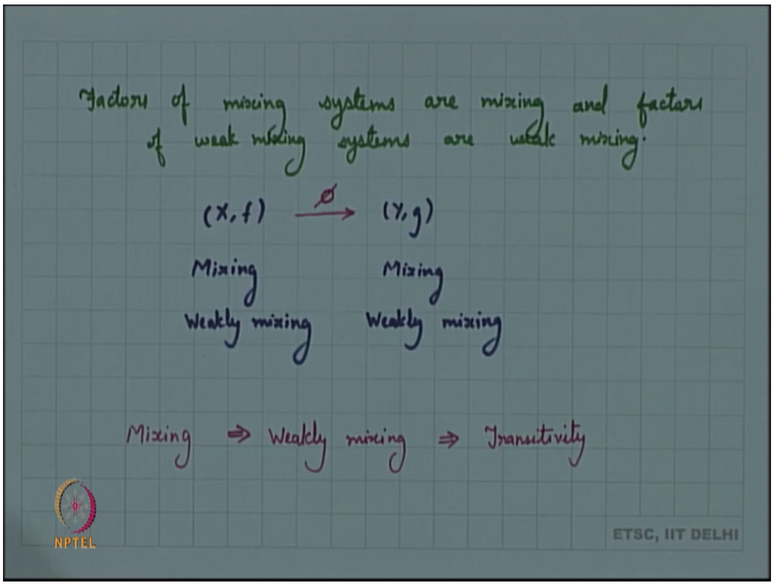

So, what we have is that I can simply say that factors of mixing systems are mixing, and factors of weak mixing systems of weak mixing. So, factors of mixing systems are mixing factors of weak mixing systems of weak mixing; what do we mean by this factors that we take up the systems, I have the system xf right I have the system yg and there is a factor in between. So, I can say that the is a continuous subjection from x to y. So, I have this which intertwines the action of f and g. So, I have this phi as a factor between this factor.

What happens here if my xf is weak mixing? So, supposing this is let me start with mixing first. So, supposing this is mixing, now you start with any 2 non-empty open sets in y right, then what happens here is you take the inverse image of that, those will be non-empty open in x. Now in x since the system is mixing right we know that there will exist in n such that, from that point onwards right there will be always an orbit from one going to the other. So, we know that this we get some n here, now again we apply fire to that particular say right we then go back over here right since this is a factor map right it intertwines the action of f and g. So, what we get here is that this also happens to be mixing.

So, the factor of a mixing system is mixing, and if we get if we assume our x to be weak mixing, then again we have the same thing we can take 4 open sets here, then again take their inverse images, what we get here is that there will exist a single n which takes right one of them f n of one of them will be going to the other right you get the same n and you get the same hitting time basically for a pair of open sets there, and then you are translating it back to this factor right what you get here is this is weakly mixing.

So, we know that this is basically closed under factors and also we have seen this earlier that your mixing is a stronger property than weak mixing. So, this implies weakly mixing and we also know that weakly mixing implies transitivity. This goes in this manner we have seen clearly that there is a difference between the way we have defined weakly mixing and mixing are they the same. So, let us try to understand that using an example right and we will try to analyze this example in various stages, but before we try to analyze this example I would want to actually explain a bit of it, because we have not studied such kind of system earlier this example earlier.

So, now you already studied the shift space right and this shift space what you had studied is that the shift map is a homomorphism. Now we are trying to look into the one sided variant of the shift map. So, we take our example here. So, maybe we get into this example here.

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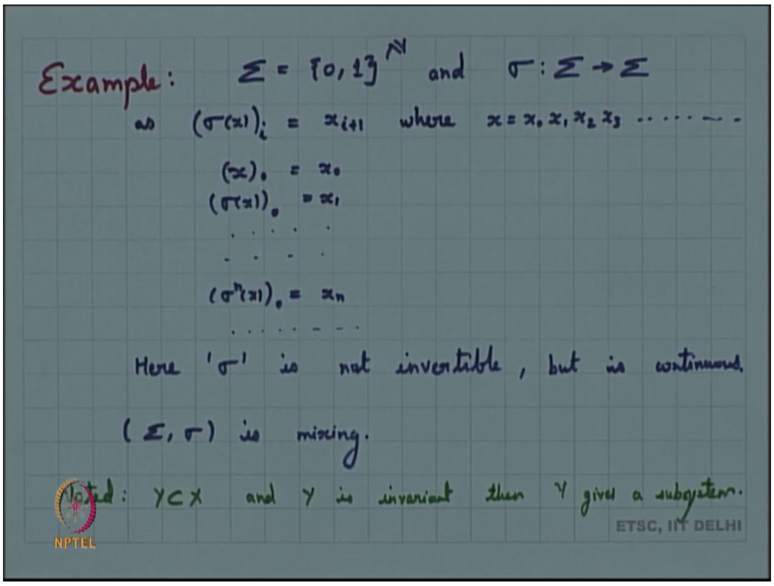

