Chaotic Dynamical Systems
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Lecture – 17
Transitivity = Chaos on Intervals

Welcome to students. So, today we will be looking into the concept that on intervals transitivity implies Devaney chaos. And one of the previous lectures we have seen that Devaney defined chaos right. So, basically if I recall this part then Devaney chaos which I write as dc comprises of 3 elements I have topologically transitivity, which I can write as TT, then I have a dense set of periodic points.

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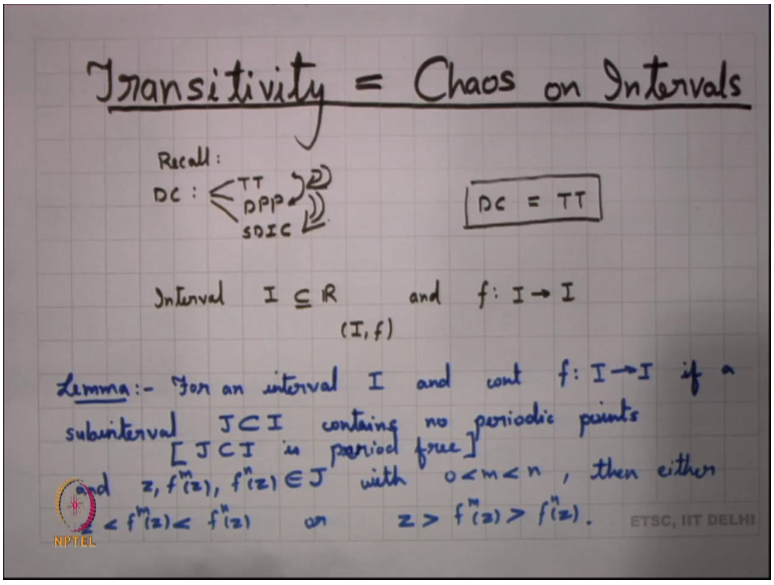

So, maybe I can just write it as dense set of periodic points dp p, right. And I have sensitive dependence to initial conditions. So, which I write as sensitive dependence to initial conditions, right.

So, these are basically the 3 elements of 3 elements that Devaney had isolated to define chaos. And this is our definition of chaos, one of the previous lectures we had seen that; if we have these 2 conditions, right. Then these 2 conditions will imply this third condition. Today we will be looking into something else over here, the main goal for the lecture today will be to prove that if we have a map on an interval. So, if we have an interval system, then basically this implies this factor.

So, topological transitivity implies dense set of periodic points. And hence we can use the earlier theorem to say that then that would imply sensitive dependence on initial conditions also. So, this is good enough to define chaos. So, what happens here is that Devaney chaos on intervals Devaney chaos becomes same as the concept of topological transitivity. So, these 2 concepts are equivalent for intervals. That is what we shall be seen today.

So, what are we going to what are our assumptions today is, we are looking for an interval I in R. And when I talk of an internal we are not looking out for the closed interval open interval, right. We are never specifying what kind of interval it is; it is whether it is a finite bounded interval or unbounded interval, it is any interval. And we have a women map f from I to I. So, we are looking out for basically that system if. And we start the small lemma here.

So, what does this lemma say? So, let me specify this lemma here. My lemma says that for an interval I and continuous f from I to I, if a sub interval J subset of I contains no periodic point, when I say that a sub interval J contains no periodic point, basically means that J subset of I is period free we call it period free interval. There are no periodic points here. So, if this does not contain any periodic points, and if I have my z fmz and fnz all this belonging to J so; that means, z and 2 of it is eat rates belonging to J with 0 less than m less than, n then either then either we have z is less than fmz is less than fnz, or we have z is greater than fmz is greater than fnz.

So that means, if I have a period free interval, and we are looking out for iterates of a single point on that particular interval, then there will be some kind of monotonicity seen, right. In the iterates of that particular point. So, either the iterate on this particular interval will be monotonically increasing, or the iterates will be monotonically decreasing. That is what we mean over here. That is our lemma, and will try to look into the proof of this particular lemma. So, start with proof here.