And in this particular example, we take our say sigma to be the set of 0 1 raise to the power N; that means, now I am not looking into by infinite sequence I am only looking into infinite sequence. So, this is one side an infinite sequence and we will be define sigma from sigma to sigma as again this is the same thing, sigma x of i equal to xi plus one. So, the definition of the shift map remains the same. So, you can define the shift map where here when I talk of my x, my x is an infinite sequence not by infinite sequence x is an infinite sequence. So, I can write my x to be x naught x 1, x 2, x 3 and so on. So, so this is just this is just a one sided infinite sequence.

Now, if we try to look into this manner, I can simply say that what is my. So, what is the initial point of x? So, the initial point of x is x naught what is sigma doing is. The action of the shift map is simply it is forgetting what was in the first place right. So, it is just forgetting or just deleting what was in the first space and then whatever remains happens to be your sequence. So, what happens at the nth state? So, what happens here is that my x right at 0 is x 0 what happens to my sigma x at 0, that is x 1 right and similarly I can say that what happens to my sigma n to sigma to the power n at x, what happens to the 0 right what is the initial what to say symbol here. So, that symbol happens to be x n right ok.

Now, the definition has changed the fact that, we are now going from by infinite to infinite sequences, but the rest remains the same can we say that still say that sigma happens to be continuous. What is the topology on this space? Space of sequence is we again have the product topology here. So, again the product topology is given by that metric right you can think of the metric same metric delta is a upon right 2 to the power i now we do not need to take more right 2 to the power i. So, you have the same metric there and under the same metric, then the same product topology, I can say that my sigma happens to be continuous right it is not difficult to check, it is the same proof that works out that sigma happens to be continuous here.

But here the difference is that sigma is just continuous right discontinuous it is subjective, but it is not a homomorphism right, because I cannot define sigma inverse here. Because if I try to take sigma inverse I will have 2 possibilities right either the sequence could start with 0 or the sequence could start with 1 right there are 2 possibilities fine. So, here sigma is not invertible.

So, here I should say that the shift map is not invertible, but is continuous. It is continuous it is objective right. So, this is our shift map here now for this particular shift map. So, this is basically called one sided shift right. So, for this shift map we now want to look into something else. Now very easy to check that those shift map will be transitive and so on right we also say that this shift map happens to be mixing right fine; because after something is over right it is the same proof that works out that this will be also mixing. So, this particular system is mixing.

Now, since it is mixing it is also weak mixing it is also transitive, that is taken for granted, but now we want to look into something else and as we have noted earlier. So, what we have noted is that, we have noted this earlier that if my Y is a subset of X right and Y is invariant then Y gives a subsystem. So, now, I am going to define a subsystem of this particular system. So, I have my sigma; sigma, this is my system here and I am trying to define a subsystem. So, let me take here again we go back to definition I am taking here P to be some subset of natural numbers.

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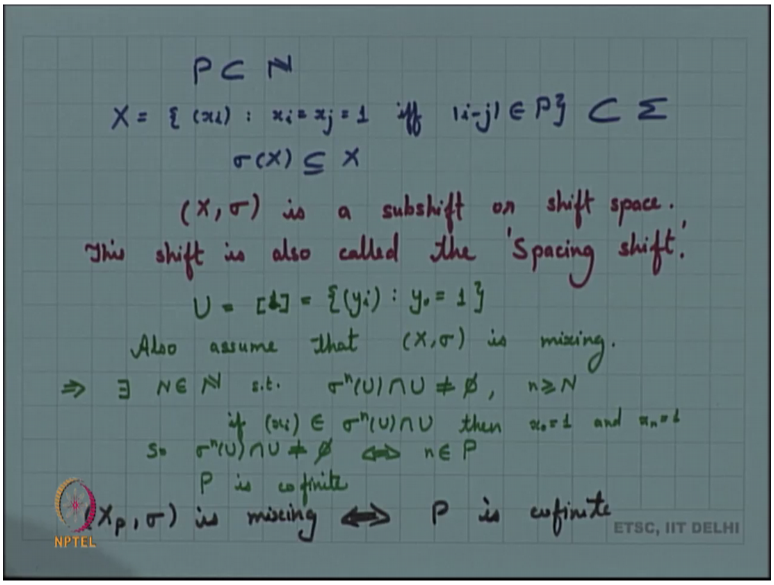