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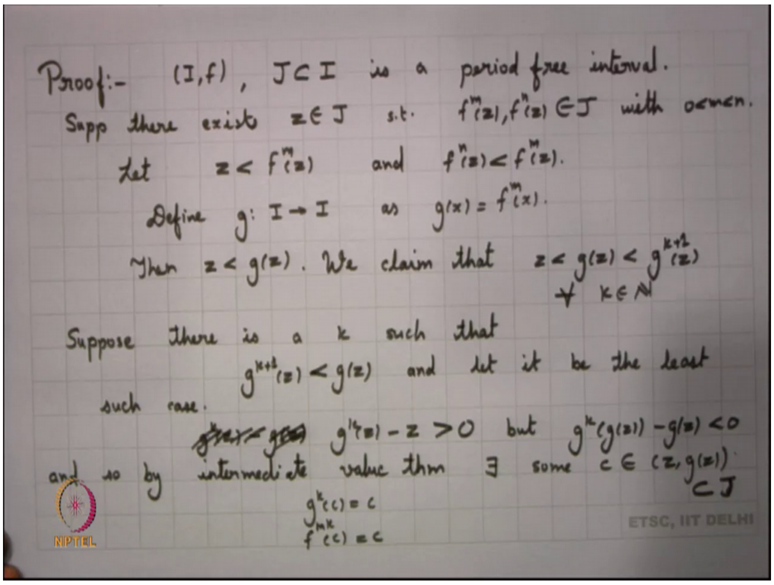

So, our assumption as I said we have our system I f, right. And we have and interval J content in I which is period free, right. And again, our assumptions says that there is suppose there exists z in J such that fmz and fnz also belongs to J, right. With 0 less than m less than n.

Now, we want to prove that either z is less than f mz is less than fnz or z is greater than fmz is lesson greater than fnz. So, what we will do is we will assume that. So, let we will assume to the contrary, let my z will less than fmz, but then I am not saying that fmz is less than fnz, let us assume the other part right. So, let us z will less than fmz, and fnz be less than fnz. So, I am not looking into what is the relation between z and fnz, but I am just saying that, fine let us z be also less than fmz, and let fnz will also less than fmz.

So, this is basically our assumption. And we start with this assumption. So, we defined g. So, we define g again from I to I, right as g of x equal to f m of x. So, that basically we are looking into the mp threat of m f and we are saying that this is same as your g. Then our assumption clearly says that z is less than g of z, right.

Now we claim that z is less than g of z and that is less than g to the power k plus 1 of z for every k belonging to N. So, that means, once you reach gz, right. After gz whatever iterate of g you take up will always be greater than gz. So, that is what our claim is and to prove our claim, right. We look into this assumption we again we want to say we want to prove this part. So, again we start with what happens if this does not happen suppose not.

So, we suppose that there is a k such that your g of k plus 1 of z is less than g of z, suppose it you get some k, right. What we are assuming is that for every k g of z should be less than g of k plus 1 z, but suppose this does not happen. Then there is a k such that this happens. And when this happens maybe it happens for many case, but what we are trying to do is we are trying to take the smallest possible k for which this happens. Suppose there is a k such that this happens, and let it be the least such case right. So, basically yes it could happen for many times, but whatever is the first instant where you get such a k, right. We are just picking up that particular k.

So now what we have this g of k plus 1 z is less than g z. What assumption I have here is that, since I have my z to be less than g z is less than gk plus 1 z, right. For all k less than this part, right. Whenever k was less than this particular say for example, this is a particular k naught, let me start filling with that part right. So, there is a particular k naught for which this happens, but let me call it just k, right. Then what happens here is what I have here is that when I take my g k of z, right. Minus g of z sorry g k of z minus z when I start with that part, right. G k of z minus z when I look into that part, right. Then what I get here is because of this relation because this is a first time it is happening. So, I get that g k of z minus z is positive right, but for this particular k again my g k of g z, right. Minus g z is negative.

Now, we already know that z is less than gz, right. And for this 2 points z and gz, what did we have is that we had this instant k such that gk z minus z is positive and gk of gz minus gz is negative. So, what does this imply, right? So, the intermediate value theorem, right. If I look into this intermediate value theorem; that means, that there is some c there exists some c belonging to z gz, right.

Some c between zn gz such that gk of c will be same as c, but what is my gk my g is f to the power m right. So, my gk is f to the power mk, right. So, that means, f to the power mk of c is equal to c. If I try to look into where does this interval z gz lie, right z and gz, right they both are elements of they both belong to J right. So, this interval is also contained in J.

So, this is basically contained this is basically contained in J. And that means, that I have a point c in J, right. Which happens to be a periodic point for f, but that is a contradiction, right.

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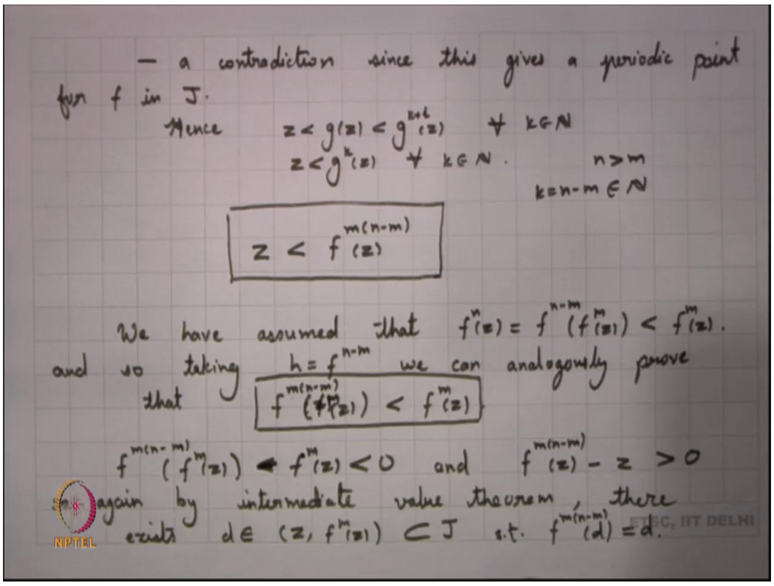

So, we have a contradiction here. This is a contradiction here, and why do we have a contradiction? Since this gives a periodic point for f in J. So, this is not possible, and hence what we have here is z is less than gz, right. Is less than g of k plus 1 z, for every k belongs to N. And in particular I can say that my z is always less than g of kz, right. For every k belonging to N.

I know that my n is greater than m right. So, my n minus m is also an element of n. So, I can say that this gives me that z is less than f to the power now because my g is f to the power m right. So, this is f to the power m into n minus m of z, right. Particularly I am interested in this particular iterate right. So, for k equal to n minus m and simply take k equal to n minus m here.

So, what I get is z is less than f to the power m into n minus n times z. So, let me mark this observation. And now we go back to the next assumption that we have taken up. We have assumed that fn of z which I can say that it is f of n minus m of fm of z, we have assume that this is less than fm of z. And so, again we can take h to be equal to f to the power n minus m, right. If I define my h from I to I as f to the power n minus m, right. We can analogously prove, now this is less than this part right.

So, it is just an analogous we just need to take the mirror arguments here, right. We can prove that f to the power m into n minus m of z, right of let me take fmz because we started with fmz, right.

This is less than fmz, we just the same arguments is giving us that I am taking my h equal to f to the power n minus m, right. So, that means, that would give me that h of fmz is less than fnz, right. We start with h of fmz is less than fmz and then we continue the same thing to show that, any power of h will be less than any power of h, right. Of fz will be less than f of fmz will be less than fmz. And that gives us again looking into the similar construction that gives us that f to the power m into n minus m of fmz is less than fmz.