Now, what I am allowing here is that I am taking my set x to be the set of all sequences xi, such that xi equal to xj equal to one right if and only if mod of i minus j belongs to P. So, what we are trying to do here is what is our system x is, it is definitely a sequence of it is an infinite sequence of zeros and ones, but what are we doing here is that we are trying to look into the system right we are trying to look into the system where mod of x minus y so; that means, we are trying to space what is the space between 2 ones. So, how many zeros are there between 2 ones, exactly all those elements which belong to P.

So, you can place zeros anywhere does not matter, but where can you place 1. So, it in place 1 only when the distance between 2 ones right is some element of P definitely is a subset of sigma. So, subset of all sequences of zeros and ones, is it invariant can I say that sigma of xs, subset of x? Because even if I take shift it right I am not changing the distance between 2 ones the distance between 2 ones remains the same right the spacing between 2 ones remains the same. So, basically I get the same thing back here right and so, I can say that this is a subsystem right. So, is a sub shift or I can say a shifts space and we have a name for this shift because of this being defined using our spacing because P is in our hands we take P in our hands. So, we call this system. So, this system is also call this shift is also called the spacing shift.

Now, it has various beautiful properties here, but we are trying to look into it from particular angle. So, now, what we do here is, we try to take up an open set here now we know that open set in shifts space are defined using cylinders and in our case the cylinder becomes very simple, because only the initial part gives you any initial word gives you a cylinder right. So, here if I take up U to be the cylinder 0; that means, I am looking into all those sequences yi such that y 0 is sorry I should take it 1 here. So, y 0 is equal to one let me start with that part y 0 is equal to 1.

So, I am starting with all those things which initially start with the one. Now I know that once I start with the one right the next instant will be. The next instant of one will be whenever your j happens to be equal to some element of P now also assumed. So, you also assume that my spacing shift is mixing of course; the properties of spacing shift depend on P right. So, we assume that this is mixing now what does it mean that this is mixing? Then for my open set U right they should be existing an n right. So, what does that imply? That implies that there exists an n in such that sigma a n U intersection U is non-empty right sigma n intersection U is non-empty for all n greater than or equal to N.

Now, think of that fact, what are the elements that can come up in sigma nU intersection V. So, if there exists an x belonging to. So, basically if x belongs to what I should say the sequence xi belongs to sigma n U intersection U since it belongs to U it is first part should be equal to 0 so that means, x naught should be equal to sorry first part should be equal to 1. So, x naught has to be 1 right and since it belongs to sigma n U right; that means, after nth stage also it should be U so; that means, xn should be equal to 1 right.

So, in that case what we have is here it is 0 is 1 and xn is 1 what are all the n for which this will be non-empty. What are all the ns for which this will be non-empty my first is always a 1 right and the next subsequent this will be non-empty only if the nth part is 1right. So, what are all this ns for which this is going to be equal to this is going to be non-empty this is if and only if is n belongs to P right. So, this is non-empty if and only if my n belongs to P. So, what do I know about this particular set? I know that this particular set n U U it should be a co finite set right. So, n U U should be a co finite set because we are in a mixing system right again recall the mixing part here. So, let me again recall this part right.

So, we say that the system is mixing if this is non-empty for all n greater than n; that means, n U U what I am starting with you. So, your n U U should be a co finite subset of n which means that now here my P should be a co finite subset because this has to be co finite right n U U has to be co finite. So, my P is co finite. So, this gives me that P is co finite. So, what we have observed here is that if my system is mixing right if my spacing shift is mixing, it is not an ordinary system it is basically a spacing shift. So, my spacing shift is mixing then; that means, that my P is that spacing set right is co finite. And I just want to say that it is not very very difficult to observe that the spacing shift right which I denote it as X p because it depends on P sigma this spacing shift is mixing if and only if right P is co finite. So, it is not very difficult to observe that the converse also holds true, but we are not getting into the proof of the converse part we just observe here is that if my spacing shift is mixing then my P has to be co finite.