Now, what does this particular thing give us, right? That we have so, let me take this also as my next observation. So, on one hand what I have here is f to the power m times n minus m, right. Of fmz is less than fmz or minus fmz is less than 0, right. And what here I have is from this factor I can say that f to the power m times n minus m of z minus z, right. This is greater than 0. I already know that my fmz is greater than z we have already taken that part right.

So, what we have is we have a d, right. And again, my intermediate value theorem. So, again by intermediate value theorem what we have is there exists dE belonging to this interval zfmz, right. And in very well know that this interval is contained in J, right. Such that f to the power m times n minus m of d is equal to d, right. But this again gives me a contradiction, right. This again gives a contradiction because; that means, that I have a periodic point for f in J right.

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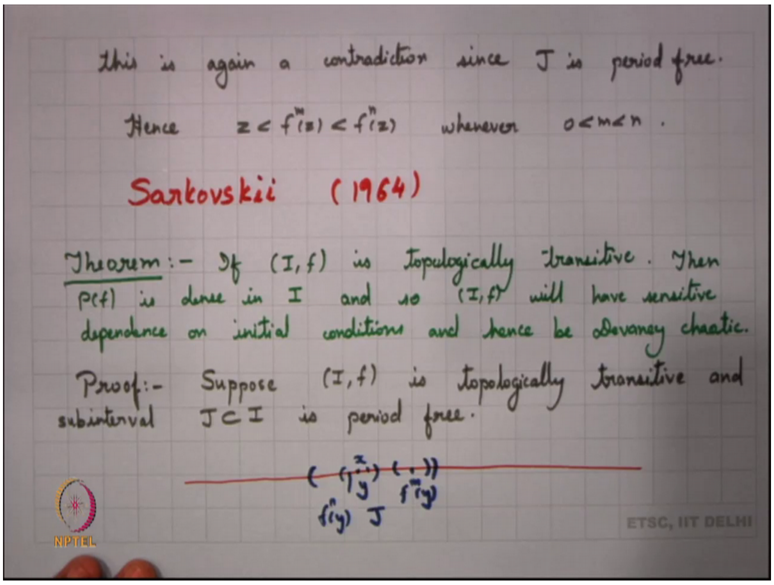

So, this is again a contradiction, since J is period free.

Now why didn't we get this contradiction? So, basically, we got this contradiction because we assume that our z is less than fmz, but fmz is also less than fmz. And hence we can say that z is less than fmz, right. It is less than fmz whenever your 0 is less than m is less than n. So, if you have a period free interval, then in the period free interval if there is any orbit entering the period free interval, then it will move monotonically. So, either it will move either it will be monotonically increasing or it will be monotonically decreasing, but it will be moving in a monotonic manner.

So, that is our observation, and basically this observation we can say that this observation was made by Swarovski. So, this is something attributed to Swarovski, maybe sometimes in 1964. And we all seen that Swarovski gave a very nice ordering of what periodic points can occur on the real line right, but it was his observation, that if the system is topologically transitive then there will be a dense set of periodic points. And this is just a part lemma is just a part of tech theorem.

So, Murkowski made is observation in 64 of course, long back that if you have if your system is topologically transitive your internal system is topologically transitive. Then it will have a dense set of periodic points, but then when people started observing Devaney's definition of chaos. And then they observe that sensitive dependence of initial condition is given by the combination of topological transitivity and dense set of periodic points, right. One could easily use Murkowski observation, and say that what happens in case of intervals that topologically transitivity gives you dense set of periodic points. And hence it gives you sensitive dependence on initial conditions. So, transitivity is good enough to define Devaney chaos on an interval system. So, will look into this observation and the proof that we are going to look into is not the proof due to Swarovski. Of course, this paper is in Russian I have not read it, but we will look into the proof given by Vellekoop and Berglund in 1994 that appeared in American mathematical monthly.

So, we are looking to that particular proof here, but let me state the main theorem here. So, the main theorem can be stated here in this manner. So, this theorem says that if this interval system is topologically transitive, then the set of periodic points is dense in I. And so, this system will have sensitive dependence on initial conditions, and hence be Devaney chaotic.

If we try to look into this particular theorem all we need to prove is that we need to assume that our system is topologically transitive, and we need to show that periodic points are dense; that means, all we are trying to show is that there is no period free interval. And what are we going to use here is that we are going to use the observation that we just make that if we have a period free interval, then your orbits will always come in a monotonic manner right. So, there will be no mix up there your orbits already come in the interval in a monotonic manner if there is a period free interval. We just going to make this observe we just going to use this observation here to show that periodic points will be dense.

So, let us start with the proof here. So, I am assuming that my system is topologically transitive. And since we want to prove that the periodic points are dense, right. We can start with the assumption that there is a period free interval right. So, suppose this is topologically transitive, and a sub interval J is period free. As usual we wish to get some kind of contradiction coming out of this assumption. So, we start with this concept. Now what is our idea here is; so, the basic idea that we will be using here is you start with say some kind of an interval here. So, you have this is basically your part of your line I, and what your starting here is your starting with say some interval here J which is period free. You start with some point x here, right. Then you can find a neighborhood of x here.

So, you start with an x here you find a neighborhood of x here, such that again in J itself you can find another open set such that these 2 open sets are disjoint.

So, this we are trying to find intervals on the you are not going beyond intervals you can find an interval around x, such that again if we look into the complement if looking to the same interval again sub interval. So, this is a basically a sub interval J you are working in J, now not going out of J. Then you find that there is some open set here which is disjoint from this spot. So, that means, now I, but by transitivity you know that there will be a point y here, right.

Such that for some m you will find fmy coming over here and then you want to look into the fact is that your y is here your fmy is here, but you will find some n again using transitivity. You will find some n greater than m such that your fmy fny again comes here. So, it comes in this particular part, but then what does not give you does not give you a monotonicity at all, right.

Because your y maybe fny is your does not matter where it is, but you do not get y less than fmy less than fmy, because that is what should happen ideally, right. If your interval is period free, but that is what does not happen here. So, there is some kind of mixture see in the orbits of this particular point y, right. And that contradicts that J is period 3. And so, you cannot find any interval J which is period free, and we know that intervals are basic open sets here.

So, there is no; that means, your periodic points happen to be dense here. So, this is basically the crux of the proof. And let us systematically write the proof now.

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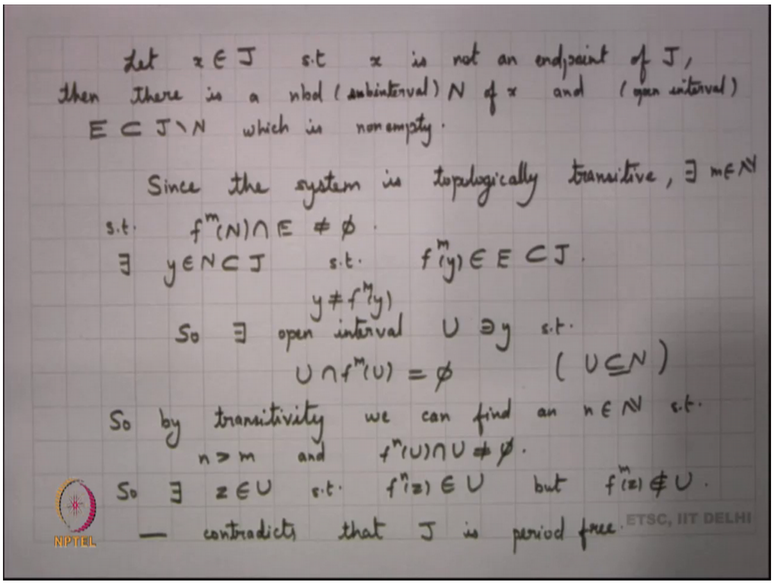

So, we start with an x in J such that your x is not an endpoint of J. Then there exists a neighborhood when I am talking of a neighborhood I would say it is an interval, right. Or should say it is a sub interval, n of x and some again some interval. So, I should say an open interval E which is contained in J minus N. So, we find a neighborhood n of x definitely n is non-empty. I can find an open interval J contained in J minus N, right. An open interval E contained in J minus N, right which is non-empty. And we know that this is always possible, right.