So, we have now we have another definite or another example of a mixing system, and now we know that a mixing system is always weak mixing and it is always transitive. So, we know that this system is weak mixing also, the system is transitive also. So, we try to go into now the weak mixing part. So, let again I would like to recall the definition of weak mixing. So, we say that this is weak mixing if for every pair of non-empty open sets U 1, U 2, V 1, V 2 that exist in n such that N U 1 V 1 intersection N U 2 V 2 is non-empty if that was basically what is meant by weak mixing, To start with this thing we go back to the wonderful observation which was pointed out by Furstenberg in 1967. So, we go to Furstenberg intersection lemma.

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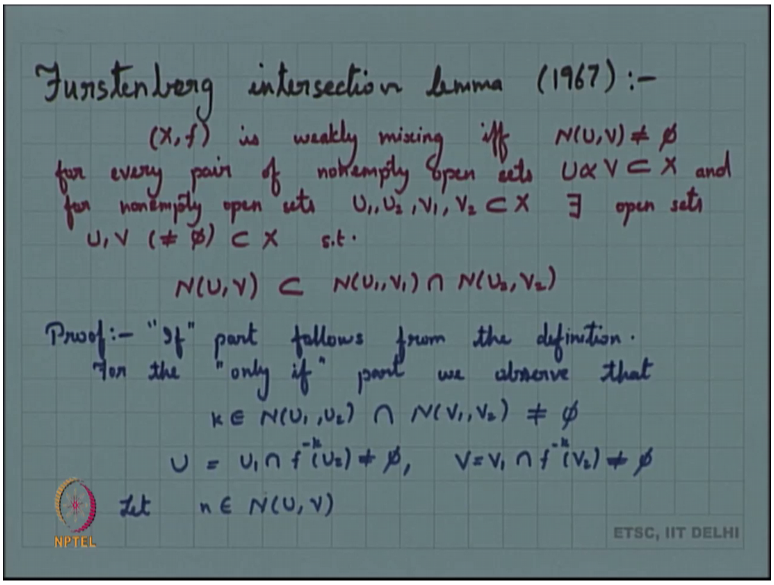

It is can be attributed to first in 1967, now this used some more mathematical definition of something which is called a filter, but we are not going to use the concept of filter here. So, I will just quickly explain what this means is.

So, I said that the system xf is weak mixing, if and only if right for N U V is non-empty for every pair of non-empty open sets and for non-empty open sets say U 1, U 2, V 1, V 2 subset of x there exists now this is not related to this one, but there exists for this U 1, U 2 V 1, V 2 there exist open sets. So, let me again take U and V definitely these are to be non-empty, such that N U, V is contained in N U 1, V 1 intersection N U 2, V 2 Furstenberg had this intersection lemma that basically if you take if you have a weak mixing system, then if you take the hitting time sets right of 2 pairs, you take their intersection then you will find that the hitting pair sites of a third pair right is contained in it. You will always find for any 2 such things you will always find open sets U and V such that this is contained inside.

Now, we tried to prove this factor and the proof here is very simple. So, try to prove this part now think of that part I am using the if part right. So, this is if and only if statement. So, if I prove the if part right. So, if part means if this part holds when anyway we know that the system is weak mixing it follows by definition right. So, the if part follows from the definition now if part follows from the definitions of what happens for the only if part.

We observe that now if I am taking N U 1, U 2 right and N V 1, V 2 then we know that for these 2 pairs right their intersection will be non-empty because my system is weak mixing. So, system is weak mixing this intersection is non empty. So, let me take a k belong into the system and what does that mean right; that means, that what I have here is U 1 intersection f to the power minus k U 2 this is non-empty right and V 1 intersection f to the power minus k V 2 this is non-empty. Now these 2 sets are non-empty these are open sets also. So, I call this open set as U and I call this open set as V; now I have 2 non-empty open sets and I know that for any pair of non-empty open sets my N U, V happens to be non-empty right. So, let n belongs to N U V.

I know that N U, V will be non-empty. So, you are always get some n belonging to N U V. So, this happens to be non-empty what we get here is. So, what does that mean?