So, what we have is; so, system since system is transitive that existent m in N, right. Such that f m of n intersection E is non-empty. So, what does that mean, right? Basically, that means, that there exists a point y belonging to N which further is a subset of J, such that f m of y, right. Belongs to E, and again we know that E is also subset of J. So, we find a y and J such that fm of y belongs to J, but now my y is in N, right. And my fm of y is in E. And we know that E is contained in the complement of N. So, we are very sure that y is not equal to fmy, right. These 2 are distinct points.

Now, since these 2 are distinct points. I can have a neighborhood of y, right. Such that if I take the m th image of this neighborhood, right. Then that does not intersect. So, there exists an open interval U containing y, right. Such that U into section fmu, right is empty. These 2 sets do not intersect. And we very well know that since U is an interval you are fmu also is a connected subset right. So, this will also be an interval.

So, your un fm U they do not intersect at all and our U is an interval in y, and I can assume that further assume that U is basically a subset of N. I can think of that also right. So, I can take because my y was from N, N is an open set, right. I can always assume that my U is an interval, right. Which is contained an xl, right in so, I am now fixing where my you happens to be, right.

And we know that E was somewhere in the complement of N. So, it is always possible I can have this kind of construction. So, can of further suppose that now U was basically contained in N. Now what do you have? You have this open send u. So, for this particular U you can have an n, which is greater than m such that U intersection f nu is non-empty. So, my transitivity we can find an n such that first of all my n is greater than m and my fn of U intersection U is non-empty since this is non-empty, right. I get a z belongs to U such that my fnz also belongs to you.

So, I have my z belonging to U fnz belonging to U right, but I know that fmu and you do not intersect at all right, but since may z belongs to U my fm of z will belong to fm of U. So, that means, I can say that my fm of z, right does not belong to U. And we have taken our U to be an interval, right. So, that means, that I have 0 less than m less than n, such that z and fm fmz they are in 1 place, but f of m z be somewhere over here. So, there is some kind of a mixture seen here, right. Which gives us a contradiction which contradicts the fact that this is J is period free right.

So, this gives a contradiction that J is pre-period free. So, this contradicts and hence J is not period free, what does that mean, right? So, you can conclude that basically, no in no subinterval of is period free, right. Which means that there is a dense set of a periodic points.

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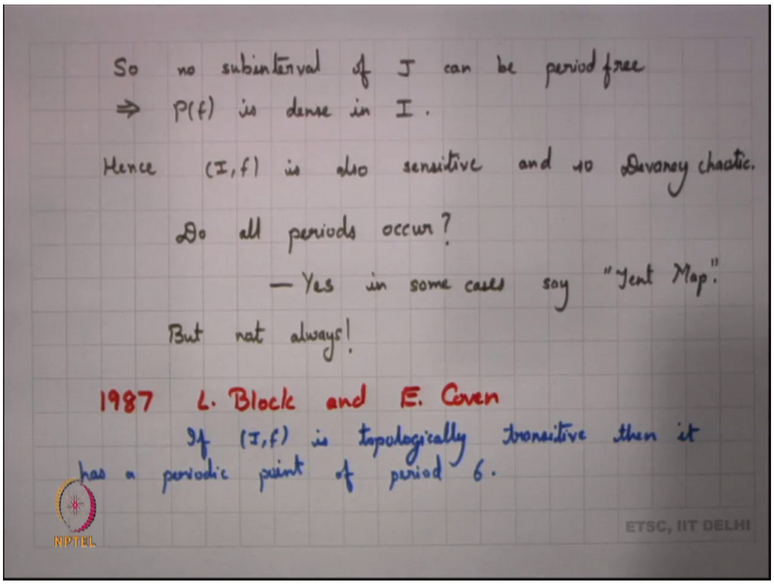

So, no sub interval of J can be period free, and since no subinterval of J can be period free this implies that a set of periodic point of f in is dense in I.