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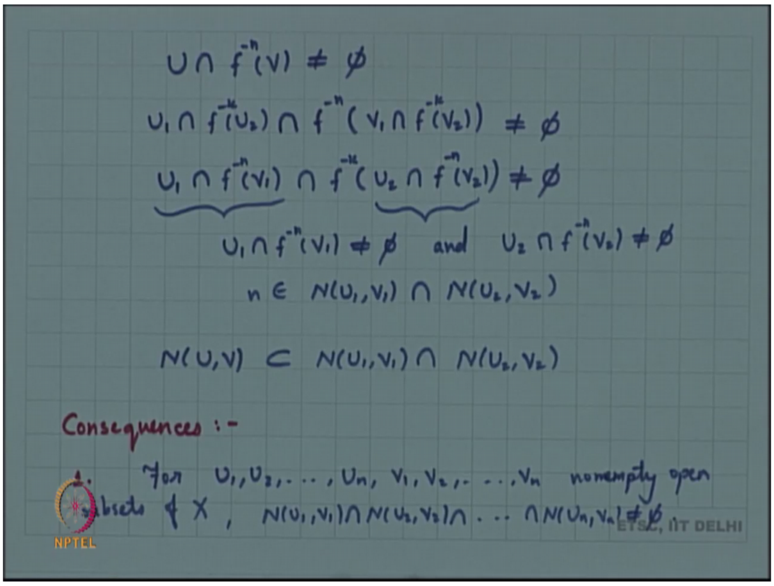

That means, that my U intersection f to the power minus n V is non-empty because my n belongs to N U, V and that gives me that V intersection U intersection if to the power minus and V is nonempty, but what is my U? My U happens to be U 1 intersection f to the power minus k U 2 intersection f to the power minus n what I have here is, I have V 1 intersection f to the power minus k V 2 I know that this is non empty.

Now, same set I can write it as U 1 intersection because this is all intersection right I can just take association here. So, this is minus n V 1 intersection f to the power minus k right I can say that this is U 2 intersection f to the power minus n V 2 now this is not empty right. So, the same set this was non empty. So, if this set is not empty what does that mean? That means, that in particular this is non-empty right and in particular because if I am looking into f minus k of this set right it means that this particular set is also non empty.

So; that means, that U 1 intersection f to the power minus and V 1 this is non-empty and U 2 intersection f to the power minus n V 2 this is non-empty now what does that imply? My n belongs to n U 1 V 1 intersection N U 2 V 2 right and the with started with n to be any element of N U, V right. So, basically means that N U, V is contained in N U 1, U V 1 intersection nu 2 V 2. So, basically if I look into the intersection of hitting time sets of 2 pairs right get that it contains the hitting time set of a third pair right that pair is definitely related in that part.

Now, try to think of this more broadly right if I take the intersection of U 1 V 1 sorry of this U V with some U 3, V 3 right then that would give me that there is another a hitting set contained in the intersection of nu 3 V 3 with U V, but N U,V is already contained in this intersection; that means, that I will have another hitting time set which is contained in N U 1, V 1 intersection N U 2 ,V 2 intersection and U 3 V 3 so; that means, now for 3 pairs U 1 U one U 2 U 1 V 1, U 2 V 2, U 3, V 3 right we find that the hitting time sets as non-empty we could take this up 2 for any finite n right.

So, what does that mean is if I look into these consequences here the consequences are very important here. So, if I look into these consequences here. So, the consequences the first consequence very interesting it says that for if I have U 1, U 2, U n right and V 1, V 2 V n non-empty open this is nonempty what is the meaning of that. In fact, I could look it very broadly what is the meaning of that.

Now, think of that I am saying that x f is weak mixing right.

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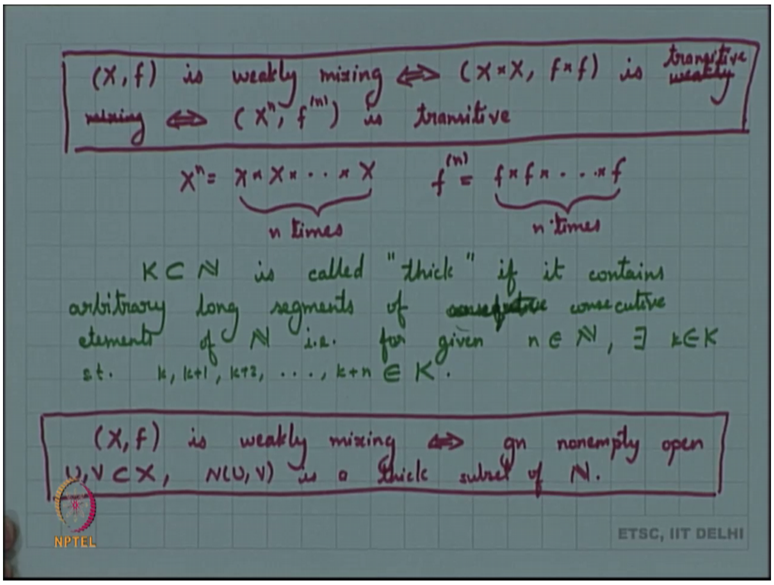

Weakly mixing right, which is same as saying that I am looking into the product of 2 sets right I am looking into the product is weakly mixing, but the recent observation says that 2 implies n right. So, if I take the hitting time sets of ten sets it is non-empty so; that means, that this is same as saying that x to the power n and now I am writing f to the power n right this is weakly mixing or this is sorry this is transitive so; that means, that this is also the n product is also transitive. So, 2 implies n right.

So, what we get here is that this is weakly mixing implies this is the product x cross x f cross f is transitive and; that means, the n product x n. So, we know that what is xn, xn is basically x product with x product with x right n times and f. Now I am writing it in them bracket n because we know we are not looking into the iterate there. So, this is basically f product with f product with f and this product goes n times. So, 2 implies. So, xf is weakly mixing if and only if the n product is transitive interesting case here now this this is one consequence and this gives rise to another consequence. So, for that another consequence let us try to take down a definition here.

So, now I am saying that if I have a subset. So, let me take k to be a subset of the natural numbers this is called thick. So, I am calling this set of natural numbers to be thick, if it contains arbitrarily long segments, it will contains are arbitrary long sequence of consecutive elements of N, what I basically mean here is that now what; that means, here is that for given any n in N right there exists a k in K such that all these elements k, k plus 1, k plus 2, k plus n will belong to k. So, what we have is we have this consecutive elements of n; that means, arbitrary large consecutive elements will be belonging to the set k. So, this set k is called thick.

What we have here is that if my system is weak mixing. So, this is the next observation that we are going to have, and let me write down here we will prove that next page, but I just write down here, that this is weak mixing it is very simple to write if and only if right given non-empty open U, V subset of x right N U, V is a thick subset. So, this is a thick subset of n.

There is another characterization here. So, this is a thick subset of n. Now we try to look into this proof of this part and the proof tends out to be very very simple here. So, what I have here is that let me start with an open pair U and V these are non-empty open subsets.

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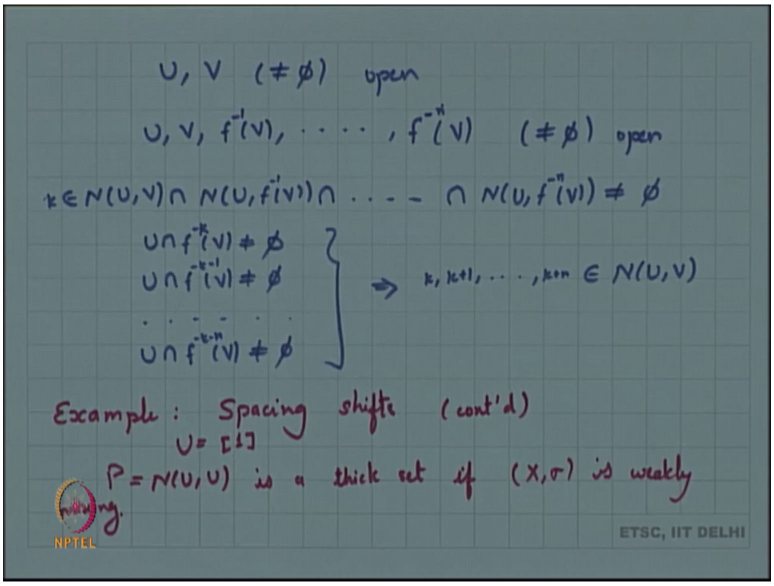

So, it is a non-empty open subsets now since these are non-empty open subsets, I can say that my U, V f inverse V right up to f minus n V or I should say f yeah f n V this is non-empty right these are all non-empty and open what does that mean? If I take my N U, V right it is intersection with nu f inverse V right intersection with N U f minus N V right this is non empty.

So, what happens if a k belongs to the set this intersection? All it means is that right U intersection f minus k V is non-empty right U intersection f minus k minus 1 V is non-empty right U intersection f minus k minus n is non-empty minus V is non-empty and that implies that k, k plus 1, k plus n belongs to N U, V. So, take any hitting set right the system is weak mixing take any hitting set what you find is that this N U,V happens to be a thick set right. So, this is one of the characterization of weak mixing system that this set is thick.