Now, we can use the theorem that we did in one of the previous lectures, that if you have transitivity, and if you have dense set of periodic point. Then that gives you sensitive dependence to initial conditions. And so, we can using that we can say that hence if is also sensitively depending or sensitive, right. Since this is also sensitive. And so, Devaney chaotic. On an interval if we have transitivity, right. That gives you chaos.

Now, what basically observation what observation basically is needed over here is that for a transitive map, right. The set of periodic points is dense in intervals. If you have an interval the set of periodic points if you have a system on an interval the set of periodic points is dense. We already know that for an interval we have a nice Swarovski ordering giving us, what kind of what periods of period is what will be the periods of the periodic points that occur. But as such we do not know what kind of periodic points could what is the possibility of periodic points that can occur. We already know that this is not this is not true right.

So, basically, we first need to know do all periods occur here. So, when I looking into a transitive system, we want to know because of Swarovski ordering we know that that I there will exist periodic point there at least there exist one periodic point, right. That is definitely clear because periodic points are dense, in finitely many periodic points, but what will be the period of this periodic point. Very important to see what kind of periods will come up. So, what are all the periods that can occur for an interval system.

Now, when we try to look into what are all the periods that can occur we know that if there is a periodic point of period 3. Then all periods will occur by Swarovski theorem. So, we have seen that all periods occur this is a possibility, right. And we have already seen that this possibility is true in some cases right. So, this is yes in some cases can if I quickly recall what kind of cases it was. So, we can say that tent map was standard example.

. So, in the tent map we did have this case that all periods occur, but still can we say something else we only know that the system is transitive. So, trying to look into so, trying to look back into our previous examples that we had seen in one of the lab last lectures, that transitivity in general does not guarantee even the existence of any periodic point. For example, if you take the rational rotation we do not have any periodic point. So, transitivity in general does not guarantee anything.

Here in transitivity does guarantee that there is a dense set of periodic points. But it cannot guarantee what periodic points will occur. Once I know that I have a periodic point of certain period, then I can guarantee that the rest of the period which are less than that Swarovski ordering will always occur. But we are not sure what kind of periods will all will occur in this particular system.

So, this is not always true, right. Not always you cannot always find out periodic points of all periods here. So, this is not true in general. So, this is still open this problem is still open that you guarantee what kind of given any system, right. You guarantee what kind of period can occur over here for example, if supposing I have one periodic point of an odd period, right. Then I know that the even periods are all the even numbers are all less than the odd period in this ordering. So, that means, I can guarantee that all even periods will also occur. And in fact, there will be many odd periods that also occur.

So, still it is not known that what could be the possible or what is a necessary and sufficient condition. That we have an odd periodic point, but what is known here is which I mentioned that was proved by block and coven in 1987. So, 1987 L block and E coven they proved a certain theorem maybe I can state it as if my interval system is topologically transitive, then it has a periodic point of period 6.

So now we know that if it has a periodic point of period 6, then 6 is the largest even number in Swarovski ordering. Since 6 is the largest even number in Swarovski ordering; that means, that all even periods occur. So, that means, when I am talking of topologically transitive interval system, right. I am saying that definitely all even periods occur, but what about the odd periods.

So, it is not that in certain circumstances you can guarantee the existence there are certain conditions under which you can guarantee the existence of an odd period. But still we cannot specify which odd period would be there, but there is definitely a guarantee, that there will be some kind of an odd period occurring. And that again that condition is related to the concept of transitivity. But we will look into that concept maybe sometime later. And today we stop here. And I hope this has been clear to all of you.