Now, let us again go back to our example of spacing shift right. So, again go back to our example spacing shifts, now when can I say now we again continue. So, we continued this example. So, our U was the cylinder set of one. So, what can we say about N U U? Now I am assuming that the shift is weak mixing right and in that case what can you say what NU U? It should be a thick set right. So, this is a thick set if my system is weakly mixing, now we know that when is this non-empty whenever you are when is that particular thing non-empty right this would be basically just my set P here right this particular for this particular U this is just the case P here right. So, my P is a thick set.

Now, we easily look into the fact that, if I have a co finite set of subset of the natural numbers the subset is co finite then the subset is always going to be thick subset is co finite. It is always going to be thick, but you may have thick subsets which are not co finite it is always possible to have thick subsets which are not co finite.

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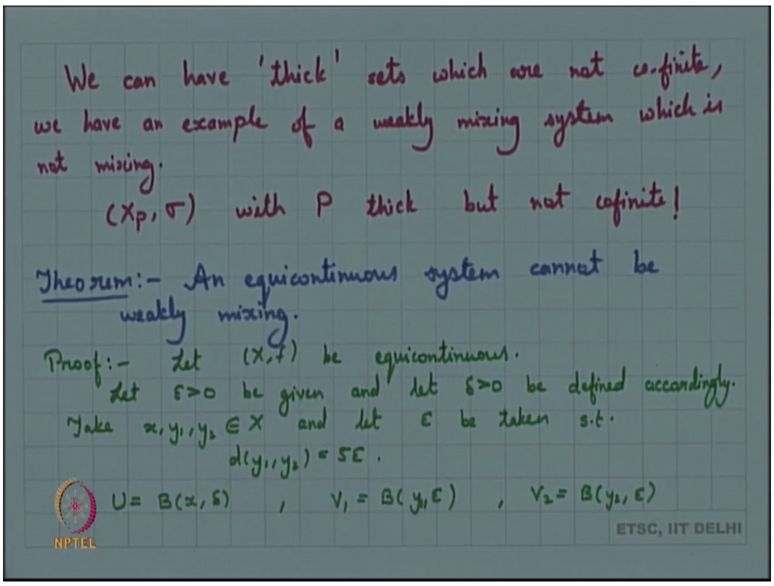

So, since we have thick subsets right we can have thick subsets. I just write thick sets which are not co finite we have an example of a mixing system, which is not weakly mixing sorry we have a system example of a weakly mixing system, which is not mixing we simply take this spacing shift right with P to be a thick set, which is not co finite right.

So, this example is simply our spacing shift Xp right with P thick right, but not co finite. So, if I take this example of spacing shift then we know that this will be weakly mixing this will be weakly mixing, but this will not be mixing right. So, in general weakly mixing is a what to say course of property right it is not stronger it is a weaker property and that is why we call it weakly mixing right. So, weaker property than mixing. So, we know that we have an example of a weakly mixing system which is not mixing.

Now, let us look into some other properties and we compare this with say minimal systems here. So, we know that the system is minimal it will want to compare that with minimal systems. So, all we look into is this just want to write a theorem here, that an equicontinuous system cannot be weakly mixing. So, an equicontinious system cannot be weakly mixing, what we want to check out here is that this would always hold. So, you start with simply proof here. So, now, we know that the system is equicontinuous. So, let x f be equicontinuous.

Now, what does it mean by x f being equi continuous we know that, whenever 2 points are at distance given any epsilon there exist a delta such that whenever 2 points are delta apart right there images will be epsilon apart right.

So, let epsilon be given and let be defined accordingly now take or I am defining my epsilon here. So, what I do is take x y 1, y 2 in x. Let us take this epsilon right such that my distance between y 1 and y 2 is phi epsilon since choosing the epsilon is in our hands we choose our epsilon such that the distance between y 1 and y 2 is phi epsilon.

Now, think of this part we take this open set U to B equal to a ball of radius delta and now with respect to this epsilon we have a delta defined right. So, we take U to be a ball of this a ball of radius delta centered at x, we take our V 1 to be a ball of radius epsilon centered at y 1 and we take our V 2 to be ball of radius epsilon centered at y 2.

And now we want to look into what happens.

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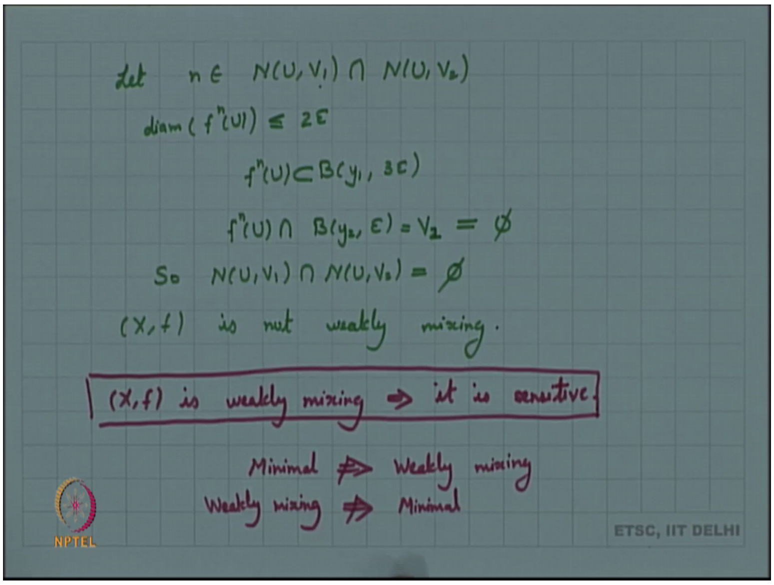

So, let n belongs to. N U, V 1 intersection N U, V 2. So, we have an n belonging to this thing; that means, we are assuming that what happens supposing our system is weakly mixing, then we know that by equicontinuity since my system is equal continuous right all I know is that diameter of f n of U right will be less than or equal to 2 epsilon because my U is having diameter delta right if we know that these are delta apart right. So, the diameter of U epsilon will be less than or equal to twice epsilon.

Now, since the diameter of f n of U is less than or equal to twice epsilon, all I can say is that supposing I am taking a ball right around y 1 of radius 3 epsilon then my f n U 1 f n of U will be contained inside this ball we have equicontinuity right, we have started with a ball of radius delta centered at x right. So, we know that whenever there delta apart, the rest will be epsilon apart right. So, at the most what we have is that the diameter of f n U because n belongs to N U, V 1 right. So, the diameter of f n U will be less than or equal to 2 epsilon in any case it will be less than or 2 epsilon because of equicontinuity will be less than 2 epsilon so; that means, that this diameter is less than 2 epsilon. So, this will be contained supposing I start with the point y 1 right I take an open ball of radius 3 epsilon right then these entire f n U will be contained inside that.

But what does that mean that would mean that f n U intersection this ball which is my V 1 sorry which is my V 2 right this would be empty, because my d of y 1 by 2 is phi epsilon right b of y 1 y 2 is phi epsilon, but what we have is at f n U is contained in this particular ball 3 epsilon around the y 1, but then that would mean that f n U can never intersect V 2 right. So, I can never have an n belonging to this factor right. So, basically that would mean that N U, V 1 intersection N U, V 2 is definitely going to be empty if f n U is comes if n belongs to this factor then n can never belong to this part because of the property of equicontinuity and so, x, f is not weakly mixing when interested more interested in the consequence here. So, we try to look into what is the consequence of this part.

Now, if I try to look into again go back to the first part, we started with x U 1 y 1 by 2 right hey; that means, my point of x right the point x right because if the system is weakly mixing, supposing I assume that the system is weakly mixing then this point x cannot be an equicontinuity point because then I would require that my N U1, V 1 intersection N U2, V 2 should be non-empty so; that means, that this point x that I start with this right this point x that we start with it cannot be an equicontinuity point so; that means, the system has to be sensitive there.

Now, think of that part the system is sensitive there we are in a compact space in a compact space every continuous, mapping is equicontinuous right. So, we can always find some sensitivity constant there right. So, basically the consequence here is that x f is weakly mixing, implies it is sensitive x f is weakly mixing implies it is sensitive. And what we have seen here is that we have minimal systems which are equicontinuous and for minimal systems which are equicontinuous these minimal systems cannot be weakly mixing right.

So, equal continious minimal systems cannot be weakly mixing; that means that my minimality right minimal does not imply weakly mixing. So, different property and again we have this example of tent map which is not a minimal system, but this is weakly mixing. So, my weakly mixing does not imply minimal. We stop here with this observation and we will continue with the same point right in the next lecture.